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Oksana SHKILNIAK, PhD (Phys. & Math.), Assoc. Prof.
ORCID ID: 0000-0003-4139-2525
e-mail: oksana.sh@knu.ua
Taras Shevchenko National University of Kyiv, Kyiv, Ukraine

FIRST-ORDER TRANSITIONAL MODAL LOGICS OF QUASIARY PREDICATES WITH EQUALITY

In this paper we study new classes of program-oriented logical formalisms of the modal type – first-order transitional modal logics (TML) of quasiary predicates with equality. Modal logics are successfully used for describing and modeling various subject areas. Epistemic logics are used to describe artificial intelligence, information, and expert systems. Temporal logics are used for modeling complex dynamic systems, program specification and verification. Traditional modal logics are based on classical predicate logic. However, such logic insufficiently accounts for the incompleteness, partiality and structuredness of information about the subject area. These limitations necessitate the development of new, program-oriented modal logical formalisms; this underscores the relevance of the proposed paper. We consider pure first-order TML of partial quasiary predicates without monotonicity condition and enriched with equality predicates. Two types of such logics can be distinguished: TML$^{Q\equiv}$ with strong equality predicates $\equiv_x$, and TML$^{Q=}$ with weak equality predicates $=_{xy}$. A feature of these logics is the use of the extended renomination composition. For quantifier elimination in TML$^{Q\equiv}$, special total indicator predicates Ez are used, while in TML$^{Q=}$ the partial indicator predicates are $=_{zz}$. Transitional modal systems (TMS) form the semantic basis of TML. For TML$^{Q\equiv}$ and TML$^{Q=}$ important varieties of such systems are specified: general (GMS$^{Q\equiv}$ and GMS$^{Q=}$), temporal (TmMS$^{Q\equiv}$ and TmMS$^{Q=}$), and multimodal (MMS$^{Q\equiv}$ and MMS$^{Q=}$). We describe semantic models and languages of these TMS and examine the properties related to equality predicates. Four types of consequence relations in TMS and logical consequence relations on sets of formulas specified with states are considered: irrefutability (IR), truth (T), falsity (F), and strong (TF). The properties of replacement of equals in TML$^{Q\equiv}$ and TML$^{Q=}$ are described. The properties of logical consequence relations on sets of formulas specified with states are the semantic basis for the further construction of the corresponding sequent type calculi.

K e y w o r d s : ***modal logic, transitional modal system, predicate, equality, logical consequence.***

AMS 2020 classification: 62H30, 34C60.

Introduction

The development of information technologies has led to the emergence and practical application of new branches of mathematical logic. Among them, modal logics hold a special place. The concepts and tools of modal logics are successfully used for describing and modeling various subject areas, artificial intelligence systems, and information and software systems (see, for example, Blackburn, van Benthem, & Wolter, 2006; Abramsky, Gabbay, & Maibaum, 1993–2000). Epistemic logics are used to describe artificial intelligence, information, and expert systems, while temporal logics are used for modeling complex dynamic systems, program specification and verification (see Bjorner, & Henson, 2008; Goranko, 2023; Fisher, 2011; Kröger, & Merz, 2008). Modal logics, especially temporal ones, are widely used in model checking (see e. g. Baier, & Katoen, 2008). Traditional modal logics (see Cocchiarella, & Freund, 2008; Fitting, & Mendelsohn, 2023) are based on the classical predicate logic. However, such logic insufficiently accounts for the incompleteness, partiality and structuredness of information about the subject area. These limitations necessitate the development of new, program-oriented modal logical formalisms; this underscores the relevance of the proposed paper.

The capabilities of traditional modal logics and program-oriented composition nominative logics of partial quasiary predicates, described, in particular, in works (Shkilniak, O., & Shkilniak, S., 2023), are combined in composition nominative modal logics (CNML). An important class of CNML is transitional modal logics (TML), which reflect the aspect of change and development of subject areas. These logics have been studied, in particular, in work (Shkilniak, O., 2015). Important subclasses of TML studied in these works include general TML (GML), temporal TML (TmML) and multimodal TML (MML), with epistemic TML types further distinguished within MML.

Traditional systems of modal, temporal and epistemic logics can naturally be considered within TML.

The goal of this paper is to investigate new classes of program-oriented modal logics – pure first-order TML of partial quasiary predicates without monotonicity (equitonicity) restriction and enriched with equality predicates. Two types of such predicates can be distinguished (see Shkilniak, 2019): the weak equality $=_{xy}$ and the strong equality $\equiv_{xy}$. Pure first-order TML without monotonicity condition are designated as TML^{Q}. In work (Shkilniak, 2015), TML^{Q} without equality predicates have been considered. In work (Shkilniak, O., & Shkilniak S., 2024) TML^{Q} with strong equality predicates have been studied. In this paper we propose TML^{Q} with weak equality predicates. TML^{Q} with strong equality predicates will be designated as TML$^{Q\equiv}$, and TML^{Q} with weak equality predicates will be designated as TML$^{Q=}$.

For quantifier elimination in the logics of non-monotone predicates, special predicates are needed to indicate the presence of a component with a corresponding name in the input data. The use of predicates-indicators is a characteristic feature of TML^{Q}. Total predicates-indicators determine the presence or absence of a component with a given name, while partial predicates-indicators only detect the presence of such a component. Total predicates-indicators *Ez* were used in works (Nikitchenko, Shkilniak, O., & Shkilniak, S., 2016; Shkilniak, 2019; Shkilniak, 2015). It should be noted that the predicates *Ez* can be expressed (see Shkilniak, 2019) as $\exists y \equiv_{xy}$. However, they appear at the renominative level, so for TML$^{Q\equiv}$, it is appropriate to define them explicitly as special 0-ary compositions. At the same time, partial predicates-indicators are already present in TML$^{Q=}$ as the $=_{xx}$ predicates.

Another feature of the studied TML^{Q} with equality is the presence of extended renominations, described in (Shkilniak, O., & Shkilniak S., 2023); they were used in (Shkilniak, O., & Shkilniak, S., 2024). Note that in the modal logics considered in (Shkilniak, 2015), traditional renominations were used.

The present paper describes semantic models and languages of TML^Q with equality and examines the properties of these logics related to equality predicates. A number of consequence relations in TML^Q and logical consequence relations on sets of formulas specified with states are considered, and the properties of replacement of equals in $TML^{Q=}$ and $TML^{Q\equiv}$ are determined. The properties of these relations form the semantic basis for the further construction of corresponding sequent type calculi for TML^Q with equality.

Concepts not defined here are interpreted in the sense of works (Shkilniak, 2015; Nikitchenko, Shkilniak, O., & Shkilniak, S., 2016; Shkilniak, 2019; Shkilniak, O., & Shkilniak, S., 2023; Shkilniak, O., & Shkilniak, S., 2024).

1. Transitional modal systems

Transitional modal logics (TML) is a very important class of CNML, they describe transitions between states of the world and thus reflect the aspect of change and evolution of subject areas. Transitional modal system (TMS) is the basic concept of TML. It is an object (see Shkilniak, O., 2015) $\boldsymbol{M} = (Cms, Ds, Dns)$, where:

– Cms is a composition modal system (CMS), it defines semantic aspects of the world;

– Ds is a descriptive system, it defines a set of standard descriptions – formulas of the corresponding TML language;

– Dns is a denotation system, it determines values of standard descriptions on semantic models: an interpretation mapping is used for this.

CMS have the form $Cms = (St, \boldsymbol{R}, Pr, C)$, where:

– St is a set of states of the world;

– $\boldsymbol{R}$ is a set of relations on St of the form $R \subseteq St \times St$; they can be interpreted as state transition relations, hence the name;

– Pr is a set of predicates on states of the world;

– C is a set of compositions on Pr.

