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До питання щодо триноміальних функціоналів

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To the question of trinomial functionals

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В статті розглядаються властивості функціоналів від триноміального розподілу, які з'являються в задачах статистичному корегування результатів IDK-тестів та оцінювання якості знань. Для одного з таких функціоналів запропоновані варіанти визначення його у «нульовій» точці та показано, як один спосіб визначення може бути корисним при обчислені математичного сподівання такого функціоналу, а інший спосіб визначення в «нульовій точці» є корисним при обчисленні реалізацій цього функціоналу в рамках задачі оцінювання якості знань. Запропонований показник якості знань оцінює впевненість студента у правильності відповідей при тестуванні і цим суттєво відрізняється від усіх існуючих методів оцінювання, які оцінюють або кількість знань, або стосуються швидше оцінювання вмінь чи навичок, але не якості набутих знань. Показано, як методика оцінювання якості знань пов'язана з політикою доброчесності в освітньому середовищі. Методи оцінювання можуть як спонукати студента до відгадування правильних відповідей, так і утримувати від порушення основ академічної доброчесності.

Ключові слова: функціонал від триноміального розподілу, оцінювання тестів, статистичне корегування, показники якості знань, політика доброчесності.

The article considers the properties of functionals from the trinomial distribution that appear in the tasks of statistical correction of the results of IDK-tests and knowledge quality estimation. For one of these functionals, variants of its definition at the "zero" point are proposed, and it is shown how one method of definition can be useful in calculating the mathematical expectation of such a functional, and another method of definition is useful in calculating the realizations of the functional within the task of knowledge quality estimation. The proposed indicator of the quality of knowledge assesses the student's confidence in the correctness of the answers during testing, and thus differs significantly from all existing assessment methods, which assess either the amount of knowledge, or rather relate to the assessment of abilities or skills, but not the quality of acquired knowledge. It is shown how the method of assessing the quality of knowledge is related to the policy of academic integrity in the educational environment. Assessment methods can both encourage the student to guess the correct answers and prevent them from violating the basics of academic integrity.

Keywords — functional of the trinomial distribution, tests assessment, statistical correction, knowledge quality estimation, the policy of academic integrity.

1 Introduction

Some ideas of statistical correction of test results have been introduced in article [1]. Two types of tests were considered. First, a classic test where there are several answer options for each question. The idea of correcting the results of such testing was to estimate the number of correctly guessed answers. Secondly, a modification of the classic test, where the "I don't know" option is added to the answer options, which the student must choose if he does not know the correct answer. We call such tests IDK-tests. The purpose of using IDK-tests is directly related to the implementation of ideas of academic integrity to our life. Using such tests, on the one hand, enables the student to honestly choose IDK-option, and on the other hand, enables the teacher to statistically evaluate not only the amount of knowledge acquired by the student, but also its strength. It is clear that quantity has a strong correlation with quality, but this is far from always the case.

Let us emphasize that when we talk about the strength of knowledge, we do not mean its quantity, but rather its quality, that is, the degree of confidence of the student in his knowledge. Instead, in most sources (e.g. [2]-[4]), when it comes to assessing the quality of knowledge, what is really meant is either the quantity of knowledge or the skills and abilities that are based on that knowledge. We will consider the quality of knowledge rather as its strength, as a measure of the student's confidence in the part of knowledge he can rely on. Method of assessing the quality of knowledge is related to the policy of academic integrity in the educational environment ([5], [6]). Whatever these methods are, they must have more than just a control function. but also encourage the student to observe the basic principles of academic integrity.

2 Notations and definitions

Suppose the test consists of N questions with $n>1$ variants of answer each while student gave C correct answers, I incorrect ones has chosen D times the IDK-option ($C+I+D=N$). Each question has got $n+1$ options of answer (n real variants and the IDK one).

Within such a model, we have the following notations.

