UDC 519.2
DOI: https://doi.org/10.17721/1812-5409.2025/2.23

Andrii CHAIKOVS'KYI, DSc (Phys. & Math.), Prof.
ORCID ID: 0000-0002-3357-8744
e-mail: andriichaikovskyi@knu.ua
Taras Shevchenko National University of Kyiv, Kyiv, Ukraine

Oleksandr LIUBIMOV, Student
ORCID ID: 0009-0009-9347-6297
e-mail: liubimov_oleksandr@knu.ua
Taras Shevchenko National University of Kyiv, Kyiv, Ukraine

Oksana LAHODA, PhD (Phys. & Math.), Assoc. Prof.
ORCID ID: 0000-0003-4515-1753
e-mail: lahoda.oksana@lll.kpi.ua
National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute", Kyiv, Ukraine

BOUNDEDNESS OF SOLUTIONS OF THE FIRST ORDER LINEAR MULTIDIMENSIONAL DIFFERENCE EQUATIONS

We investigate the boundedness of solutions of the first order linear difference equation of the form $x_{n+1} = Ax_n + y_n,\ n \geqslant 1$ where A is a square matrix with complex entries, sequence $\{y_n\}_{n\geqslant 1}$ and initial value x_1 are supposed to be known. Firstly, we discuss the one-dimensional case of this equation $x_{n+1} = ax_n + y_n,\ n \geqslant 1$ where a is a complex number. In particular, we obtain the sufficient conditions for boundedness or unboundedness of the solutions in case $|a| = 1$(the critical case) by considering the exponential sums of the forms $\sum y_n e(n\varphi)$ and $\sum e(f(n))$.

Then we proceed to the investigation of the equation in the multidimensional case and reduce our problem to the analysis of the spectrum and Jordan cells of the matrix A. The problem is especially interesting when the spectrum of A contains eigenvalues λ with $|\lambda| = 1$. At the end of the article we obtain a theorem that reveals the connection between equations $x_{n+1} = ax_n + y_n,\ n \geqslant 1$ with $|a| = 1$ and $x_{n+1} = Jx_n + y_n,\ n \geqslant 1$ with J being a Jordan cell of an eigenvalue λ, $|\lambda| = 1$.

K e y w o r d s: ***Linear difference equation, bounded solution, exponential sums.***

AMS 2020 classification: 39A06, 11L03, 11L07.

Introduction

In space $\mathbb{C}^m$ with Euclidean norm we consider the following difference equation

$$x_{n+1} = Ax_n + y_n,\ n \geqslant 1 \tag{1}$$

with respect to unknown sequence $\{x_n : n \geqslant 1\} \subset \mathbb{C}^m$. The first element of this sequence x_1, sequence $\{y_n : n \geqslant 1\} \subset \mathbb{C}^m$ and square matrix $A \in \mathcal{M}_m(\mathbb{C})$ of order m are supposed to be known. Various necessary and sufficient conditions for the existence and uniqueness of the bounded solution of this equation were previously investigated by different authors in finite-dimensional and infinite-dimensional cases (see (Gomilko, Gorodnii, & Lagoda, 2003), (Horodnii, & Lahoda, 2001), (Chaikovs'kyi, & Lagoda, 2022), (Gorodnii, & Kravets', 2020) and cited literature there).

Difference equation (1) has the following well-studied cases:

1) If spectrum of A is such that $\sigma(A) \subset \{z \in \mathbb{C} \mid |z| < 1\}$ then for any $x_1 \in \mathbb{C}^m$ the solution is bounded;

2) If $\sigma(A) \subset \{z \in \mathbb{C} \mid |z| > 1\}$ then only for $x_1 = -\sum_{k=1}^{\infty} A^{-k-1} y_k$ the solution is bounded;

3) If $\sigma(A) \cap \{z \in \mathbb{C} \mid |z| = 1\} = \varnothing$ then it is possible to decompose the space into a cartesian product of two spaces, which satisfy the conditions of cases 1 and 2.

On the other hand, the case $\sigma(A) \cap \{z \in \mathbb{C} \mid |z| = 1\} \neq \varnothing$ always requires some additional investigation because the simple characterization for it is not known. Existence of the bounded solution of the difference equation (1) in this case depends on the input sequence $\{y_n : n \geqslant 1\}$ in a rather complicated way in comparison with the cases 1)-3). Our aim is to establish some interesting and significant instances of such a dependency.

1. One-dimensional equation

We consider the difference equation of the form

$$x_{n+1} = ax_n + y_n \tag{2}$$

with respect to unknown sequence $\{x_n\}_{n\geqslant 1}$. Numbers $x_1,\ a \in \mathbb{C}$ and the sequence $\{y_n\}_{n\geqslant 1} \subset \mathbb{C}$ are supposed to be known.

By induction, it is not difficult to deduce that the solution of the equation (2) is

$$x_n = a^{n-1}x_1 + \sum_{k=1}^{n-1} a^{n-k-1} y_k,\quad n \geqslant 1.$$

For the critical case $|a| = 1$ we investigate the conditions for $\{y_n\}_{n\geqslant 1}$ and x_1 under which the solution $\{x_n\}_{n\geqslant 1}$ is bounded or unbounded.

We begin this section with the following observation.

Proposition 1. *If* $|a| = 1$*, then the solution of the difference equation* (2) *is bounded* $\Longleftrightarrow \exists M > 0\ \forall K \geqslant 0$ *:*

$$\left|\sum_{k=1}^{K} \frac{y_k}{a^k}\right| \leqslant M.$$

Proof. The proof is a simple consequence of the solution formula above. □

The question of boundedness of solution $\{x_n\}_{n\geqslant 1}$ in case $|a| = 1$ is rather difficult. Unlike in other cases, the simple and full characterization is still not known. So, particular results with sufficient conditions of boundedness are interesting.

