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SPACE-TIME ANISOTROPY AND ASTRONOMICAL OBSERVATIONS ON THE EXAMPLE OF THE KASNER SOLUTION[1]

The pattern of motion of test bodies in space-time described by the anisotropic Kasner solution is studied. Distances and velocities are determined by the methods used in astronomical observations. The motion can be described by the anisotropic Hubble parameter, which depends on the position of the object in the sky. The observed anisotropy decreased over time inversely proportional to the age of the universe, falling below any given threshold for all celestial bodies within a sphere of fixed radius.

K e y w o r d s : ***general relativity, cosmology, astronomy, Kasner metricm anisotropy.***

Background

General relativity is one of the pinnacles of theoretical physics. Among other advantages, it allows you to use any coordinate system to describe space-time. However, this also has its negative side. The physical meaning of the obtained solution is not always clear. This is due to ignorance of the coordinates in which a particular metric is written. Thus, it took several decades to understand the properties of a solution that describes even the simplest Schwarzschild black hole, not to mention the Kerr and Reissner-Nordström metrics.

The most important for modern cosmology is the Friedmann-Lemaître-Robertson-Walker metric, which describes a homogeneous isotropic space-time. A similar role for anisotropic cosmology is played by the Kasner solution obtained back in 1921 (Kasner, 1921). It describes the solution in the absence of matter and dark energy. Its derivation is described in particular in the textbook (Landau, & Lifshitz, 1980). It looks like

$$ds^2 = dt^2 - t^{2p_1}dx^2 - t^{2p_2}dy^2 - t^{2p_3}dz^2 . \quad (1)$$

A system of units is used in which the speed of light is equal to one c=1. We denoted spatial coordinates by letters x^i=(x,y,z). The Latin indices run from 1 to 3 (e.g. i = 1, 2, 3), while the Greek ones change from 0 to 3. All other designations and signs coincide with those used in the book (Landau, & Lifshitz, 1980). The set of constants p_i , called Kasner indices, satisfies two conditions, namely

$$p_1 + p_2 + p_3 = 1, \qquad p_1{}^2 + p_2{}^2 + p_3{}^2 = 1 . \quad (2)$$

One of the indices must lie in the range from -1/3 to 0, the second from 0 to 2/3, the third from 2/3 to 1.

Space-time (1) has a singularity at t = 0. The invariants of the curvature tensor diverge in it, unless one of the indices is equal to 1, and the other two are 0, then this is part of the Minkowski space.

Many exact solutions of the Einstein equations have solutions near their singularities that are a simple generalization of the metric (1) (Stephani et al., 2015). The most general form of the generalized Kasner solution near its singularity was studied in (Lifshitz, & Khalatnikov, 1963). In more complex cases, for example, for homogeneous spaces of Bianchi types IX and VIII, the oscillatory solution near the singularity (Belinskii, Khalatnikov, & Lifshitz, 1970) consists of an infinite number of epochs, during which the space-time is well described by the generalized Kasner solution. The presence of matter or the cosmological constant changes the form of the solution, but not its Kasner asymptotics (see for example (Parnovsky, 2016)).

All of the above draws special attention to the properties of the space-time described by the metric (1). How well do we understand the meaning of Kasner's metric? Here is what is written about this in (Landau, & Lifshitz, 1980): "If we arrange them in the order $p_1<p_2<p_3$, <...> the metric corresponds to a homogeneous but anisotropic space whose total volume increases (with increasing t) proportionally to t; the linear distances along two of the axes (y and z) increase, while they decrease along the third axis (x)." Let us show that this statement, although true, does not give a complete idea of the solution.

Choosing a coordinate system for analyzing solution properties

This interpretation is related to the use of x^α coordinates in which the metric (1) is written. Their choice is natural. The coordinate $x^0 = t$ is the cosmological time; the directions of the vectors of the axes of the x^i coordinates are related to the Killing vectors of the space-time (1). Since it is homogeneous, the coordinate system is not only synchronous, but also comoving. In addition to homogeneity, this is also seen after the addition of test particles with 4-velocity with components u^α =(1,0,0,0), which create a negligibly weak gravitational field, serving exclusively as indicators of motion. They maintain their 4-velocity while remaining fixed in the x^i coordinate system. The metric (1) is simple and depends on only one variable.