- N Number of questions in a test.
- n Number of options (variants) of answer per question, $n>1$, apart from IDK-option.
- C Number of given correct answers in a test; a random variable.
- I Number of given incorrect answers in a test; a random variable.
- D Number of given IDK-answers in a test; a random variable.
- α The share of the material studied by the student (testee), $\alpha \in [0;1]$. In other words, it is the probability that the student knows the answer to a particular question.
- β An indicator of student honesty, $\beta \in [0;1]$. This is the probability that the student, not knowing the correct answer, will try to guess the answer, ignoring the IDK option.

Now, it is important to distinguish the random variables of this model from ordinary coefficients, although often realizations of random variables or functions of them can be coefficients. At the same time, the coefficients under other conditions can be random values.

It is easy to see that random variables C, I, D have got binomial distributions with corresponding parameters

$$C \sim Bin\left(N,\, p_C\right),\ p_C = \alpha + \frac{\beta(1-\alpha)}{n},$$

$$D \sim Bin\left(N,\, p_D\right),\ p_D = (1-\alpha)(1-\beta),$$

$$I \sim Bin\left(N,\, p_I\right),\ p_I = \frac{\beta(1-\alpha)(n-1)}{n},$$

$$p_I + p_C + p_D = 1,$$

$$C + I + D = N,$$

and the random vector $\left(C, I, D\right)$ has a multinomial, more precisely, a trinomial distribution. In article [1], a quite logical coefficient

$$K = \frac{Cn-(N-D)}{(n-1)(N-D)},\ 0 < \frac{N-D}{n} < C \le N-D,$$

was proposed for assessing the quality of knowledge, but seven years of its application in practice revealed some shortcomings. In particular, the expectation $E\left[K\right]$ of this coefficient for some values of $\alpha \in [0;1]$ was not a monotonic function in $\beta \in [0;1]$ as shown in the following figure 1.

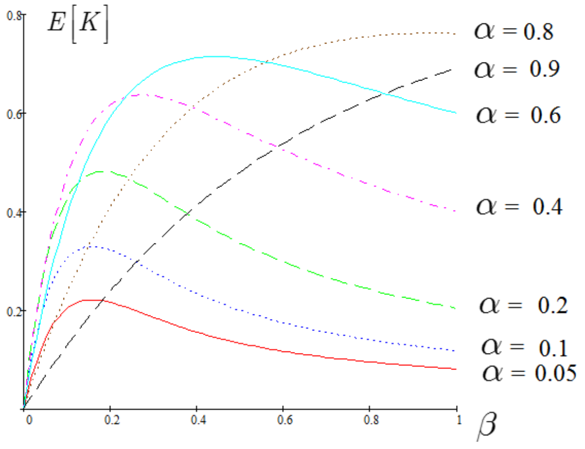

Fig. 1

This meant that at a low level of knowledge, it was more profitable for the student to try to guess the correct answers, rather than honestly choose IDK option. And this is not just a mathematical problem, in fact, the previously proposed model inclines students to dishonest behavior, which is not entirely good. To solve this problem, note that most of the mentioned coefficient variations will be linear functions of the fraction $\dfrac{C}{N-D} = \dfrac{C}{I+C}$. But the fraction itself is not defined when $D = N$. An attempt to define this fraction by zero on the basis of logical considerations led to the mentioned problem of a moral and ethical nature. Oddly enough, now we will try to extend this fraction by $x = \dfrac{p_C}{1-p_D}$. This extended definition has a rather specific character, because the expression $\dfrac{p_C}{1-p_D} = \dfrac{\alpha n + \beta(1-\alpha)}{n\left(\alpha + \beta(1-\alpha)\right)}$ depends on unknown variables α and β, but formally these variables are simply parameters of our model and we have the right to take such a step. Will the logic suffer? The answer is negative. If a student essentially says that he knows nothing, then he is at least honest, and at the maximum he also understands that he does not know the material. Does this improve our model? The answer is positive. Indeed, the conditional distribution

$$C_D \sim Bin\left(N-D, \frac{p_C}{1-p_D}\right)$$

therefore, according to the conditional expectation property,

$$\tilde{E}\left[\frac{C}{N-D}\right] = \tilde{E}\left[\tilde{E}\left[\frac{C}{N-D}\middle| D\right]\right] =$$

$$= \tilde{E}\left[\frac{\left(N-D\right)\dfrac{p_C}{1-p_D}}{N-D}\right] = \frac{p_C}{1-p_D} =$$

$$= \frac{\alpha n + \beta(1-\alpha)}{n\left(\alpha + \beta(1-\alpha)\right)}.$$

The tilde-sign over the mathematical expectation symbol here means that when calculating, we define the value of the fraction at $N = D$ by the value x. If these calculations do not explicitly show exactly where and how we extended the value of the fraction, then we can get the same result in a longer way by explicitly calculating the mathematical expectation.