We write $a = e^{-2\pi i\varphi}$, where $\varphi \in [0,1)$. Therefore according to the Proposition 1 solution $\{x_n\}_{n\geqslant 1}$ of the difference equation (2) is bounded if and only if the sequence of sums

$$\left\{\sum_{n=1}^{N} y_n e^{2\pi i n\varphi} : N \geqslant 1\right\}$$

is bounded.

Thus, we see that our problem is equivalent to the study of the boundedness of the exponential sums written above.

For real number x define $e(x) := e^{2\pi i x}$.

In this subsection, we are solely concerned with the boundedness of exponential sums

$$\left\{\sum_{n=1}^{N} e(f(n))\ :\ N \geqslant 1\right\}, \tag{3}$$

where $f : \mathbb{N} \to \mathbb{R}$ is some function.

Definition 1. For the function $f : \mathbb{N} \to \mathbb{R}$ we define the **finite difference of** f to be

$$\Delta f(n) := f(n+1) - f(n)$$

and we define the **second order finite difference of** f to be

$$\Delta^2 f(n) := \Delta f(n+1) - \Delta f(n) = f(n+2) - 2f(n+1) + f(n).$$

One of the interesting results connected with the question of boundedness of exponential sums is following.

Theorem 1 (Kusmin-Landau Inequality)**.** *Suppose function* f *and real number* θ *are such that*

$$0 < \theta \leqslant \Delta f(1) \leqslant \Delta f(2) \leqslant \ldots \leqslant \Delta f(N-1) < 1 - \theta.$$

Then

$$\left|\sum_{n=1}^{N} e\left(f(n)\right)\right| \leqslant \cot\left(\frac{\pi\theta}{2}\right).$$

Proof. For the proof and detailed discussion, see the paper of Mordell (Mordell, 1958). □

We would like to remark that if function f has monotone derivative f' on segment $[1, N]$ and $\theta < f'(x) < 1 - \theta$, then from Lagrange's mean value theorem it follows that f satisfies the conditions of Theorem 1.

The proof of Kusmin-Landau Inequality written in the book of Graham and Kolesnik (Graham, & Kolesnik, 1991) presents us an interesting and useful identity.

Proposition 2.

$$-\sum_{n=n_0}^{N} e(f(n)) = \frac{1}{2}\left(1 + i\cdot\cot(\pi\Delta f(N+1))\right)\cdot e(f(N+1)) - \frac{1}{2}\left(1 + i\cdot\cot(\pi\Delta f(n_0))\right)\cdot e(f(n_0)) -$$

$$-\frac{1}{2}\sum_{n=n_0}^{N} e(f(n+1))\cdot i\left(\cot(\pi\Delta f(n+1)) - \cot(\pi\Delta f(n))\right).$$

Proof. We write

$$e(f(n)) = \frac{e(f(n+1)) - e(f(n))}{e(\Delta f(n)) - 1} = -\frac{1}{2}(e(f(n+1)) - e(f(n)))\cdot(1 + i\cdot\cot(\pi\Delta f(n)))$$

and using Abel's summation by parts formula, we obtain the desired identity. □

Boundedness of exponential sums $\sum_{n\geqslant 1} e(n\alpha),\ \alpha \in (0,1)$ and geometric intuition suggest us a natural question:
If $\Delta f(n) \to \psi \notin \mathbb{Z}$ *as* $n \to \infty$*, is it true that the sequence of exponential sums* (3) *is bounded?*

And the answer is Yes! if $\Delta f(n)$ converges fast enough. With the aid of Proposition 2, we prove the following Theorem.

Theorem 2. *Let f be such that $\sum_{n\geqslant 1}|\Delta^2 f(n)| < \infty$. Then:*

1. $\Delta f(n) \to \psi \in \mathbb{R}$ *as* $n \to \infty$.
2. *If $\psi \notin \mathbb{Z}$ then sequence of exponential sums*

$$\left\{\sum_{n=1}^{N} e(f(n)) \ : \ N \geqslant 1\right\}$$

is bounded.

Proof. Firstly, since $\sum_{n\geqslant 1}|\Delta^2 f(n)| < \infty$ series $\sum_{n\geqslant 1}\Delta^2 f(n)$ is convergent. Therefore, because

$$\Delta f(N) = \Delta f(1) + \sum_{n=1}^{N-1}\Delta^2 f(n),$$

we deduce that $\exists \lim_{n\to\infty}\Delta f(n) =: \psi \in \mathbb{R}$.

Now suppose that $\psi \notin \mathbb{Z}$. There exist some $m \in \mathbb{Z}$, $n_0 \in \mathbb{N}$ and $\theta \in (0,1)$ such that for all $n \geqslant n_0$ we have

$$m + \theta \leqslant \Delta f(n) \leqslant m + 1 - \theta.$$

It means that $|\sin(\pi\Delta f(n))| \geqslant \sin(\pi\theta) =: T$ for all $n \geqslant n_0$.