Pure first-order TMS (TMS of quantifier level) will be designated as TMS^Q. Here St is specified as a set of algebraic structures (systems) $\alpha = (A_\alpha, Pr_\alpha)$, where A_α is a set of basic data of the state α, Pr_α is a set of quasiary predicates ${}^VA_\alpha \to \{T, F\}$, called *predicates of the state* α. Let us call *global* the predicates ${}^VA \to \{T, F\}$, where $A = \bigcup_{\alpha \in S} A_\alpha$.

V-A-quasiary predicate is (see Nikitchenko, Shkilniak, O., & Shkilniak, S., 2016) a partial function of the form $Q : {}^VA \to \{T, F\}$, where V is a set of subject names (variables), A is a set of subject values (basic data), $\{T, F\}$ is the set of truth values, ${}^VA$ is a set of all V-A-nominative sets. V-A-nominative set (V-A-NS) is defined as a single-valued function of the form $d : V \to A$. V-A-NS is presented as a set of pairs $[v_1 \mapsto a_1,...,v_n \mapsto a_n,...]$, where $v_i \in V$, $a_i \in A$, $v_i \neq v_j$ when $i \neq j$.

For V-A-NS we introduce an operation $\|_{-Z}$, where $Z \subseteq V$: $d\|_{-Z} = \{v \mapsto a \in d \mid v \notin Z\}$. Also a parametric operation of the extended renomination is specified $r^{\bar{v},\bar{u}}_{\bar{x},\perp} : {}^VA \to {}^VA$ (see Shkilniak O., & Shkilniak S., 2023).

Each V-A-quasiary predicate Q can be determined by two sets:

– the *truth* domain $T(Q) = \{d \in {}^VA \mid Q(d) = T\}$;

– the *falsity* domain $F(Q) = \{d \in {}^VA \mid Q(d) = F\}$.

Predicate Q is *single-valued*, or *P-predicate*, if $T(Q) \cap F(Q) = \varnothing$.

The class of V-A-quasiary single-valued predicates will be denoted by PrP^{V-A}.

For modal logics we specifically consider single-valued predicates.

Predicate Q is:

– *valid* (*irrefutable*), if $F(Q) = \varnothing$;

– *satisfiable*, if $T(Q) \neq \varnothing$.

In the PrP^{V-A} class, there are 3 constant predicates denoted by T, F and $\perp$:

– Q is identically true, if $T(Q) = {}^VA$ and $F(Q) = \varnothing$;

– Q is identically false, if $T(Q) = \varnothing$ and $F(Q) = {}^VA$;

– Q is totally undefined, if $T(Q) = \varnothing$ and $F(Q) = \varnothing$.

For single-valued functions (and predicates), the concept of monotonicity is specified as equitonicity – preservation of the already accepted value after data expansion (see Nikitchenko, Shkilniak, O., & Shkilniak, S., 2016).

P-predicate Q is *equitone*, or *PE*-predicate, if ($Q(d)\downarrow$ and $d \subseteq d'$) $\Rightarrow Q(d')\downarrow = Q(d)$.

Subject name $x \in V$ is *unessential* for the predicate Q, if $d\|_{-x} = h\|_{-x} \Rightarrow Q(d) = Q(h)$.

The set of compositions in TMS is defined by the basic *logical* compositions and basic *modal* compositions.

Basic logical compositions for TMS^Q are logical connectives $\neg$ and $\vee$, renomination $R^{\bar{v},\bar{u}}_{\bar{x},\perp}$ and existential quantification $\exists x$. For TMS^Q with equality, equality predicates are added as specific 0-ary compositions. Two types of equality predicates are considered: weak equality $=_{xy}$ and strong equality $\equiv_{xy}$. Thus, TMS^Q with predicates $\equiv_{xy}$ will be called $TMS^{Q\equiv}$, and TMS^Q with predicates $=_{xy}$ will be called $TMS^{Q=}$.

Other compositions like &, $\to$, $\forall x$ can be definable from these.

Basic compositions $\neg$, $\vee$, $R^{\bar{v},\bar{u}}_{\bar{x},\perp}$, $\exists x$ and derivative compositions &, $\to$, $\forall x$ are defined in (Nikitchenko, Shkilniak, O., & Shkilniak, S., 2016; Shkilniak, O., & Shkilniak, S., 2023).

Let us specify two types of equality predicates according to work (Shkilniak, 2019):

– weak (up to definability) equality predicates $=_{\{x,y\}}$;

– strong (strict) equality predicates $\equiv_{\{x,y\}}$.

We define $=_{\{x,y\}}$ and $\equiv_{\{x,y\}}$ through their truth and falsity domains:

$T(=_{\{x,y\}}) = \{d \mid d(x)\downarrow, d(y)\downarrow \text{ and } d(x) = d(y)\}$,

$F(=_{\{x,y\}}) = \{d \mid d(x)\downarrow, d(y)\downarrow \text{ and } d(x) \neq d(y)\}$;

$T(\equiv_{\{x,y\}}) = \{d \mid d(x)\downarrow, d(y)\downarrow \text{ and } d(x) = d(y)\} \cup \{d \mid d(x)\uparrow \text{ and } d(y)\uparrow\}$,

$F(\equiv_{\{x,y\}}) = \{d \mid d(x)\downarrow, d(y)\downarrow, d(x) \neq d(y)\} \cup \{d \mid d(x)\downarrow, d(y)\uparrow \text{ or } d(x)\uparrow, d(y)\downarrow\}$.

In case x and y match, predicates $=_{\{x,y\}}$ and $\equiv_{\{x,y\}}$ become $=_{\{x\}}$ and $\equiv_{\{x\}}$.

Further on, predicates $=_{\{x,y\}}$, $=_{\{x\}}$ and $\equiv_{\{x,y\}}$, $\equiv_{\{x\}}$ will be denoted more traditionally by $=_{xy}$, $=_{xx}$ and $\equiv_{xy}$, $\equiv_{xx}$ respectively. From the definitions implies that pairs $=_{xy}$ and $=_{yx}$, $\equiv_{xy}$ and $\equiv_{yx}$ represent the same predicates.

Predicates $\equiv_{xy}$ and $\equiv_{xx}$ are total and non-monotone (non-equitone), while predicates $=_{xy}$ and $=_{xx}$ are partial and equitone.

Special predicates-indicators which detect whether a component with a corresponding name has a value in the input data are needed for quantifier elimination. Total predicates-indicators can distinguish either the subject variable *has* an assigned value or *not*, while partial predicates-indicators can only *ascertain* that a value is assigned.

Total predicates-indicators Ez are non-monotone and defined (see Nikitchenko, Shkilniak, O., & Shkilniak, S., 2016) as follows:

$T(Ez) = \{d \mid d(z)\downarrow\}$;

$F(Ez) = \{d \mid d(z)\uparrow\}$.

$TMS^{Q=}$ don't need separate predicates-indicators as predicates $=_{zz}$ *already* work that way:

$T(=_{zz}) = \{d \mid d(z)\downarrow\} = T(Ez)$;

$F(=_{zz}) = \{d \mid d(z)\downarrow \text{ and } d(z) \neq d(z)\} = \varnothing$.

Equality predicates and predicates-indicators can be interpreted as special 0-ary parametric compositions.

We have (see Shkilniak, S., 2019) $Ez = \exists y \equiv_{zy}$, thus predicates Ez are derivative in $TMS^{Q\equiv}$. However, Ez can be introduced on the renominative level, hence it is convenient to specify them explicitly as 0-ary compositions.

Therefore, we obtain the following sets of basic logical compositions: $\{\neg, \vee, R_{\bar{x},\perp}^{\bar{v},\bar{u}}, \exists x, \equiv_{xy}, Ez\}$ for $TMS^{Q\equiv}$ and $\{\neg, \vee, R_{\bar{x},\perp}^{\bar{v},\bar{u}}, \exists x, =_{xy}\}$ for $TMS^{Q=}$. Their properties are described in (Nikitchenko, Shkilniak, O., & Shkilniak, S., 2016; Shkilniak, S., 2019; Shkilniak, O., & Shkilniak, S., 2023).