3 Main results and their proofs

Lemma The sum

$$\sum_{c=1}^{N}\sum_{d=0}^{N-c}\frac{N!}{(N-d)(c-1)!\,d!(N-c-d)!}\times$$
$$\times y^{c-1}t^d\left(1-y-t\right)^{N-c-d} = \frac{1-t^N}{1-t}$$

doesn't depend upon y for all $N \in \mathbb{N}$, $y,t \geq 0$, $y+t<1$.

Proof. Not complicated but rather lengthy calculations show that the derivative with respect to y will be equal to 0. Consider a more elegant proof. Since

$$\sum_{c=1}^{N}\sum_{d=0}^{N-c} f_{c,d} = \sum_{d=0}^{N-1}\sum_{c=1}^{N-d} f_{c,d},$$

then

$$\sum_{c=1}^{N}\sum_{d=0}^{N-c}\frac{N!}{(N-d)(c-1)!\,d!(N-c-d)!}\times$$
$$\times y^{c-1}t^d\left(1-y-t\right)^{N-c-d} =$$
$$= \sum_{d=0}^{N-1}\sum_{c=1}^{N-d}\frac{N!}{(N-d)(c-1)!\,d!(N-c-d)!}\times$$
$$\times y^{c-1}t^d\left(1-y-t\right)^{N-c-d} =$$
$$= N!\left(1-y-t\right)^{N-1}\sum_{d=0}^{N-1}\frac{t^d\left(1-y-t\right)^{-d}}{(N-d)d!}\times$$
$$\times\sum_{c=1}^{N-d}\frac{\left(\dfrac{y}{1-y-t}\right)^{c-1}}{(c-1)!(N-c-d)!}.$$

Let us temporarily denote $s = \dfrac{y}{1-y-t}$, only for the inner sum and continue the transformations

$$N!\left(1-y-t\right)^{N-1}\sum_{d=0}^{N-1}\frac{t^d\left(1-y-t\right)^{-d}}{(N-d)d!}\times$$
$$\times\sum_{c=1}^{N-d}\frac{s^{c-1}}{(c-1)!(N-c-d)!}=$$
$$=N!\left(1-y-t\right)^{N-1}\sum_{d=0}^{N-1}\frac{t^d\left(1-y-t\right)^{-d}}{(N-d)d!(N-d)!}\times$$
$$\times\frac{d}{ds}\left(\left(1+s\right)^{N-d}-1\right)=$$
$$=N!\left(1-y-t\right)^{N-1}\sum_{d=0}^{N-1}\frac{t^d\left(1-y-t\right)^{-d}}{d!(N-d)!}\times$$
$$\times\left(\frac{1-t}{1-y-t}\right)^{N-d-1}=$$
$$=N!\left(1-t\right)^{N-1}\sum_{d=0}^{N-1}\frac{t^d}{d!(N-d)!}\left(1-t\right)^{-d}=$$
$$=\left(1-t\right)^{N-1}\left(\left(\frac{1}{1-t}\right)^{N}-\left(\frac{t}{1-t}\right)^{N}\right)=\frac{1-t^N}{1-t},$$

which was required to be shown.