Thus for all $n \geqslant n_0$ we have

$$|\cot\pi(\Delta f(n+1)) - \cot(\Delta f(n))| = \frac{|\sin(\pi\Delta f(n+1) - \pi\Delta f(n))|}{|\sin(\pi\Delta f(n+1))||\sin(\pi\Delta f(n))|} =$$

$$= \frac{|\sin(\pi\Delta^2 f(n)|}{|\sin(\pi\Delta f(n+1))||\sin(\pi\Delta f(n))|} \leqslant \frac{\pi|\Delta^2 f(n)|}{T^2}.$$

We conclude that for all $N \geqslant n_0$ we have:

$$\left|\sum_{n=n_0}^{N} e(f(n+1))\cdot i\left(\cot(\pi\Delta f(n+1)) - \cot(\pi\Delta f(n))\right)\right| \leqslant$$

$$\leqslant \sum_{n=n_0}^{N}|\cot(\pi\Delta f(n+1)) - \cot(\pi\Delta f(n))| \leqslant \frac{\pi}{T^2}\sum_{n=n_0}^{N}|\Delta^2 f(n)| \leqslant \frac{\pi}{T^2}\sum_{n\geqslant 1}|\Delta^2 f(n)| < \infty.$$

Also

$$|(1 + i\cdot\cot(\pi\Delta f(N+1)))\cdot e(f(N+1))| \leqslant 1 + |\cot(\pi\theta)|.$$

Thus, according to Proposition 2, sequence of exponential sums $\left\{\sum_{n=1}^{N} e(f(n))\right\}_{N\geqslant 1}$ is bounded. □

With the help of this Theorem, we can, for example, prove boundedness of the following sequences of exponential sums.

Example. 1. For all $\varphi \in \mathbb{R}\setminus\mathbb{Z}$

$$\left\{\sum_{n=1}^{N} e\left(n\varphi + \sqrt{n}\log(n)\right) : N \geqslant 1\right\}.$$

2. For arbitrary $s > 1$ and $\varphi \in (0,1)$, $\varphi + \sum_{k\geqslant 2}\frac{\sin(k)}{k\log^s(k)} \notin \mathbb{Z}$

$$\left\{\sum_{n=2}^{N} e\left(n\varphi + \sum_{r=2}^{n}\sum_{k=2}^{r}\frac{\sin(k)}{k\log^s(k)}\right) : N \geqslant 2\right\}.$$

We proceed to another observation, which is also inspired by geometric intuition.

Theorem 3. *If $\frac{1}{N}\sum_{n=1}^{N}|\Delta f(n)| \to 0,\ N \to \infty$, then the sequence of exponential sums*

$$\left\{\sum_{n=1}^{N} e(f(n)) \ : \ N \geqslant 1\right\}$$

is unbounded.

Proof. Put $S(N) := \sum_{n=1}^{N} e(f(n))$ for $N \geqslant 1$. We suppose by contradiction that $\exists M > 0\ \forall N \geqslant 1\ :\ |S(N)| \leqslant M$.

Using Abel's summation by parts formula, we get:

$$|S(N) - N| = \left|\sum_{n=1}^{N}(e(f(n)) - 1)\right| = \left|\sum_{n=1}^{N} e(f(n))(1 - e(-f(n)))\right| =$$

$$= \left| (1-e(-f(N+1)))S(N+1) - (1-e(-f(1)))S(1) + \sum_{n=1}^{N}(e(-f(n+1)) - e(-f(n)))S(n+1) \right| \leqslant$$

$$\leqslant 4M + \sum_{n=1}^{N} |e(-f(n+1)) - e(-f(n))||S(n+1)| \leqslant 4M + M\sum_{n=1}^{N} |-f(n+1)+f(n)| =$$

$$= 4M + M\sum_{n=1}^{N} |\Delta f(n)|.$$

There exists some N_0 such that for all $N \geqslant N_0 : \sum_{n=1}^{N} |\Delta f(n)| \leqslant \frac{N}{2M}$ and therefore

$$N - |S(N)| \leqslant |N - |S(N)|| \leqslant |S(N) - N| \leqslant 4M + \frac{1}{2}N,$$

which implies that $|S(N)| \geqslant \frac{1}{2}N - 4M$. Contradiction. □

Corollary 1. *If $\Delta f(n) \to \psi \in \mathbb{Z},\ n \to \infty$, then the sequence of exponential sums*

$$\left\{ \sum_{n=1}^{N} e(f(n))\ :\ N \geqslant 1 \right\}$$

is unbounded.

Proof. Since $e(f(n)) = e(f(n) - n\psi)$ and $\Delta(f(n) - n\psi) = \Delta f(n) - \psi$ without loss of generality, we may suppose that $\Delta f(n) \to 0,\ n \to \infty$. Hence, by Stolz-Cesàro Theorem, $\frac{1}{N}\sum_{n=1}^{N} |\Delta f(n)| \to 0,\ N \to \infty$. The corollary follows from the previous Theorem. □

Now we write $a = e^{-2\pi i \cdot \varphi} = e(-\varphi)$, where $\varphi \in [0,1)$, and study boundedness of the solutions of difference equation:

$$x_{n+1} = e(-\varphi)x_n + y_n,\ n \geqslant 1. \tag{4}$$

We would like to remark that Proposition 1 implies that the boundedness of solutions of (4) depends only on φ and $\{y_n\}_{n\geqslant 1}$ and does not depend on the initial value x_1.

Proposition 3. *Every solution of the difference equation* (4) *is bounded if and only if the following sequence of sums*

$$\left\{ \sum_{n=1}^{N} y_n \cdot e(n\varphi)\ :\ N \geqslant 1 \right\}$$

is bounded.

Proof. Follows from the Proposition 1. □

Definition 2. By $\mathbb{W}_\varphi$ we denote the set of all sequences $\{y_n\}_{n\geqslant 1} \subset \mathbb{C}$ for which every solution of difference equation (4) is bounded.

We remark that $\mathbb{W}_\varphi$ is not empty because $\{0\}_{n\geqslant 1} \in \mathbb{W}_\varphi$. We make the following easy observation.