Important special cases of TMS are (see Shkilniak, 2015):

– *general* TMS (GMS), with $\boldsymbol{R} = \{\rhd\}$ and a basic modal composition $\Box$ ("necessary");

– *temporal* TMS (TmMS), with $\boldsymbol{R} = \{\rhd\}$ and basic modal compositions $\Box\uparrow$ ("it will always be the case") i $\Box\downarrow$ ("it has always been the case");

– *multimodal* TMS (MMS), with $\boldsymbol{R} = \{\rhd_i \mid i \in I\}$ and basic modal compositions M_i, $i \in I$ in which each $\rhd_i \in \boldsymbol{R}$ is matched with a corresponding modal composition M_i; each M_i is basically $\Box$ with respect to its relation $\rhd_i$.

Derivative compositions are also usually defined. For GMS, $\Diamond$ ("possibly"): $\Diamond P$ means $\neg\Box\neg P$.

For TmMS, $\Diamond\uparrow$ ("it will sometime be the case") and $\Diamond\downarrow$ ("it was sometime the case"): $\Diamond\uparrow P$ means $\neg\Box\uparrow\neg P$, $\Diamond\downarrow P$ means $\neg\Box\downarrow\neg P$.

The following types of TMS^{Q} with equality can be obtained. For $TMS^{Q\equiv}$, we have $GMS^{Q\equiv}$, $TmMS^{Q\equiv}$, $MMS^{Q\equiv}$; for $TMS^{Q=}$, we have $GMS^{Q=}$, $TmMS^{Q=}$, $MMS^{Q=}$.

2. Languages of transitional modal systems

Let us describe the language of $GMS^{Q\equiv}$. The alphabet: a set V of subject names (variables); a set Ps of predicate symbols (language signature); a set of basic logical compositions' symbols $\{\neg, \vee, R_{\bar{x},\perp}^{\bar{v},\bar{u}}, \exists x, \equiv_{xy}, Ex\}$; a set of basic modal compositions' symbols (modal signature) $Ms = \{\Box\}$. The set Fr of language formulas is determined inductively:

Fa) $Ps \subseteq Fr$;

Fc) $\{Ex \mid x \in V\} \subseteq Fr$ and $\{\equiv_{xy} \mid x, y \in V\} \subseteq Fr$;

FpR) $\Phi, \Psi \in Fr \Rightarrow \neg\Phi \in Fr$, $\vee\Phi\Psi \in Fr$, and $R_{\bar{x},\perp}^{\bar{v},\bar{u}}\Phi \in Fr$;

F∃) $\Phi \in Fr \Rightarrow \exists x\Phi \in Fr$;

F□) $\Phi \in Fr \Rightarrow \Box\Phi \in Fr$.

Formulas of the type $p \in Ps$, Ex, $\equiv_{xy}$ are *atomic*.

We will use traditional derivative compositions and thus write $\Phi\&\Psi$, $\Phi\to\Psi$, $\forall x\Phi$, $\Diamond\Phi$ in place of formulas $\vee\neg\Phi\Psi$, $\neg\vee\neg\Phi\neg\Psi$, $\neg\exists x\neg\Phi$, $\neg\Box\neg\Phi$.

Sets of guaranteed to be unessential names for formulas are defined by the function $\nu : Fr \to 2^V$ (see Shkilniak, 2019; Shkilniak, O., & Shkilniak, S., 2023).

A $GMS^{Q\equiv}$ type is determined by an extended signature $\sigma = (Ps, \nu)$ and properties of the relation $\rhd$.

An interpretation mapping Im is specified for formulas for TMS^{Q}. compositions' symbols are interpreted as corresponding compositions. For basic predicates, we have $Im : PS \times ST \to PR$, with condition $IT(P, \alpha) \in PR_{\alpha}$. the mapping Im is continued to $FT \times ST \to PR$ as follows:

IpR) $Im(\neg, \alpha) = \neg(Im(\Phi, \alpha))$; $Im(\vee\Phi\Psi, \alpha) = \vee(Im(\Phi, \alpha), Im(\Psi, \alpha))$; $Im(R_{\bar{x},\perp}^{\bar{v},\bar{u}}(\Phi), \alpha) = R_{\bar{x},\perp}^{\bar{v},\bar{u}}(Im(\Phi, \alpha))$;

F∃) $\Phi \in Fr \Rightarrow \exists x\Phi \in Fr$;

F□) $\Phi \in Fr \Rightarrow \Box\Phi \in Fr$.

Formulas of the type $p \in Ps$, Ex, $\equiv_{xy}$ are *atomic*.

We will use traditional derivative compositions and thus write $\Phi\&\Psi$, $\Phi\to\Psi$, $\forall x\Phi$, $\Diamond\Phi$ in place of formulas $\vee\neg\Phi\Psi$, $\neg\vee\neg\Phi\neg\Psi$, $\neg\exists x\neg\Phi$, $\neg\Box\neg\Phi$.

Sets of guaranteed to be unessential names for formulas are defined by the function $\nu : Fr \to 2^V$ (see Shkilniak, 2019; Shkilniak, O., & Shkilniak, S., 2023).

A $GMS^{Q\equiv}$ type is determined by an extended signature $\sigma = (Ps, \nu)$ and properties of the relation $\rhd$.

An interpretation mapping Im is specified for formulas for TMS^Q. compositions' symbols are interpreted as corresponding compositions. For basic predicates, we have $Im: PS\times ST\to PR$, with condition $IT(P,\alpha)\in PR_\alpha$. the mapping Im is continued to $FT\times ST\to PR$ as follows:

IpR) $Im(\neg,\alpha)=\neg(Im(\Phi,\alpha))$; $Im(\vee\Phi\Psi,\alpha)=\vee(Im(\Phi,\alpha), Im(\Psi,\alpha))$; $Im(R^{\bar{v},\bar{u}}_{\bar{x},\perp}(\Phi),\alpha)=\mathrm{R}^{\bar{v},\bar{u}}_{\bar{x},\perp}(Im(\Phi,\alpha))$;

I∃) $$Im(\exists x\Phi,\alpha)(d)=\begin{cases}T, \text{ if } Im(\Phi,\alpha)(d\,\|_{-x}\cup x\mapsto a)=T \text{ for some } a\in A_\alpha,\\ F, \text{ if } Im(\Phi,\alpha)(d\,\|_{-x}\cup x\mapsto a)=F \text{ for any } a\in A_\alpha,\\ \text{else undefined;}\end{cases}$$

I□) $$Im(\Box\Phi,\alpha)(d)=\begin{cases}T, \text{ if } Im(\Phi,\delta)(d)=T \text{ for any } \delta\in St:\ \alpha\rhd\delta,\\ F, \text{ if exists } \delta\in St:\ \alpha\rhd\delta \text{ and } Im(\Phi,\delta)(d)=F,\\ \text{else undefined.}\end{cases}$$

Given for $\alpha\in St$ there is no β such that $\alpha\rhd\beta$, then for each $d\in {}^VA$ we define $Im(\Box{\uparrow}\Phi,\alpha)(d)\uparrow$.

The interpretation mapping Im can be specified in the similar manner for the derivative formulas $\Phi\&\Psi$, $\Phi\to\Psi$, $\forall x\Phi$, $\Diamond\Phi$.

Predicates which are values of formulas that do not use modalities (i.e. contain no symbols from Ms) belong to predicates of states. Predicates which are values of modalized formulas are global.

The language of GMS$^{Q=}$ is defined similarly. It has a different set of basic logical compositions' symbols: $\{\neg,\vee,R^{\bar{v},\bar{u}}_{\bar{x},\perp},\exists x,=_{xy}\}$. The rule Fc for language formulas has a form $\{=_{xy}\mid x,y\in V\}\subseteq Fr$; the interpretation mapping is modified accordingly.

The languages of TmMS$^{Q\equiv}$ and TmMS$^{Q=}$ are also specified similarly to the languages of GMS$^{Q\equiv}$ and GMS$^{Q=}$.