Theorem Under the introduced notation, the following equality holds

$$\tilde{E}\left[\frac{C}{N-D}\right]=\frac{p_C}{1-p_D}=\frac{\alpha n+\beta(1-\alpha)}{n\left(\alpha+\beta(1-\alpha)\right)}.$$

Proof

$$\tilde{E}\left[\frac{C}{N-D}\right]=\sum_{c=0}^{N}\sum_{d=0}^{N-c}\left\{\begin{matrix}\frac{c}{N-d}, & d\neq N\\ x, & d=N\end{matrix}\right\}\times$$
$$\times\frac{N!}{c!d!(N-c-d)!}p_C^c p_D^d p_I^{N-c-d}=$$
$$=\sum_{d=0}^{N}\left\{\begin{matrix}0, & d\neq N\\ x, & d=N\end{matrix}\right\}\frac{N!}{d!(N-d)!}p_D^d p_I^{N-d}+$$
$$+\sum_{c=1}^{N}\sum_{d=0}^{N-c}\frac{c}{N-d}\frac{N!}{c!d!(N-c-d)!}p_C^c p_D^d p_I^{N-c-d}=$$
$$=xp_D^N+\sum_{c=1}^{N}\sum_{d=0}^{N-c}\frac{c}{N-d}\frac{N!}{c!d!(N-c-d)!}p_C^c p_D^d p_I^{N-c-d}=$$
$$=xp_D^N+N!\,p_I^N\sum_{c=1}^{N}\frac{p_C^c p_I^{-c}}{(c-1)!}\sum_{d=0}^{N-c}\frac{p_D^d p_I^{-d}}{(N-d)d!(N-c-d)!}.$$

Let us prove that the following equality holds.

$$xp_D^N+\sum_{c=1}^{N}\sum_{d=0}^{N-c}\frac{c}{N-d}\frac{N!}{c!d!(N-c-d)!}p_C^c p_D^d p_I^{N-c-d}=$$
$$=\frac{p_C}{1-p_D}.$$
$$\sum_{c=1}^{N}\sum_{d=0}^{N-c}\frac{N!}{(N-d)(c-1)!d!(N-c-d)!}\times$$
$$\times p_C^{c-1}p_D^d\left(1-p_C-p_D\right)^{N-c-d}=\frac{1-p_D^N}{1-p_D}.$$

It follows from Lemma that the left-hand side of this equality does not depend on p_C and is equal to the right-hand side of this equality. The theorem is proved.

Corollary If the value of the fraction $F=\frac{C}{N-D}=\frac{C}{I+C}$ is considered only at its domain, that is, only when the denominator is not equal to zero, then its mathematical expectation

$$E\left[\frac{C}{N-D}\right]=\sum_{c=1}^{N}\sum_{d=0}^{N-c}\frac{c}{N-d}\frac{N!}{c!d!(N-c-d)!}\times$$
$$\times p_C^c p_D^d p_I^{N-c-d}=\tilde{E}\left[\frac{C}{N-D}\right]-xp_D^N=$$
$$=\frac{p_C}{1-p_D}-xp_D^N=\frac{p_C\left(1-p_D^N\right)}{1-p_D}=$$
$$=\frac{\alpha n+\beta(1-\alpha)}{n\left(\alpha+\beta(1-\alpha)\right)}\left(1-\left((1-\alpha)(1-\beta)\right)^N\right)$$

and for sufficiently large N on condition $\alpha^2+\beta^2\neq 0$ it is easy to see that

$$E\left[\frac{C}{N-D}\right]\approx\tilde{E}\left[\frac{C}{N-D}\right].$$

Note that the fraction $F=\frac{C}{N-D}=\frac{C}{I+C}$ has got rather good properties as a function of $C=\overline{0,N-D}$, for a fixed D it monotonically grows from 0 to 1. And, as a function of two variables $F(I,C)=\frac{C}{I+C}$, $I,C\geq 0$, $0<I+C\leq N$ it also has the “suitable” monotonicity (see fig. 2), and also takes values from 0 up to 1.