Proposition 4. *For every $\varphi \in [0,1)$ the following holds:*

- *If $\{z_n\}_{n\geqslant 1}, \{w_n\}_{n\geqslant 1} \in \mathbb{W}_\varphi$ then $\{z_n + w_n\}_{n\geqslant 1} \in \mathbb{W}_\varphi$.*
- *If $\alpha \in \mathbb{C}$ and $\{z_n\}_{n\geqslant 1} \in \mathbb{W}_\varphi$ then $\{\alpha z_n\}_{n\geqslant 1} \in \mathbb{W}_\varphi$.*
- *If $\{z_n\}_{n\geqslant 1} \in \mathbb{W}_\varphi$, then $\{z_n\}_{n\geqslant 1}$ is a bounded sequence.*
- *If $\{b_n\}_{n\geqslant 1} \subset \mathbb{R}$ is monotone bounded sequence and $\{z_n\}_{n\geqslant 1} \in \mathbb{W}_\varphi$, then $\{b_n z_n\}_{n\geqslant 1} \in \mathbb{W}_\varphi$.*

Corollary 2. *For every $\varphi \in [0,1)$ $\mathbb{W}_\varphi$ is a vector space over $\mathbb{C}$. In addition, it is the vector subspace of the vector space of all bounded sequences in $\mathbb{C}$.*

Theorem 4. *Let f be such that $\sum_{n\geqslant 1} |\Delta^2 f(n)| < \infty$. Then $\Delta f(n) \to \psi \in \mathbb{R}$ as $n \to \infty$. If $\psi + \varphi \notin \mathbb{Z}$ and $y_n = e(f(n))$, $n \geqslant 1$, then every solution of* (4) *is bounded, that is, $\{e(f(n))\}_{n\geqslant 1} \in \mathbb{W}_\varphi$.*

Proof. Put $g(n) := f(n) + n\varphi$. Then $\Delta^2 g(n) = \Delta^2 f(n),\ n \geqslant 1$ which means that $\sum_{n\geqslant 1} |\Delta^2 g(n)| < \infty$. In addition $\Delta g(n) \to \psi + \varphi,\ n \to \infty$ and

$$\left\{ \sum_{n=1}^{N} y_n \cdot e(n\varphi)\ :\ N \geqslant 1 \right\} = \left\{ \sum_{n=1}^{N} e(g(n))\ :\ N \geqslant 1 \right\}.$$

By Theorem 2 sequence of exponential sums $\left\{ \sum_{n=1}^{N} e(g(n)) \right\}_{N\geqslant 1}$ is bounded. Hence, by Proposition 3, every solution of (4) is bounded. □

Example. For all $\varphi \in (0,1)$ every solution of the difference equation

$$x_{n+1} = e(\varphi)x_n + e(n^{\frac{1}{2}} - n^{\frac{1}{3}} + \log(n)),\ n \geqslant 1$$

is bounded.

Theorem 5. *Let f be such that $\Delta f(n) \to \psi \in \mathbb{R}$ as $n \to \infty$. If $\frac{1}{N}\sum_{n=1}^{N} |\Delta f(n)+\varphi| \to 0,\ N \to \infty$ and $y_n = e(f(n)),\ n \geqslant 1$, then every solution of* (4) *is unbounded, that is, $\{e(f(n))\}_{n\geqslant 1} \notin \mathbb{W}_{\varphi}$.*

Proof. Put $g(n) := f(n) + n\varphi$. Then $\Delta g(n) = \Delta f(n) + \varphi,\ n \geqslant 1$. So $\frac{1}{N}\sum_{n=1}^{N} |\Delta g(n)| \to 0,\ N \to \infty$. Therefore, by Theorem 3 sequence of exponential sums $\left\{\sum_{n=1}^{N} e(g(n))\right\}_{N\geqslant 1}$ is unbounded.

Because $\left\{\sum_{n=1}^{N} y_n \cdot e(n\varphi)\right\}_{N\geqslant 1} = \left\{\sum_{n=1}^{N} e(g(n))\right\}_{N\geqslant 1}$, Proposition 3 implies that every solution of the difference equation (4) is unbounded. □

Corollary 3. *Let f be such that $\Delta f(n) \to \psi \in \mathbb{R}$ as $n \to \infty$. If $\psi + \varphi \in \mathbb{Z}$ and $y_n = e(f(n)),\ n \geqslant 1$, then every solution of* (4) *is unbounded, that is, $\{e(f(n))\}_{n\geqslant 1} \notin \mathbb{W}_{\varphi}$.*

Proof. Put $g(n) := f(n) + n\varphi$. Then $\Delta g(n) = \Delta f(n) + \varphi,\ n \geqslant 1$. So $\Delta g(n)$ converges to an integer. Therefore, by Corollary 1 sequence of exponential sums $\left\{\sum_{n=1}^{N} e(g(n))\right\}_{N\geqslant 1}$ is unbounded.

Because $\left\{\sum_{n=1}^{N} y_n \cdot e(n\varphi)\right\}_{N\geqslant 1} = \left\{\sum_{n=1}^{N} e(g(n))\right\}_{N\geqslant 1}$, Proposition 3 implies that every solution of the difference equation (4) is unbounded. □

Example. For all $\alpha \in (0,1)$ all the solutions of the following difference equation are unbounded

$$x_{n+1} = x_n + e(n^{\alpha}),\ n \geqslant 1.$$

Example. All the solutions of the difference equation

$$x_{n+1} = e^{i\frac{\pi}{2}}x_n + e^{i\sum_{k=1}^{n}\arctan(k)},\ n \geqslant 1$$

are unbounded.

2. Multidimensional equation

In space $\mathbb{C}^M$ with Euclidean norm we consider following difference equation

$$x(n+1) = Ax(n) + y(n),\ n \geqslant 1 \tag{5}$$

with respect to unknown sequence $\{x(n) : n \geqslant 1\} \subset \mathbb{C}^M$.

The first element of the sequence $x(1)$, a sequence $\{y(n) : n \geqslant 1\} \subset \mathbb{C}^M$ and $M \times M$ square matrix A with complex entries are supposed to be known.