The set of basic modal compositions' symbols is $Ms=\{\Box\uparrow, \Box\downarrow\}$. In the definition of language formulas Fm, the rule F□ changes to:

F□↑↓) $\Phi\in Fr\ \Rightarrow\ \Box{\uparrow}\Phi\in Fr$ and $\Box{\downarrow}\Phi\in Fr$.

Short notation $\Diamond{\uparrow}\Phi$ AND $\Diamond{\downarrow}\Phi$ can be used instead of $\neg\Box{\uparrow}\neg\Phi$ AND $\neg\Box{\downarrow}\neg\Phi$.

Interpretation mapping Im should take into account time direction:

I□↑↓) $$Im(\Box{\uparrow}\Phi,\alpha)(d)=\begin{cases}T, \text{ if } Im(\Phi,\delta)(d)=T \text{ for any } \delta\in St:\ \alpha\rhd\delta,\\ F, \text{ if exists } \delta\in St:\ \alpha\rhd\delta \text{ and } Im(\Phi,\delta)(d)=F,\\ \text{else undefined.}\end{cases}$$

$$Im(\Box{\downarrow}\Phi,\alpha)(d)=\begin{cases}T, \text{ if } Im(\Phi,\delta)(d)=T \text{ for any } \delta\in St:\ \delta\rhd\alpha,\\ F, \text{ if exists } \delta\in St:\ \delta\rhd\alpha \text{ and } Im(\Phi,\delta)(d)=F,\\ \text{else undefined.}\end{cases}$$

Given for $\alpha\in St$ there is no β such that $\alpha\rhd\beta$, then for each $d\in {}^VA$ we define $Im(\Box{\uparrow}\Phi,\alpha)(d)\uparrow$.

Given for $\alpha\in St$ there is no β such that $\beta\rhd\alpha$, then for each $d\in {}^VA$ we define $Im(\Box{\downarrow}\Phi,\alpha)(d)\uparrow$.

Similarly, the languages of MMS$^{Q\equiv}$ and MMS$^{Q=}$ are defined in the same fashion as the languages of GMS$^{Q\equiv}$ and GMS$^{Q=}$. The set of basic modal compositions' symbols changes to $Ms=\{\mathrm{M}_i\mid i\in I\}$. In the definition of language formulas Fm, the rule F□ changes to FM) $\Phi\in Fr\Rightarrow \mathrm{M}_i\Phi\in Fr$.

For interpretation mapping Im, for modalities we have:

IM) $$Im(\mathrm{M}_i\Phi,\alpha)(d)=\begin{cases}T, \text{ if } Im(\Phi,\delta)(d)=T \text{ for any } \delta\in St:\ \alpha\rhd_i\delta,\\ F, \text{ if exists } \delta\in St:\ \alpha\rhd_i\delta \text{ and } Im(\Phi,\delta)(d)=F,\\ \text{else undefined.}\end{cases}$$

Given for $\alpha\in St$ there is no β such that $\alpha\rhd_i\beta$, then for each $d\in {}^VA$ we define $Im(\mathrm{M}_i\Phi,\alpha)(d)\uparrow$.

In case the interpretation Im is defined explicitly, TMS can be represented as $\boldsymbol{M}=(St,\boldsymbol{R},A,Im)$.

Predicate $Im(\Phi,\alpha)$ which is a value of the formula Φ in the state α is denoted by Φ_α.

Formula Φ is *valid* (*irrefutable*) *on state* α (denoted $\alpha\models\Phi$), if Φ_α is a valid (irrefutable) predicate.

Formula Φ is *valid in* TMS $\boldsymbol{M}$ (denoted $\boldsymbol{M}\models\Phi$), if for each $\alpha\in St$ the predicate Φ_α is valid.

Let $\mathcal{M}$ be a TMS class of a given type. Formula Φ is $\mathcal{M}$-*valid* (denoted $\mathcal{M}\models\Phi$), if $\boldsymbol{M}\models\Phi$ for all $\boldsymbol{M}\in\mathcal{M}$.

Depending on conditions imposed on the relation $\rhd$, different classes of TMS^Q can be specified. Traditionally, cases where $\rhd$ can be reflexive, symmetric or transitive are considered. In case $\rhd$ is reflexive/transitive/symmetric, then the corresponding symbol R/T/S will appear in the TMS^Q name. Thus, the following classes can be obtained:

R-GMS$^{Q\equiv}$, *T*-GMS$^{Q\equiv}$, *S*-GMS$^{Q\equiv}$, *RT*-GMS$^{Q\equiv}$, *RS*-GMS$^{Q\equiv}$, *TS*-GMS$^{Q\equiv}$, *RTS*-GMS$^{Q\equiv}$;
R-GMS$^{Q=}$, *T*-GMS$^{Q=}$, *S*-GMS$^{Q=}$, *RT*-GMS$^{Q=}$, *RS*-GMS$^{Q=}$, *TS*-GMS$^{Q=}$, *RTS*-GMS$^{Q=}$;
R-TmMS$^{Q\equiv}$, *T*-TmMS$^{Q\equiv}$, *S*-TmMS$^{Q\equiv}$, *RT*-TmMS$^{Q\equiv}$, *RS*-TmMS$^{Q\equiv}$, *TS*-TmMS$^{Q\equiv}$, *RTS*-TmMS$^{Q\equiv}$;
R-TmMS$^{Q=}$, *T*-TmMS$^{Q=}$, *S*-TmMS$^{Q=}$, *RT*-TmMS$^{Q=}$, *RS*-TmMS$^{Q=}$, *TS*-TmMS$^{Q=}$, *RTS*-TmMS$^{Q=}$.

For MMS, the situation is more complicated. Given that all $\rhd_i$ have the same type, we can obtain pure MMS$^{Q\equiv}$ classes similar to the corresponding GMS$^{Q\equiv}$ classes. However, more complex mixed types are also possible: for example, transitive $\rhd_1$, symmetric $\rhd_2$, reflexive and transitive $\rhd_3$ etc.

Let us consider how the value of $\Phi_\delta(d)$ can be set in case $d\notin {}^VA_\delta$. Two types of TMS are specified (see also Shkilniak, O., & Shkilniak, S., 2024): with *strong* condition of definedness on states and with *general* condition of definedness on states. The *strong* condition requires $\Phi_\delta(d)\uparrow$ for $d\notin {}^VA_\delta$. Hence: $\Phi_\delta(d)\downarrow\ \Rightarrow d\in {}^VA_\delta$. Then $(\Box\Phi)_\alpha(d)=T$ implies that $d\in {}^VA_\delta$ for all δ such that $\alpha\rhd\beta$. This means that basic objects (basic data) cannot disappear when transitioning to a successor state, which imposes strong restrictions on semantic models. Equitonicity of predicates with modalities is also violated. Therefore, this condition turns out too restrictive.

The *general* condition of definedness on states looks more organic. Let d_δ denote the name set $[v \mapsto a \in d \mid a \in A_\delta]$. For non-modalized formulas for all $d \in {}^VA$ and $\delta \in St$ we specify $\Phi_\delta(d) = \Phi_\delta(d_\delta)$. Given $d \notin {}^VA_\delta$, we obtain $\Phi_\delta(d) = \Phi_\delta(d_\delta)$. This means that predicates on states δ "perceive" only such components $v \mapsto a$ that $a \in A_\delta$. Moreover, in TMS with the general condition of definedness on states basic modal compositions preserve equitonicity. This is an additional reason to concentrate on TMS with such condition of definedness on states.

Further on, we will consider the case of GMS^Q. Similar statements can be formulated for TmMS and MMS.

Let us consider the interaction of modalities with renominations and quantifiers.

Symbols of modal compositions *Ms* can be carried through renominations:

Theorem 1. $R^{\bar{v},\bar{u}}_{\bar{x},\perp}(\Box\Phi)(d) = \Box(R^{\bar{v},\bar{u}}_{\bar{x},\perp}(\Phi))(d)$ *for arbitrary* $\Phi \in Fr$, $d \in {}^VA$.