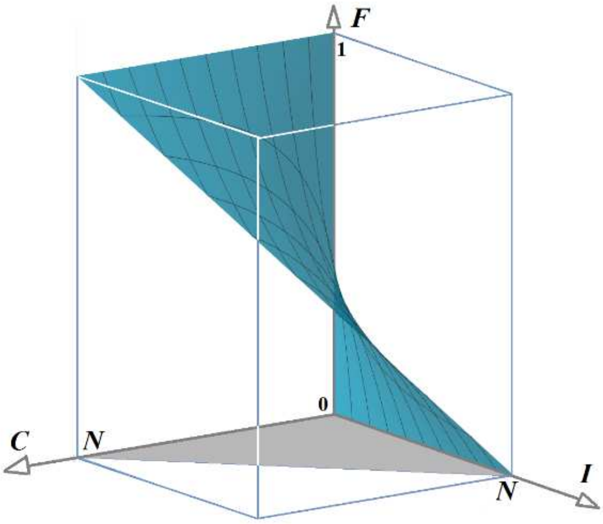

Fig. 2

Let us remind once again that $F(I,C)=\dfrac{C}{I+C}$ is not just undefined at the point $\left(C,I\right)=\left(0,0\right)$, but also cannot be determined by monotonicity, because the limit $\lim\limits_{\substack{C\to 0\\ I\to 0}}\dfrac{C}{C+I}$ does not exist. On the other hand, if the rate of convergence to zero is chosen in proportion to the corresponding probabilities $C=t\cdot p_C$, and $I=t\cdot p_I$, $t\to 0$, then

$$\lim_{\substack{C\to 0\\ I\to 0}}\frac{C}{C+I}=\lim_{t\to 0}\frac{t\cdot p_C}{t\cdot p_C+t\cdot p_I}=\frac{p_C}{p_C+p_I}=\frac{p_C}{1-p_D}.$$

We will not justify the appropriateness of choosing such speeds because, from a practical point of view, we only need to learn to choose the value of $F(0,0)$. In contrast to the way it was done in the theorem, we cannot choose $F(0,0)=\dfrac{p_C}{1-p_D}=\dfrac{\alpha n+\beta(1-\alpha)}{n\left(\alpha+\beta(1-\alpha)\right)}$ at least because the parameters α, β are unknown in advance. The fact that we can estimate α, β by values $\hat{\alpha}=0$ and $\hat{\beta}=0$ also does not improve the situation. However, we suggest the following approach.

- If we consider the **indicator of knowledge strength** (IKS) $k=F(I,C)=\dfrac{C}{I+C}$ as the final result of research, then in the case of $\left(C,I\right)=\left(0,0\right)$ we can assign it a non-numerical value "no knowledge detected", or put it equal to 0, it will no longer have a fundamental meaning, it is enough to understand what exactly this value means according to our agreement.
- If k should take a numerical value and will be used in the construction of further assessments, for example, when combining with an assessment of the amount of knowledge, then we will “take a step towards the student” and choose $\beta=0$, $\alpha=\varepsilon>0$, then $F(0,0)=1$.

The graph of function $\tilde{E}[k]$ is shown in Figure 3. A gap of $\dfrac{1}{n}$ does not interest us at all, although it brings in a certain imbalance.

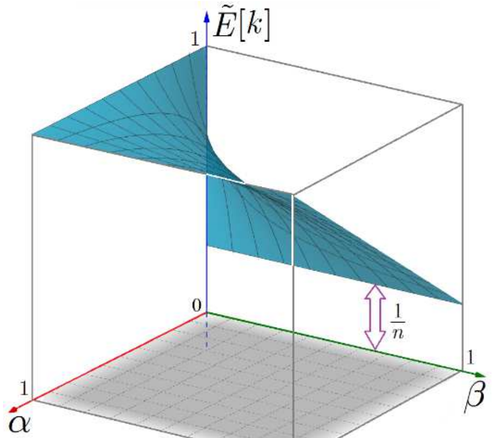

Fig. 3

This nuance can be reconciled, taking into account more important aspects.

- First, for each fixed $\alpha\in[0;1]$, the function $f(\alpha,\beta)=\tilde{E}[k(\alpha,\beta)]$ monotonically decreases along β. *This means that guessing becomes simply not profitable*.
- Second, the closer α gets to zero, the more sharply the function decreases along β. *This means that with a low level of knowledge, it is absolutely not profitable to try to guess the answers*. The closer α gets to 1, the smaller the dependence on β becomes, i.e. *with a high level of knowledge, it makes no sense to guess the answers* (see fig. 4).