Our goal is to find sufficient conditions for the sequence $\{y(n) : n \geqslant 1\} \subset \mathbb{C}^M$ and value $x(1)$ which guarantee boundedness of the solution in the critical case $\sigma(A) \cap \{z \in \mathbb{C} \mid |z| = 1\} \neq \varnothing$.

Remark. We will use Hilbert-Shmidt norm of matrix $A = (a_{ij}) \in \mathcal{M}_M(\mathbb{C})$ which is defined as $\|A\|_2 = \sqrt{\sum_{i,j}|a_{ij}|^2}$.

Note that the Cauchy-Bunyakovskyi-Schwartz inequality implies

$$\forall x \in \mathbb{C}^M\ \forall A \in \mathcal{M}_M(\mathbb{C})\ : \|Ax\| \leqslant \|A\|_2\|x\|.$$

Now we introduce the following notation:

$$(x)^{[n]} := \prod_{i=0}^{n-1}(x+i), \qquad (x)_{[n]} := \prod_{i=0}^{n-1}(x-i).$$

These polynomials are widely known as *factorial polynomials* or, more precisely, *raising* and *falling factorials* respectively. They satisfy the useful identities listed below.

Proposition 5. $\binom{x}{n} = \frac{(x)_{[n]}}{n!}$, $\binom{x+n-1}{n} = \frac{(x)^{[n]}}{n!}$ *for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$.*

Proposition 6. $(-x)_{[n]} = (-1)^n \cdot (-x)^{[n]}$.

Proposition 7. $(x+y)_{[n]} = \sum_{i=0}^{n}\binom{n}{i}(x)_{[i]} \cdot (y)_{[n-i]}$.

Proof. Induction on n. □

These properties immediately imply the following lemma, which will be of great importance for us later.

Lemma 1.

$$\binom{n-k}{m} = \sum_{j=0}^{m} (-1)^{m-j} \binom{n}{j} \frac{(k)^{[m-j]}}{(m-j)!}.$$

We proceed to the analysis of the multidimensional equation. We remark that the computations below are well known; however, for completeness, we would like to include them in our article.

Let J be a Jordan Normal Form (JNF) of the matrix A. There exists an invertible matrix T such that $A = TJT^{-1} \iff J = T^{-1}AT$. Multiply the equation (5) at left by matrix T^{-1} and obtain:

$$T^{-1}x(n+1) = T^{-1}Ax(n) + T^{-1}y(n) \iff T^{-1}x(n+1) = T^{-1}AT\left(T^{-1}x(n)\right) + T^{-1}y(n) \iff$$

$$\iff T^{-1}x(n+1) = J\left(T^{-1}x(n)\right) + T^{-1}y(n).$$

Put $\{x'(n) : n \geqslant 1\} := \{T^{-1}x(n)\}$, $\{y'(n) : n \geqslant 1\} := \{T^{-1}y(n)\}$ and obtain the difference equation

$$x'(n+1) = Jx'(n) + y'(n), \ n \geqslant 1. \tag{6}$$

The nonsingularity of the matrix T implies that the solution of the difference equation (5) is bounded if and only if the solution of the difference equation (6) is bounded; thus, we will investigate the equation (6).

In this way, our problem is reduced to the study of the difference equation of the form:

$$x(n+1) = J_\lambda x(n) + y(n), \ n \geqslant 1, \tag{7}$$

where $J_\lambda = \begin{pmatrix} \lambda & 1 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$ is the Jordan cell that corresponds to the eigenvalue λ of the matrix A of the equation (5).

If the order of J_λ is 1, then equation (7) is the First Order Linear Difference Equation in $\mathbb{C}$.

If the order of J_λ is $M \geqslant 2$, then we do the following.

Let $x(n) = \begin{pmatrix} x_1(n) \\ x_2(n) \\ \vdots \\ x_M(n) \end{pmatrix}$, $y(n) = \begin{pmatrix} y_1(n) \\ y_2(n) \\ \vdots \\ y_M(n) \end{pmatrix}$, then equation (7) can be rewritten as

$$\begin{pmatrix} x_1(n+1) \\ x_2(n+1) \\ \vdots \\ x_{M-1}(n+1) \\ x_M(n+1) \end{pmatrix} = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} x_1(n) \\ x_2(n) \\ \vdots \\ x_{M-1}(n) \\ x_M(n) \end{pmatrix} + \begin{pmatrix} y_1(n) \\ y_2(n) \\ \vdots \\ y_{M-1}(n) \\ y_M(n) \end{pmatrix} \iff$$

$$\iff \begin{pmatrix} x_1(n+1) \\ x_2(n+1) \\ \vdots \\ x_{M-1}(n+1) \\ x_M(n+1) \end{pmatrix} = \begin{pmatrix} \lambda x_1(n) + x_2(n) + y_1(n) \\ \lambda x_2(n) + x_3(n) + y_2(n) \\ \vdots \\ \lambda x_{M-1}(n) + x_M(n) + y_{M-1}(n) \\ \lambda x_M(n) + y_M(n) \end{pmatrix}, \ n \geqslant 1.$$

The solution of the difference equation $x_M(n+1) = \lambda x_M(n) + y_M(n)$ is

$$x_M(n) = \lambda^{n-1} x_M(1) + \sum_{k=1}^{n-1} \lambda^{n-k-1} y_M(k).$$

Then the solution of difference equation $x_{M-1}(n+1) = \lambda x_{M-1}(n) + x_M(n) + y_{M-1}(n)$ is

$$x_{M-1}(n) = \lambda^{n-1} x_{M-1}(1) + \sum_{k=1}^{n-1} \lambda^{n-k-1} \left(x_M(k) + y_{M-1}(k)\right).$$

Hence, the solution of equation $x_i(n+1) = \lambda x_i(n) + x_{i+1}(n) + y_i(n)$ is

$$x_i(n) = \lambda^n x_i(1) + \sum_{k=1}^{n-1} \lambda^{n-k-1} \left(x_{i+1}(k) + y_i(k)\right).$$

Thus, we have found the solution of the difference equation (7). Now we turn our attention to the investigation of the case $|\lambda| = 1$, and our idea is to find the connection between one-dimensional and multidimensional equations in this case. At the end of this section, we will describe an interesting statement that, in fact, reveals this connection.