Corollary 1. *Formulas* $R^{\bar{v},\bar{u}}_{\bar{x},\perp}(\Box\Phi) \to \Box(R^{\bar{v},\bar{u}}_{\bar{x},\perp}(\Phi))$ *and* $\Box(R^{\bar{v},\bar{u}}_{\bar{x},\perp}(\Phi)) \to R^{\bar{v},\bar{u}}_{\bar{x},\perp}(\Box\Phi)$ *are valid.*

The interaction of modal compositions and quantifiers is more complex. For modalities $\Box$ and $\Diamond$ and quantifiers $\exists x$ and $\forall x$ we get:

Theorem 2 (see also Shkilniak, O., & Shkilniak, S., 2024). *In a GMS of equitone predicates*:

1) *formulas* $\Box\exists x\Phi \to \exists x\Box\Phi$, $\forall x\Box\vartheta \to \Box\forall x\vartheta$, $\forall x\Diamond\Psi \to \Diamond\forall x\Psi$, $\Diamond\exists x\theta \to \exists x\Diamond\theta$ *are refutable;*

2) *formulas* $\exists x\Box\Phi \to \Box\exists x\Phi$, $\Box\forall x\vartheta \to \forall x\Box\vartheta$, $\Diamond\forall x\Psi \to \forall x\Diamond\Psi$, $\exists x\Diamond\theta \to \Diamond\exists x\theta$ *are valid (irrefutable).*

At the same time, formulas $\exists x\Box\Phi \to \Box\exists x\Phi$, $\Box\forall x\vartheta \to \forall x\Box\vartheta$, $\Diamond\forall x\Psi \to \forall x\Diamond\Psi$, $\exists x\Diamond\theta \to \Diamond\exists x\theta$ are refutable in GMS^Q without monotonicity restrictions.

3. Properties of GMS^Q with equality

Let us consider properties of equality predicates in GMS^Q.

Assertion 1. 1) *Formulas* $\equiv_{xy}$ *and* $\Box\equiv_{xy}$ *are refutable in* $GMS^{Q\equiv}$ *with 1-element states*;

2) *Formulas* $=_{xy}$ *and* $\Box=_{xy}$ *are irrefutable in* $GMS^{Q=}$ *with 1-element states*.

Proof. Let us specify such $GMS^{Q\equiv}$. We define $St = \{\alpha, \beta\}$, $\boldsymbol{R} = \{\alpha \rhd \beta\}$, $A_\alpha = A_\beta = \{a\}$. Let us take $d = [x \mapsto a]$. Then $d_\alpha(y)\uparrow$ and $d_\beta(y)\uparrow$, hence $(\equiv_{xy})_\alpha(d) = F$ and $(\equiv_{xy})_\beta(d) = F$, therefore $(\Box\equiv_{xy})_\alpha(d) = F$. So, $\alpha \not\models \equiv_{xy}$ and $\alpha \not\models \Box\equiv_{xy}$.

For 1-element interpretations with $A = \{a\}$ for each $d \in {}^V\{a\}$ impossible $d(x)\downarrow$, $d(y)\downarrow$ and $d(x) \neq d(y)$, hence impossible $=_{xy}(d) = F$. Therefore, $=_{xy}$ and $\Box=_{xy}$ are irrefutable in $GMS^{Q=}$ with 1-element states.

Note that formulas $x = y$ are irrefutable under 1-element interpretations in classical logic as well.

For equality predicates we have $F(\equiv_{xx}) = \varnothing$ and $F(=_{xx}) = \varnothing$. We obtain

Assertion 2. 1) $\boldsymbol{M} \models \equiv_{xx}$ *and* $\boldsymbol{M} \models \Box\equiv_{xx}$ *for each* $GMS^{Q\equiv}$ $\boldsymbol{M}$;

2) $\boldsymbol{M} \models =_{xx}$ *and* $\boldsymbol{M} \models \Box=_{xx}$ *for each* $GMS^{Q=}$ $\boldsymbol{M}$.

Proof. Predicate $=_{xx}$ is partial, therefore formulas $=_{xx}$ and $\Box=_{xx}$ cannot be interpreted as a constant predicate T.

Predicate $\equiv_{xx}$ is a constant T, hence formula $\equiv_{xx}$ is always interpreted as T. At the same time, formula $\Box\equiv_{xx}$ is not always interpreted as T. Indeed, let us consider $GMS^{Q\equiv}$ with $St = \{\alpha, \beta\}$ and $\boldsymbol{R} = \{\alpha \rhd \beta\}$. As there does not exist a state η such that $\beta \rhd \eta$, therefore $(\Box\equiv_{xx})_\beta(d)\uparrow$ for each d.

Theorem 3. *Formulas* $=_{xy} \to \Box=_{xy}$ and $\Box=_{xy} \to =_{xy}$ *are irrefutable in* $GMS^{Q=}$.

Proof. Let us prove that $=_{xy} \to \Box=_{xyz}$ by contradiction. Suppose $=_{xy} \to \Box=_{xy}$ is refuted in a $GMS^{Q=}$ $\boldsymbol{M} = (St, \boldsymbol{R}, A, Im)$. Then for some $\alpha \in St$ and $d \in {}^VA$ we have $(=_{xy} \to \Box=_{xy})_\alpha(d_\alpha) = F$, hence $(=_{xy})_\alpha(d_\alpha) = T$ and $(\Box=_{xy})_\alpha(d_\alpha) = F$. The latter implies that there exists $\beta \in St$ such that $\alpha \rhd \beta$ and $(=_{xy})_\beta(d_\beta) = F$, hence $d_\beta(x)\downarrow$, $d_\beta(y)\downarrow$ and $d_\beta(x) \neq d_\beta(y)$. Therefore, there exist $a, b \in A_\beta$ such that $a \neq b$, $d_\beta(x) = a$ and $d_\beta(y) = b$, thence $d(x) = a$ and $d(y) = b$.

$(=_{xy})_\alpha(d_\alpha) = T$ implies that $d_\alpha(x)\downarrow$, $d_\alpha(y)\downarrow$ and $d_\alpha(x) = d_\alpha(y)$. Given $d_\alpha(x)\downarrow$ and $d_\alpha(y)\downarrow$, we have $d_\alpha(x) = d(x) = a$ and $d_\alpha(y) = d(y) = b$. However, $a \neq b$, which contradicts $d_\alpha(x) = d_\alpha(y)$.

Let us prove the irrefutability of $\Box=_{xy} \to =_{xy}$. Assume the opposite: $\Box=_{xy} \to =_{xy}$ is refuted in a $GMS^{Q=}$ $\boldsymbol{M} = (St, \boldsymbol{R}, A, Im)$. Then for some $\alpha \in St$ and $d \in {}^VA$ we have $(\Box=_{xy} \to =_{xy})_\alpha(d_\alpha) = F$, hence $(\Box=_{xy})_\alpha(d_\alpha) = T$ and $(=_{xy})_\alpha(d_\alpha) = F$. From $(\Box=_{xy})_\alpha(d_\alpha) = T$ we obtain: for all $\beta \in St$ such that $\alpha \rhd \beta$, we have $(=_{xy})_\beta(d_\beta) = T$, therefore for all such β we have $d_\beta(x)\downarrow$, $d_\beta(y)\downarrow$ and $d_\beta(x) = d_\beta(y)$. Such β that $\alpha \rhd \beta$ should exist, else $(\Box=_{xy})_\alpha(d)\uparrow$ for all $d \in {}^VA$. Let us fix such a β. Then there exists $a \in A_\beta$ such that $d_\beta(x) = d_\beta(y) = a$, hence $d(x) = d(y) = a$. From $(=_{xy})_\alpha(d_\alpha) = F$ we obtain $d_\alpha(x)\downarrow$, $d_\alpha(y)\downarrow$ and $d_\alpha(x) \neq d_\alpha(y)$. From $d_\alpha(x)\downarrow$ and $d_\alpha(y)\downarrow$ we obtain $d_\alpha(x) = d(x)$ and $d_\alpha(y) = d(y)$. However, $d(x) = d(y) = a$, thence $d_\alpha(x) = d_\alpha(y) = a$, which contradicts $d_\alpha(x) \neq d_\alpha(y)$.