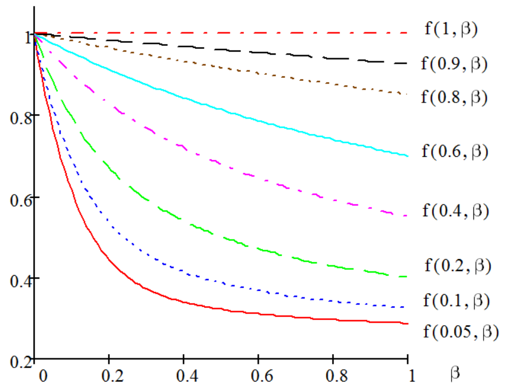

Fig. 4

The described methods and obtained results can be generalized in the future within the framework of setting new tasks for evaluating tests. The first and main such task is the construction of a functional from quantitative and qualitative indicators of knowledge. Such an integral indicator can be built, taking into account the extent to which we combine the indicator of the amount of knowledge and the indicator of the strength of knowledge. The second task of future research is the study of the statistical and probabilistic characteristics of the constructed functionals, in particular, the calculation of their point and interval estimates.

Список використаних джерел

1. *Sharapov M.* Statistical correction of test results, Bulletin of Taras Shevchenko National University of Kyiv. Series Physics & Mathematics, 2015, 2, pp. 199-202.

2. *Aas G.H., Askling B, Dittrich K., Froestad W., Haug P., Lycke K.H., Moitus S., Pyykkö R., Sørskår A.K.* Assessing educational quality: Knowledge production and the role of experts. ENQA Workshop report 6, 2009, Helsinki, Finland.

3. *Rech J., Decker B., Ras E., Jedlitschka A., Feldmann R.L.* The Quality of Knowledge: Knowledge Patterns and Knowledge Refactorings. International Journal of Knowledge Management (IJKM), 2009, IGI Publishing, Vol. 3, Iss. 3, pp. 74-103. DOI: 10.4018/jkm.2007070105

4. *Kai Mertins, Peter Heisig (Ed.); Vorbeck, Jens (Ed.)*. Knowledge Management. Concepts and Best Practices (2nd ed.). 2003, Berlin: Springer-Verlag.

5. *Holden O.L., Norris M.E. and Kuhlmeier V.A.* Academic Integrity in Online Assessment: A Research Review. 2021, Front. Educ. 6:639814. doi: 10.3389/feduc.2021.639814

6. *Morris E.J.* Academic integrity matters: five considerations for addressing contract cheating. Int J Educ Integr 14, 15 (2018). https://doi.org/10.1007/s40979-018-0038-5

References

1. SHARAPOV, M. (2015) Statistical correction of test results, *Bulletin of Taras Shevchenko National University of Kyiv. Series: Physics & Mathematics*. (2). pp. 199-202.

2. AAS, G.H., ASKLING, B., DITTRICH, K., FROESTAD, W., HAUG, P., LYCKE, K.H., MOITUS, S., PYYKKÖ, R., SØRSKÅR, A.K. (2009) *Assessing educational quality: Knowledge production and the role of experts*. ENQA Workshop report 6, Helsinki, Finland.

3. RECH, J., DECKER, B., RAS, E., JEDLITSCHKA, A., FELDMANN, R.L. (2009) *The Quality of Knowledge: Knowledge Patterns and Knowledge Refactorings*. International Journal of Knowledge Management (IJKM), IGI Publishing, Vol. 3, Iss. 3, pp. 74-103. DOI: 10.4018/jkm.2007070105

4. MERTINS, K., HEISIG, P. (Ed.); VORBECK, J. (Ed.) (2003) Knowledge Management. Concepts and Best Practices (2nd ed.). Berlin: Springer-Verlag.

5. HOLDEN, O.L., NORRIS, M.E. and KUHLMEIER, V.A. (2021) *Academic Integrity in Online Assessment: A Research Review*. Front. Educ. 6:639814. doi: 10.3389/feduc.2021.639814

6. MORRIS, E.J. (2018) *Academic integrity matters: five considerations for addressing contract cheating*. Int J Educ Integr 14, 15. https://doi.org/10.1007/s40979-018-0038-5

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