By induction, it is not difficult to deduce that the solution of (7) can be written in the following way:

$$x(n) = J_\lambda^{n-1} \cdot x(1) + \sum_{k=1}^{n-1} J_\lambda^{n-k-1} y(k).$$

Thus, for $1 \leqslant m \leqslant M$:

$$x_m(n) = \sum_{i=0}^{M-m} \left[\binom{n-1}{i} \lambda^{n-1-i} \cdot x_{i+m}(1) + \sum_{k=1}^{n-1} \binom{n-k-1}{i} \lambda^{n-k-1-i} \cdot y_{i+m}(k) \right].$$

We apply Lemma 1:

$$\binom{n-k-1}{i} = \sum_{r=0}^{i} (-1)^{i-r} \binom{n-1}{r} \frac{(k)^{[i-r]}}{(i-r)!}$$

and obtain

$$x_m(n) = \sum_{i=0}^{M-m} \left[\binom{n-1}{i} \lambda^{n-1-i} \cdot x_{i+m}(1) + \sum_{k=1}^{n-1} \left(\sum_{r=0}^{i} (-1)^{i-r} \binom{n-1}{r} \frac{(k)^{[i-r]}}{(i-r)!} \right) \cdot \lambda^{n-k-1-i} \cdot y_{i+m}(k) \right] =$$

$$= \sum_{i=0}^{M-m} \left[\binom{n-1}{i} \lambda^{n-1-i} \cdot x_{i+m}(1) + \sum_{r=0}^{i} \left\{ (-1)^{i-r} \binom{n-1}{r} \cdot \left(\sum_{k=1}^{n-1} \frac{(k)^{[i-r]}}{(i-r)!} \cdot \lambda^{n-k-1-i} \cdot y_{i+m}(k) \right) \right\} \right].$$

We change the order of summation:

$$\sum_{i=0}^{M-m} \sum_{r=0}^{i} \left\{ (-1)^{i-r} \binom{n-1}{r} \cdot \left(\sum_{k=1}^{n-1} \frac{(k)^{[i-r]}}{(i-r)!} \cdot \lambda^{n-k-1-i} \cdot y_{i+m}(k) \right) \right\} =$$

$$= \sum_{r=0}^{M-m} \sum_{i=r}^{M-m} \left\{ (-1)^{i-r} \binom{n-1}{r} \cdot \left(\sum_{k=1}^{n-1} \frac{(k)^{[i-r]}}{(i-r)!} \cdot \lambda^{n-k-1-i} \cdot y_{i+m}(k) \right) \right\}.$$

For convenience, we swap the indices i and r:

$$\sum_{i=0}^{M-m} \sum_{r=i}^{M-m} \left\{ (-1)^{r-i} \binom{n-1}{i} \cdot \left(\sum_{k=1}^{n-1} \frac{(k)^{[r-i]}}{(r-i)!} \cdot \lambda^{n-k-1-r} \cdot y_{r+m}(k) \right) \right\}.$$

Therefore,

$$x_m(n) = \sum_{i=0}^{M-m} \left[\binom{n-1}{i} \lambda^{n-1-i} \cdot x_{i+m}(1) + \sum_{r=i}^{M-m} \binom{n-1}{i} \left\{ (-1)^{r-i} \cdot \left(\sum_{k=1}^{n-1} \frac{(k)^{[r-i]}}{(r-i)!} \cdot \lambda^{n-k-1-r} \cdot y_{r+m}(k) \right) \right\} \right] =$$

$$= \sum_{i=0}^{M-m} \binom{n-1}{i} \left[\lambda^{n-1-i} \cdot x_{i+m}(1) + \sum_{r=i}^{M-m} \left\{ (-1)^{r-i} \cdot \sum_{k=1}^{n-1} \left(\frac{(k)^{[r-i]}}{(r-i)!} \cdot \lambda^{n-k-1-r} \cdot y_{r+m}(k) \right) \right\} \right] =$$

$$= \sum_{i=0}^{M-m} \binom{n-1}{i} \left[\lambda^{n-1-i} \cdot x_{i+m}(1) + \sum_{r=0}^{M-m-i} \left\{ (-1)^{r} \cdot \sum_{k=1}^{n-1} \left(\frac{(k)^{[r]}}{r!} \cdot \lambda^{n-k-1-r-i} \cdot y_{r+i+m}(k) \right) \right\} \right].$$

Proposition 8. *The solution of* (7) *can be written in a following way:*

$$x_m(n) = \sum_{i=0}^{M-m} \binom{n-1}{i} \lambda^{n-1-i} \left[x_{i+m}(1) + \sum_{r=0}^{M-(i+m)} \left\{ (-1)^{r} \cdot \sum_{k=1}^{n-1} \left(\frac{(k)^{[r]}}{r!} \cdot \lambda^{-k-r} \cdot y_{r+i+m}(k) \right) \right\} \right], \ 1 \leqslant m \leqslant M.$$

The main theorem of this section follows.