Corollary 2. *Formulas* $=_{xx} \to \Box=_{xx}$ *and* $\Box=_{xx} \to =_{xx}$ *are irrefutable in* $GMS^{Q=}$.

For $GMS^{Q\equiv}$, the situation is different (see (Shkilniak O. and Shkilniak S., 2024)):

Theorem 4. *Formulas* $\equiv_{xy} \to \Box\equiv_{xy}$ *and* $\Box\equiv_{xy} \to \equiv_{xy}$ *are refutable in* $GMS^{Q\equiv}$.

The following statement is similar to Theorem 4, but formulated for predicates-indicators *Ex*.

Theorem 5. *Formulas* $Ex \to \Box Ex$ *and* $\Box Ex \to Ex$ *are refutable in* $GMS^{Q\equiv}$.

Proof. Let us consider $GMS^{Q\equiv}$ with $St = \{\alpha, \beta\}$, $\boldsymbol{R} = \{\alpha \rhd \beta\}$, $A_\alpha = \{a\}$, $A_\beta = \{b\}$. Let $d = [x \mapsto a, y \mapsto b] \Rightarrow d_\alpha = [x \mapsto a]$ and $d_\beta = [y \mapsto b]$. Hence $d_\alpha(x) = a$ and $d_\beta(x)\uparrow \Rightarrow Ex_\alpha(d) = Ex_\alpha(d_\alpha) = T$ and $Ex_\beta(d) = Ex_\beta(d_\beta) = F \Rightarrow (\Box Ex)_\alpha(d) = F$. Therefore, $(Ex \to \Box Ex)_\alpha(d) = F \Rightarrow \alpha \not\models Ex \to \Box Ex$.

Let $h = [x \mapsto b, y \mapsto a] \Rightarrow d_\alpha = [y \mapsto a]$ and $d_\beta = [x \mapsto b]$. Hence $d_\alpha(x)\uparrow$ and $d_\beta(x) = b \Rightarrow Ex_\alpha(d) = = Ex_\alpha(d_\alpha) = F$ and $Ex_\beta(d) = Ex_\beta(d_\beta) = T \Rightarrow (\Box Ex)_\alpha(d) = T$. Therefore, $(\Box Ex \to Ex)_\alpha(d) = F \Rightarrow \alpha \not\models \Box Ex \to Ex$.

At the same time, everything is different for partial predicates-indicators $=_{xx}$ (see Corollary 2).

Theorem 6. *Formula $\equiv_{xy} \,\&\, \Box{\equiv_{xz}} \to \Box{\equiv_{yz}}$ is refutable in GMS$^{Q\equiv}$.*

Proof. Let us consider GMS$^{Q\equiv}$ with $St = \{\alpha, \beta\}$, $\boldsymbol{R} = \{\alpha \rhd \beta\}$, $A_\alpha = \{a\}$, $A_\beta = \{b, c\}$. Let $d = [\, x \mapsto c, y \mapsto b, z \mapsto c, s \mapsto a \,] \Rightarrow d_\alpha = [\, s \mapsto a \,]$ and $d_\beta = [\, x \mapsto c, y \mapsto b, z \mapsto c \,]$. Hence $d_\alpha(x)\uparrow$, $d_\alpha(y)\uparrow$; $d_\beta(x) = c$, $d_\beta(y) = b$, $d_\beta(z) = c$.

Therefore, $(\equiv_{xy})_\alpha(d) = (\equiv_{xy})_\alpha(d_\alpha) = T$; $(\equiv_{xz})_\beta(d) = (\equiv_{xz})_\beta(d_\beta) = T$, $(\equiv_{yz})_\beta(d) = (\equiv_{yz})_\beta(d_\beta) = F$.

Thence we obtain: $(\equiv_{xz})_\beta(d) = T$ and $\alpha \rhd \beta \Rightarrow (\Box{\equiv_{xz}})_\alpha(d) = T$; $(\equiv_{yz})_\beta(d) = F$ and $\alpha \rhd \beta \Rightarrow (\Box{\equiv_{yz}})_\alpha(d) = F$.

Therefore, $(\equiv_{xy} \,\&\, \Box{\equiv_{xz}} \to \Box{\equiv_{yz}})_\alpha(d) = F \Rightarrow \alpha \,|\!\neq \equiv_{xy} \,\&\, \Box{\equiv_{xz}} \to \Box{\equiv_{yz}}$.

Theorem 7. *Formula $=_{xy} \,\&\, \Box{=_{xz}} \to \Box{=_{yz}}$ is irrefutable in GMS$^{Q=}$.*

Proof. Let us assume the opposite: $=_{xy} \,\&\, \Box{=_{xz}} \to \Box{=_{yz}}$ is refuted in a GMS$^{Q=}$ $\boldsymbol{M} = (St, \boldsymbol{R}, A, Im)$. Then for some $\alpha \in St$ and $d \in {}^V\!A$ we have $(=_{xy} \,\&\, \Box{=_{xz}} \to \Box{=_{yz}})_\alpha(d_\alpha) = F$, hence $(=_{xy})_\alpha(d_\alpha) = T$, $(\Box{=_{xz}})_\alpha(d_\alpha) = T$ and $(\Box{=_{yz}})_\alpha(d_\alpha) = F$. From $(\Box{=_{yz}})_\alpha(d_\alpha) = F$ we obtain: there exists $\beta \in St$ such that $\alpha \rhd \beta$ and $(=_{yz})_\beta(d_\beta) = F$, thence $d_\beta(y)\downarrow$, $d_\beta(z)\downarrow$ and $d_\beta(y) \neq d_\beta(z)$. Therefore, $a, b \in A_\beta$ should exist such that: $a \neq b$, $d_\beta(y) = a$ and $d_\beta(z) = b$, hence $d(y) = a$ and $d(z) = b$.

$(\Box{=_{xz}})_\alpha(d_\alpha) = T$ implies that for all $\gamma \in St$ such that $\alpha \rhd \gamma$, we have $(=_{xz})_\gamma(d_\gamma) = T$, in particular, for a state β we have $(=_{xz})_\beta(d_\beta) = T$. Hence $d_\beta(x)\downarrow$, $d_\beta(z)\downarrow$ and $d_\beta(x) = d_\beta(z)$, which implies $d(x) = d(z)$. However, $d(z) = b$, therefore $d(x) = b$.

$(=_{xy})_\alpha(d_\alpha) = T$ implies $d_\alpha(x)\downarrow$, $d_\alpha(y)\downarrow$ and $d_\alpha(x) = d_\alpha(y)$, thence $d(x) = d(y)$. However, $d(x) = b$ and $d(y) = a$, and $a \neq b$. We came to contradiction.

Therefore as we see, there are significant differences in semantic properties of GMS$^{Q\equiv}$ and GMS$^{Q=}$.

4. Logical consequence relations on a set of formulas specified with states

We define logical consequence relations in TMS on a set of formulas specified with names of states (or states of formulas). The specification indicates the state of the world on which the formula is considered. A formula specified with a state is denoted by Φ^α, where Φ is a formula of the language, $\alpha \in \mathrm{S}$ is its specification, S is a set of names of states of the world.

Let Σ be a set of formulas specified with states with a specifications' set S.

We will call a set Σ *consistent* with TMS $\boldsymbol{M} = (St, \boldsymbol{R}, A, Im)$, provided that an injection of S into St is defined.

For sets of formulas specified with states, the notation $\Gamma\, {}_M|\!=_* \Delta$ assumes the consistency of Γ and Δ with TMS $\boldsymbol{M}$.

On sets of formulas specified with states, let us introduce (see Shkilniak, O., & Shkilniak, S., 2024) the relations of irrefutability (*IR*), truth (*T*), falsity (*F*), and strong (*TF*) logical consequence. Such relations are specified similarly to the corresponding relations in the traditional logics of quasiary predicates (see Nikitchenko, Shkilniak, O., & Shkilniak, S., 2016; Shkilniak, 2019).