These coordinates have only one disadvantage. The curvature invariants drop to zero at $t \to \infty$ and the space-time (1) is asymptotically flat. However, the metric (1) does not tend to the Minkowski metric. To obtain a metric that tends to Minkowski one, the cosmological time t should be supplemented with a set of three spatial coordinates

$$\xi^i = t^{p_i} x^i . \quad (3)$$

Here and below there is no summation over repeated indices. The metric (1) takes the form

$$ds^2 = dt^2 - \sum_{i=1}^{3} \left(d\xi^i - p_i \xi^i t^{-1} dt \right)^2 . \quad (4)$$

[1] The article is presented in the author's edition, since "Recommendations for Authors" were developed after the submission of the manuscript to the editorial office.

The coordinate system t, ξ^i is neither synchronous nor comoving. The metric (4) depends on all 4 coordinates. However, it is a natural system for an observer studying asymptotically flat space-time at $t\to\infty$. In this case, the distances are described by the spatial metric

$$\gamma_{ik} = -g_{ik} + \frac{g_{0i}g_{0k}}{g_{00}} = \delta_{ik} + \frac{p_i p_k \xi^i \xi^k}{t^2 - \sum_{l=1}^{3}\left(p_l \xi^l\right)^2}, \qquad (5)$$

where δ_{ik} is the Kronecker symbol.

These distances are determined by location, i.e. by the time of propagation of light to the object and its return to the starting point. This is one of the most accurate methods used in real measurements of distances to celestial bodies. It is clear that it is used to determine the distance to nearby objects, for example, in laser ranging of the Moon and planets, where the influence of the expansion of the Universe can be neglected. But we could measure the shift of the spectra during the radial motion of nearby test bodies in the space-time exactly described by the metric (1), without taking into account the errors associated with their peculiar velocities. Additionally, at least in a gedanken (thought) experiment, it would be possible to determine the distance even to distant objects using their location.

Let's find the length of the segment along the axis with coordinate ξ^1 ($\xi^2 = \xi^3 = 0$). One end of it is located at the origin of coordinates ξ^i, the second at a point with $\xi^1 = l$. In this case, we assume that the age of the universe, i.e. the cosmological time t from the anisotropic Big Bang at t=0 significantly exceeds the length of the segment t>>l. Then in the denominator of the second term in (5) we can neglect the sum $\sum_{l=1}^{3}\left(p_l \xi^l\right)^2$ compared to t^2. In this case, the length of the segment is

$$L \approx \int_0^l \sqrt{1 + p_1^2(\xi^1)^2 t^{-2}}\, d\xi^1 \approx l + \frac{p_1^2}{6t^2} l^3 \xrightarrow[t\to\infty]{} l. \qquad (6)$$

Observed motion of test particles immobile in the comoving frame

The x^α coordinates are the preferred choice in the theoretical study of space-time properties near the singularity t = 0. But the ξ^i coordinates naturally arise in the study of distances to celestial bodies and their motion, i.e. in astronomy.

It is not difficult to find the motion of matter motionless in the comoving frame x^α with u^α = (1,0,0,0) in the coordinates (t, ξ^i), using the transformation, which in this case is described by the 4-velocity

$$u'^\alpha = \left(1, \frac{\partial \xi^i}{\partial t}\right) = \left(1, \frac{p_i \xi^i}{t}\right). \qquad (7)$$

The speed of movement of the far end of the segment described above is approximately equal to $V \approx p_1 t^{-1} l \approx p_1 t^{-1} L$. So for t >>l we can introduce something like the anisotropic Hubble parameter. It is approximately equal to $p_i t^{-1}$ along the ξ^i-th axis. So it is positive along two axes and negative along the third. It is easy to find this value along any direction. This parameter decreases with time and disappears at $t\to\infty$.