Theorem 6. *Let* $M \geqslant 2$, $\{\widetilde{y}_m(n) : n \geqslant 0\} \subset \mathbb{C},\ 1 \leqslant m \leqslant M$ *and the following conditions hold:*

- *For all* $1 \leqslant m \leqslant M\ : y_m(n) = \frac{\widetilde{y}_m(n)}{n^{m-1}},\ n \geqslant 1$
- *For each* $1 \leqslant m \leqslant M$ *there exists some* $\alpha_m \in \mathbb{C}$*, such that the solution of the difference equation* $z_{n+1} = \lambda z_n + \widetilde{y}_m(n)$, $n \geqslant 1$, $z_1 = \alpha_m$ *is bounded. Equivalently, the sequence of sums* $\left\{ \sum_{n=1}^{N} \widetilde{y}_m(n) \lambda^{-n} \right\}_{N \geqslant 1}$ *is bounded.*

Then the solution of the difference equation (7) *is bounded if and only if* $\forall m \in \{2, \ldots, M\}$:

$$x_m(1) = -\sum_{r=0}^{M-m} \left\{ (-1)^r \cdot \sum_{k=1}^{\infty} \left(\frac{(k)^{[r]}}{r!} \cdot \lambda^{-k-r} \cdot y_{r+m}(k) \right) \right\}.$$

Proof of The Main Theorem

Firstly, we would like to remark that each series

$$\sum_{k=1}^{\infty} \left(\frac{(k)^{[r]}}{r!} \cdot \lambda^{-k-r} \cdot y_{r+m}(k) \right),\ 2 \leqslant m \leqslant M,\ 0 \leqslant r \leqslant M-m$$

is convergent by Dirichlet's test. Indeed,

$$\frac{(k)^{[r]}}{r!} \cdot \lambda^{-k-r} \cdot y_{r+m}(k) = \frac{\lambda^{-r}}{r!} \cdot \frac{\widetilde{y}_{r+m}(k)\lambda^{-k}}{\left(\frac{k^{r+m-1}}{(k)^{[r]}}\right)}. \qquad (8)$$

The sequence $\left\{ \left(\frac{k^{r+m-1}}{(k)^{[r]}} \right)^{-1} : k \geqslant 1 \right\}$ is eventually monotone and is convergent to 0 because

$$\frac{k^{r+m-1}}{(k)^{[r]}} \sim \frac{k^{r+m-1}}{k^r} = k^{m-1},\ k \to \infty.$$

Also, by the assumption, the sequence of sums $\left\{ \sum_{k=1}^{K} \widetilde{y}_m(k)\lambda^{-k} \right\}_{N \geqslant 1}$ is bounded.

Proof of sufficiency

We need the following Lemma:

Lemma 2. *Let* $\{a_n : n \geqslant 1\} \subset \mathbb{R}$ *be monotone, convergent to 0 sequence. And let* $\{b_n : n \geqslant 1\} \subset \mathbb{C}$ *be a sequence, for which there exists a constant* $C > 0$*, such that* $\forall N \geqslant 1 : \left| \sum_{n=1}^{N} b_n \right| \leqslant C$*. Then* $\left| \sum_{k=n}^{\infty} a_k b_k \right| \leqslant C|a_n|$ *for all* $n \geqslant 1$.

This lemma follows from Abel's summation by parts formula.

Now we use that $\forall m \in \{1, 2, \ldots, m\}\ \exists C_m > 0\ \forall N \geqslant 1 : \left| \sum_{n=1}^{N} \widetilde{y}_m(n)\lambda^{-n} \right| \leqslant C_m$.

Therefore, by (8), for all $2 \leqslant m \leqslant M,\ 0 \leqslant r \leqslant M-m$ we have:

$$\left| \sum_{k=N}^{\infty} \left(\frac{(k)^{[r]}}{r!} \cdot \lambda^{-k-r} \cdot y_{r+m}(k) \right) \right| \leqslant \frac{C_m}{r!} \cdot \frac{(N)^{[r]}}{N^{r+m-1}} \leqslant \frac{C_m}{r!} \cdot \frac{(N+r-1)^r}{N^{r+m-1}} = O\left(\frac{1}{N^{m-1}} \right),\ N \to \infty.$$

Hence, since $\forall m \in \{2, \ldots, M\}$:

$$x_m(1) = -\sum_{r=0}^{M-m} \left\{ (-1)^r \cdot \sum_{k=1}^{\infty} \left(\frac{(k)^{[r]}}{r!} \cdot \lambda^{-k-r} \cdot y_{r+m}(k) \right) \right\}.$$

We deduce that for all $1 \leqslant m \leqslant M$ and for all $0 \leqslant i \leqslant M-m$ we have:

$$x_{i+m}(1) + \sum_{r=0}^{M-m-i} \left\{ (-1)^r \cdot \sum_{k=1}^{n-1} \left(\frac{(k)^{[r]}}{r!} \cdot \lambda^{-k-r} \cdot y_{r+i+m}(k) \right) \right\} = O\left(\frac{1}{n^{i+m-1}} \right),\ n \to \infty.$$

Since $\binom{n-1}{i} = O\left(n^i\right),\ n \to \infty$ using Proposition 8 we obtain that the solution of (7) is indeed bounded.

Proof of necessity

Suppose that the solution $\{x(n) : n \geqslant 1\}$ of the equation (7) is bounded. In particular, the sequence $\{x_1(n) : n \geqslant 1\}$ is bounded.

$$x_1(n) = \sum_{i=0}^{M-1} A_i(n) \cdot \binom{n-1}{i},$$

where

$$A_i(n) := \lambda^{n-1-i} \left[x_{i+m}(1) + \sum_{r=0}^{M-m-i} \left\{ (-1)^r \cdot \sum_{k=1}^{n-1} \left(\frac{(k)^{[r]}}{r!} \cdot \lambda^{-k-r} \cdot y_{r+i+m}(k) \right) \right\} \right],\ n \geqslant 1.$$

Then $A_{M-1}(n) \cdot \binom{n-1}{M-1} = O\left(n^{M-2}\right)$, so $A_{M-1}(n) = O\left(\frac{1}{n}\right) \Longrightarrow A_{M-1}(n) \to 0,\ n \to \infty$. Using the same argument from the proof of the *sufficiency* we deduce that $A_{M-1} = O\left(\frac{1}{n^{(M-1)+1-1}} \right) \Longrightarrow A_{M-1}(n) \cdot \binom{n-1}{M-1} = O(1)$.