Let Δ and Γ be sets of formulas specified with states.

Δ is an *IR*-consequence of Γ in a consistent with them TMS $\boldsymbol{M}$ (denoted $\Gamma\, {}_M|\!=_{IR} \Delta$), if for all $d \in {}^V\!A$:

$\Phi_\alpha(d) = T$ for all $\Phi^\alpha \in \Gamma \Rightarrow \Psi_\beta(d) \neq F$ for some $\Psi^\beta \in \Delta$.

Δ is a *T*-consequence of Γ in a consistent with them TMS $\boldsymbol{M}$ (denoted $\Gamma\, {}_M|\!=_T \Delta$), if for all $d \in {}^V\!A$:

$\Phi_\alpha(d) = T$ for all $\Phi^\alpha \in \Gamma \Rightarrow \Psi_\beta(d) = T$ for some $\Psi^\beta \in \Delta$.

Δ is a *F*-consequence of Γ in a consistent with them TMS $\boldsymbol{M}$ (denoted $\Gamma\, {}_M|\!=_F \Delta$), if for all $d \in {}^V\!A$:

$\Psi_\beta(d) = F$ for all $\Psi^\beta \in \Delta \Rightarrow \Phi_\alpha(d) = F$ for some $\Phi^\alpha \in \Gamma$

Δ is a *TF*-consequence of Γ in a consistent with them TMS $\boldsymbol{M}$ (denoted $\Gamma\, {}_M|\!=_{TF} \Delta$), if $\Gamma\, {}_M|\!=_T \Delta$ and $\Gamma\, {}_M|\!=_F \Delta$.

The relation of logical $*$-consequence with respect to a TMS of a certain type is introduced according to the following scheme (here $*$ denotes one of the *IR*, *T*, *F*, *TF*):

Δ is a *logical $*$-consequence* of Γ with respect to a TMS of a type $\mathcal{M}$ (denoted $\Gamma\, {}^{\mathcal{M}}|\!=_* \Delta$), if $\Gamma\, {}_M|\!=_* \Delta$ for all $\boldsymbol{M} \in \mathcal{M}$.

The non-modal properties of the specified relations repeat the corresponding properties of the same-named relations for sets of formulas of the traditional logic of quasiary predicates (see Nikitchenko, Shkilniak, O., & Shkilniak, S., 2016; Shkilniak, O., & Shkilniak, S., 2023).

Let us consider the properties related to equality predicates.

The *symmetry* properties of equality follow from the definitions, because the pairs $=_{xy}$ and $=_{yx}$, $\equiv_{xy}$ and $\equiv_{yx}$ represent same predicates.

The *reflexivity* properties: predicates $\equiv_{xx}$ are constants T, while predicates $=_{xx}$ are irrefutable.

The *transitivity* property for the weak equality is violated (see Shkilniak, S., 2019) for the *T*- and *F*-consequence relations: $=_{xy}, =_{yz}\, {}_M|\!\neq_F =_{xy}, =_{yz}, =_{xz}$ and $\neg{=_{xy}}, \neg{=_{yz}}, \neg{=_{xz}}\, {}_M|\!\neq_T \neg{=_{xy}}, \neg{=_{yz}}$.

Indeed, for $d = [\, x \mapsto a, z \mapsto b \,]$ we have $d \notin F(=_{xy}) \cup F(=_{yz}) = F(=_{xy} \,\&\, =_{yz})$, while $d \in F(=_{xz}) \subseteq F(=_{xy} \,\&\, =_{yz} \,\&\, =_{xz})$.

Therefore, if the transitivity property for the weak equality is violated for the *T*- and *F*-consequence relations in GMS$^{Q=}$, then it is violated for the stronger *TF*-consequence relation. Hence, in GMS$^{Q=}$ these relations are *incorrect*.

We obtain

Assertion 3. *For GMS$^{Q=}$, only the IR-consequence relation is non-degenerate and correct.*

Properties of the transitivity of equality in GMS$^{Q\equiv}$ and GMS$^{Q=}$ are represented as follows:

Tr≡) $\equiv_{xy}{}^\alpha, \equiv_{yz}{}^\alpha, \Gamma\, {}_M|\!=_* \Delta \Leftrightarrow \equiv_{xy}{}^\alpha, \equiv_{yz}{}^\alpha, \equiv_{xz}{}^\alpha, \Gamma\, {}_M|\!=_* \Delta$;

Tr=) $=_{xy}{}^\alpha, =_{yz}{}^\alpha, \Gamma\, {}_M|\!=_{IR} \Delta \Leftrightarrow =_{xy}{}^\alpha, =_{yz}{}^\alpha, =_{xz}{}^\alpha, \Gamma\, {}_M|\!=_{IR} \Delta$.

Properties of substitution of equals are important for logics with equality. For GMS$^{Q=}$ they are specified as follows:

$=$rp$_L$) $=_{xy}{}^\alpha, R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi)^\alpha, \Gamma\, {}_M|\!=_{IR} \Delta \;\Leftrightarrow\; =_{xy}{}^\alpha, R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi)^\alpha, R^{\bar{v},\bar{u},z}_{\bar{w},\perp,y}(\Phi)^\alpha, \Gamma\, {}_M|\!=_{IR} \Delta$;

$=$rp$_R$) $=_{xy}{}^\alpha, \Gamma\, {}_M|\!=_{IR} R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi)^\alpha, \Delta \;\Leftrightarrow\; =_{xy}{}^\alpha, \Gamma\, {}_M|\!=_{IR} R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi)^\alpha, R^{\bar{v},\bar{u},z}_{\bar{w},\perp,y}(\Phi)^\alpha, \Delta$.

At the same time, the substitution of equals in GMS$^{Q\equiv}$ is not straightforward. Let us consider some examples.

Example 1. $\equiv_{xy}{}^{\alpha}, \Box\equiv_{xz}{}^{\alpha}, \Box\equiv_{yz}{}^{\alpha}\ {}_M|\!=_{*}\ \Box\equiv_{yz}{}^{\alpha}$ *is true.*

However, $\equiv_{xy}{}^{\alpha}, \Box\equiv_{xz}{}^{\alpha}\ {}_M|\!=_{*}\ \Box\equiv_{yz}{}^{\alpha}$ is refuted from the proof of Theorem 6: in GMS$^{Q\equiv}$ for $d = [\,x \mapsto c, y \mapsto b, z \mapsto c, s \mapsto a\,]$ we have $(\equiv_{xy})_{\alpha}(d) = T$, $(\Box\equiv_{xz})_{\alpha}(d) = T$, $(\Box\equiv_{yz})_{\alpha}(d) = F$; hence $\equiv_{xy}{}^{\alpha}, \Box\equiv_{xz}{}^{\alpha}\ {}_M|\!\neq_{IR}\ \Box\equiv_{yz}{}^{\alpha}$, therefore $\equiv_{xy}{}^{\alpha}, \Box\equiv_{xz}{}^{\alpha}\ {}_M|\!\neq_{*}\ \Box\equiv_{yz}{}^{\alpha}$. However, $\Box\equiv_{xz} = \mathrm{R}^{v}_{x}(\Box\equiv_{vz})$ and $\Box\equiv_{yz} = \mathrm{R}^{v}_{y}(\Box\equiv_{vz})$, which implies $\equiv_{xy}{}^{\alpha}, R^{v}_{x}(\Box\equiv_{vz})^{\alpha}\ {}_M|\!\neq_{*}\ R^{v}_{y}(\Box\equiv_{vz})^{\alpha}$. But from $\equiv_{xy}{}^{\alpha}, R^{v}_{x}(\Box\equiv_{vz})^{\alpha}, R^{v}_{y}(\Box\equiv_{vz})^{\alpha}\ {}_M|\!=_{*}\ R^{v}_{y}(\Box\equiv_{vz})^{\alpha}$ follows that $\equiv_{xy}{}^{\alpha}, R^{v}_{x}(\Phi)^{\alpha}, \Gamma\ {}_M|\!=_{*}\ \Delta$ and $\equiv_{xy}{}^{\alpha}, R^{v}_{x}(\Phi)^{\alpha}, R^{v}_{y}(\Phi)^{\alpha}\Gamma\ {}_M|\!=_{*}\ \Delta$ are not equivalent.