Can we assume that since the velocities of motions of test bodies decrease to zero in the coordinate system (t, ξ^i), then the space-time is isotropized? This is largely determined by what exactly we mean by this term. For example, the anisotropy index entered in (Parnovsky, 2016) remains constant in time. According to this indicator, there is no isotropization of space-time.

Discussion and conclusions

The metric (4) at large values of the cosmological time t tends to the Minkowski metric. Therefore, from the point of view of measuring distances to celestial bodies, the coordinate system (t, ξ^i) is distinguished. Let's call it conditionally "astronomical". This system is very inconvenient for studying the properties of space-time at small t. Among other things, it changes the sign of the g_{00} component of the metric tensor and the components of the distance tensor (5). Therefore, it is more convenient to carry out all calculations in the x^i coordinate system used in the pioneering paper by Kasner.

However, if a conditional observer studies the motion of bodies (test particles in our case) in the space-time (1), then the natural choice of the coordinate system will be either an "astronomical" system or one tending to it. Problems with the metric at $t^2 = \sum_{l=1}^{3}\left(p_l \xi^l\right)^2$ should not worry him or her, because this condition determines the cosmological horizon. Light from more distant objects will not have time to reach the observer during the existence of the universe.

The velocities of test bodies located inside the cosmological horizon, measured in the "astronomical" system, are anisotropic. Some bodies move away from the observer, some approach depending on their position in the sky. But the velocities and the redshifts of any of these bodies decrease as t^{-1}. Therefore, the velocities of all bodies inside a sphere of fixed radius will eventually fall below any given threshold value. For real observations, this threshold value is related to the measurement accuracy and to the characteristic velocity of peculiar motions of bodies. So the hypothetical astronomer would come to the conclusion that the velocity anisotropy is falling or is below the limit of observational accuracy.

When using the coordinate system x^i, the conclusion about the constant anisotropy of the motion of the bodies naturally arises. However, despite the convenience of this system for calculations, it is clearly different from the one based on astronomical observations. It seems to me that the "astronomical" frame of reference is more preferable for the analysis of the observed movements. Therefore, we can conclude that the manifestations of space-time anisotropy observed by astronomers decrease with time.

I add that we considered the vacuum solution (1). If there is a matter or a cosmological constant in the universe, then this leads to a real isotropization of the expansion. The degree of anisotropy in this case decreases with time (Parnovsky, 2016). As a result, deviations from isotropic expansion become unobservable even faster. We are confident that there is both matter and dark energy in the Universe around us, which lead to rapid isotropization during the epoch of inflation. Does this mean that consideration of the vacuum solution (1) is of purely speculative interest? On the one hand, the demonstration of the observed differences between isotropic and anisotropic cosmological models makes the very concept of anisotropy scientific in accordance with Popper's falsifiability criterion. On the other hand, in some scientific articles, for example, (Sarmah, & Goswami, 2022), they try to pass off the effects associated with gravitational instability and the large-scale structure of the Universe formed as a result of it (the Local Cluster, the Great Attractor, the Perseus-Pisces Supercluster, the Shapley concentration, etc), for the observed manifestations of anisotropy.

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АНІЗОТРОПІЯ ПРОСТОРУ-ЧАСУ Й АСТРОНОМІЧНІ СПОСТЕРЕЖЕННЯ НА ПРИКЛАДІ РОЗВ'ЯЗКУ КАЗНЕРА

Вивчено картину руху пробних тіл у просторі-часі, що описується анізотропним розв'язком Казнера. Відстані та швидкості визначено методами, які застосовують в астрономічних спостереженнях. Рух може бути описаний анізотропним параметром Габбла, який залежить від положення об'єкта на небі. Швидкість руху зменшується пропорційно віку Всесвіту, тому анізотропія, що спостерігається, спадає із часом і стає нижчою за будь-яке задане граничне значення для всіх небесних тіл усередині сфери фіксованого радіуса.

Ключові слова: ***загальна теорія відносності; космологія, астрономія; метрика Казнера, анізотропія.***

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