Thus $A_{M-2}(n) \cdot \binom{n-1}{M-2} = O\left(n^{M-3}\right) \Longrightarrow A_{M-2}(n) \to 0,\ n \to \infty$ and analogously $A_{M-2}(n) \cdot \binom{n-1}{M-2} = O(1)$. Continuing this argument we show that $A_m(n) \to 0,\ n \to \infty$ for all $2 \leqslant m \leqslant M$.

Discussion and conclusions

In this work, we established novel, insightful sufficient conditions for the existence of a bounded solution of the first order linear difference equation in the one-dimensional case and further applied these conditions to the multidimensional case of this equation. The obtained results enrich the existing body of research in the field and complement previous studies.

Authors' contribution: Andrii Chaikovskyi — conceptualization; Oleksandr Liubimov — methodology; Oksana Lahoda — analysis of the results, preparation of literature review.

Sources of funding. Funding is partially provided by Taras Shevchenko National University of Kyiv.

References

Chaikovs'kyi, A., & Lagoda, O. (2022). Bounded solutions of difference equations in a banach space with input data from subspaces. *Ukranian Mathematical Journal*, 73, 1810–1824. https://doi.org/10.1007/s11253-022-02031-3

Gomilko, A., Gorodnii, M., & Lagoda, O. (2003). On the boundedness of a recurrence sequence in a banach space. *Ukrainian Mathematical Journal*, 55(10), 1699–-1708. https://doi.org/10.1023/B:UKMA.0000022074.96704.7e

Gorodnii, M., & Kravets', V. (2020). On bounded solutions of one difference equation of the second order. *Journal of Mathematical Sciences*, 249, 601–608. https://doi.org/10.1007/s10958-020-04960-5

Graham, S., & Kolesnik, G. (1991). *Van der corput's method of exponential sums*. Cambridge University press.

Horodnii, M., & Lahoda, O. (2001). Bounded solutions for some classes of difference equations with operator coefficients. *Ukrainian Mathematical Journal*, 53(11), 1817–1824. https://doi.org/10.1023/A:1015298712652

Mordell, J. L. (1958). On the kusmin-landau inequality for exponential sums. *Acta Arithmetica*, 4, 3–9.

Отримано редакцією журналу / Received: 05.02.25
Прорецензовано / Revised: 26.03.25
Схвалено до друку / Accepted: 10.10.25

Андрій ЧАЙКОВСЬКИЙ, д-р фіз.-мат. наук, проф.
ORCID ID: 0000-0002-3357-8744
e-mail: andriichaikovskyi@knu.ua
Київський національний університет імені Тараса Шевченка, Київ, Україна

Олександр ЛЮБІМОВ, студ.
ORCID ID: 0009-0009-9347-6297
e-mail: liubimov_oleksandr@knu.ua
Київський національний університет імені Тараса Шевченка, Київ, Україна

Оксана ЛАГОДА, канд. фіз.-мат. наук, доц.
ORCID ID: 0000-0003-4515-1753
e-mail: lahoda.oksana@lll.kpi.ua
НТУУ "КПІ ім. Ігоря Сікорського", Київ, Україна

ОБМЕЖЕНІСТЬ РОЗВ'ЯЗКІВ ЛІНІЙНОГО РІЗНИЦЕВОГО РІВНЯННЯ ПЕРШОГО ПОРЯДКУ У БАГАТОВИМІРНОМУ ВИПАДКУ

Досліджено обмеженість розв'язків лінійного різницевого рівняння першого порядку вигляду $x_{n+1} = Ax_n + y_n,\ n \geqslant 1$, де A – комплексна квадратна матриця, послідовність $\{y_n\}_{n\geqslant 1}$ та початкове значення x_1 вважаються відомими. Розглянуто одновимірний випадок цього рівняння $x_{n+1} = ax_n + y_n,\ n \geqslant 1$, де a – комплексне число. Зокрема, отримано достатні умови, за яких розв'язки рівняння є обмеженими або необмеженими у випадку $|a| = 1$(критичний випадок), розглядаючи експоненціальні суми вигляду $\sum y_n e(n\varphi)$ та $\sum e(f(n))$.

Потім здійснено дослідження рівняння у багатовимірному випадку та зведено нашу задачу до аналізу спектра та клітинок Жордана матриці A. Задача є особливо цікавою, коли спектр матриці A містить власні числа λ, $|\lambda| = 1$. У підсумку отримано теорему, яка показує зв'язок між рівняннями $x_{n+1} = ax_n + y_n,\ n \geqslant 1$, $|a| = 1$ та $x_{n+1} = Jx_n + y_n,\ n \geqslant 1$, де J – клітинка Жордана, що відповідає власному числу λ, $|\lambda| = 1$.

К л ю ч о в і с л о в а: ***лінійне різницеве рівняння, обмежений розв'язок, експоненціальні суми.***

Автори заявляють про відсутність конфлікту інтересів. Спонсори не брали участі в розробленні дослідження (у зборі, аналізі чи інтерпретації даних, якщо це мало місце), у написанні рукопису та в рішенні про публікацію результатів.

The authors declare no conflicts of interest. The funders had no role in the design of the study (in the collection, analyses or interpretation of data if applicable), in the writing of the manuscript as well as in the decision to publish the results.