As a result we obtain that the following statements are *not* equivalent ($*$ denotes one of the *T*, *F*, *TF*):

$\equiv_{xy}{}^{\alpha}, R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi)^{\alpha}, \Gamma\ {}_M|\!=_{*}\ \Delta$ and $\equiv_{xy}{}^{\alpha}, R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi)^{\alpha}, R^{\bar{v},\bar{u},z}_{\bar{w},\perp,y}(\Phi)^{\alpha}, \Gamma\ {}_M|\!=_{*}\ \Delta$;

$\equiv_{xy}{}^{\alpha}, \Gamma\ {}_M|\!=_{*}\ \Delta, R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi)^{\alpha}$ and $\equiv_{xy}{}^{\alpha}, \Gamma\ {}_M|\!=_{*}\ \Delta, R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi)^{\alpha}, R^{\bar{v},\bar{u},z}_{\bar{w},\perp,y}(\Phi)^{\alpha}$.

We can see that validity of $\equiv_{xy}{}^{\alpha}$ is not enough for the substitution of equals in GMS$^{Q\equiv}$.

Example 2. $\equiv_{xy}{}^{\alpha}, Ex^{\alpha}, Ey^{\alpha}, \Box\equiv_{xz}{}^{\alpha}\ |\!=_{*}\Box\equiv_{yz}{}^{\alpha}$ *is true.*

Suppose the opposite: for a GMS $\boldsymbol{M} = (St, \boldsymbol{R}, A, Im)$ and $\alpha \in St$, $d \in {}^{V}\!A$ we have $(\equiv_{xy})_{\alpha}(d) = T$, $Ex^{\alpha}(d) = T$, $Ey^{\alpha}(d) = T$, $(\Box\equiv_{xz})_{\alpha}(d) = T$ and $(\Box\equiv_{yz})_{\alpha}(d) = F$; hence $d_{\alpha}(x){\downarrow} = d_{\alpha}(y){\downarrow} = a$ for some $a \in A \Rightarrow d(x) = d(y) = a$. Following $(\Box\equiv_{yz})_{\alpha}(d) = F$ we have $(\equiv_{yz})_{\beta}(d) = F$ for some β: $\alpha \triangleright \beta$.

Condition $(\Box\equiv_{xz})_{\alpha}(d) = T$ implies $(\equiv_{xz})_{\beta}(d) = T$, therefore two cases are possible:

1) $d_{\beta}(x){\uparrow}$ and $d_{\beta}(z){\uparrow}$; hence $d(x) \notin d_{\beta}$; from $d(x) = d(y) = a$ we get $d_{\beta}(y){\uparrow}$, therefore $(\equiv_{yz})_{\beta}(d) = T$, which contradicts the condition $(\equiv_{yz})_{\beta}(d) = F$.

2) $d_{\beta}(x){\downarrow} = d_{\beta}(z){\downarrow}$; from $d(x) = d(y) = a$ we obtain $d_{\beta}(z) = d_{\beta}(x) = d_{\beta}(y) = a$, which contradicts the condition $(\equiv_{yz})_{\beta}(d) = F$.

We came to contradiction in both cases.

Therefore, the substitution of equals in GMS$^{Q\equiv}$ requires an addition of conditions of Ex^{α} and Ey^{α} validity to the condition of validity of $\equiv_{xy}{}^{\alpha}$.

Theorem 8. *We obtain the following conditions for the substitution of equals in GMS$^{Q\equiv}$:*

$\equiv$rp$_{L}$) $\equiv_{xy}{}^{\alpha}, Ex^{\alpha}, Ey^{\alpha}, R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi)^{\alpha}, \Gamma\ {}_M|\!=_{*}\ \Delta \;\Leftrightarrow\; \equiv_{xy}{}^{\alpha}, Ex^{\alpha}, Ey^{\alpha}, R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi)^{\alpha}, R^{\bar{v},\bar{u},z}_{\bar{w},\perp,y}(\Phi)^{\alpha}, \Gamma\ {}_M|\!=_{*}\ \Delta$;

$\equiv$rp$_{R}$) $\equiv_{xy}{}^{\alpha}, Ex^{\alpha}, Ey^{\alpha}, \Gamma\ {}_M|\!=_{*}\ R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi)^{\alpha}, \Delta \;\Leftrightarrow\; \equiv_{xy}{}^{\alpha}, Ex^{\alpha}, Ey^{\alpha}, \Gamma\ {}_M|\!=_{*}\ R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi)^{\alpha}, R^{\bar{v},\bar{u},z}_{\bar{w},\perp,y}(\Phi)^{\alpha}, \Delta$;

$\equiv\neg$rp$_{L}$) $\equiv_{xy}{}^{\alpha}, Ex^{\alpha}, Ey^{\alpha}, \neg R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi)^{\alpha}, \Gamma\ {}_M|\!=_{*}\ \Delta \;\Leftrightarrow\; \equiv_{xy}{}^{\alpha}, Ex^{\alpha}, Ey^{\alpha}, \neg R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi)^{\alpha}, \neg R^{\bar{v},\bar{u},z}_{\bar{w},\perp,y}(\Phi)^{\alpha}, \Gamma\ {}_M|\!=_{*}\ \Delta$;

$\equiv\neg$rp$_{R}$) $\equiv_{xy}{}^{\alpha}, Ex^{\alpha}, Ey^{\alpha}, \Gamma\ {}_M|\!=_{*}\ \neg R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi)^{\alpha}, \Delta \;\Leftrightarrow\; \equiv_{xy}{}^{\alpha}, Ex^{\alpha}, Ey^{\alpha}, \Gamma\ {}_M|\!=_{*}\ \neg R^{\bar{v},\bar{u},z}_{\bar{w},\perp,x}(\Phi)^{\alpha}, \neg R^{\bar{v},\bar{u},z}_{\bar{w},\perp,y}(\Phi)^{\alpha}, \Delta$.

Properties of carrying modalities over renominations and modality elimination belong to the modal properties of logical consequence relations. For TMS^{Q} with equality, such properties basically repeat the corresponding properties for TMS^{Q} without equality (see Shkilniak, 2015).

In case of additional conditions imposed on the transition relation, the elimination properties for modalities will be modified accordingly. The properties for the relation ${}_M|\!=_{IR}$ and dual properties for the relation ${}_M|\!\neq_{IR}$ for the cases of transitive, reflexive or symmetric transition relation in GMS^{Q} can be found in work (Shkilniak, 2015).

The described properties of relations ${}_M|\!=_{*}$ for sets of formulas induce analogous properties of the corresponding logical consequence relations $|\!=_{*}$ for a given type of TMS^{Q}.

Discussion and conclusions

In this paper we studied pure first-order transitional modal logics of partial non-monotonic quasiary predicates with equality which are program-oriented logical formalisms of the modal type. Two types of such logics were introduced: logics with strong equality predicates and with weak equality predicates. Depending on types of modal compositions and transition relations, classes of general, temporal and multimodal logics can be considered. Semantic models and languages of these logics were described with special attention to the properties of equality predicates. We determined a number of consequence relations in TMS and logical consequence relations on sets of formulas specified with states; the substitution of equals was described in detail.

Properties of these relations will be the semantic basis for constructing the corresponding sequent type calculi.

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Оксана ШКІЛЬНЯК, канд. фіз.-мат. наук, доц.
ORCID ID: 0000-0003-4139-2525
e-mail: oksana.sh@knu.ua
Київський національний університет імені Тараса Шевченка, Київ, Україна

ПЕРШОПОРЯДКОВІ ТРАНЗИЦІЙНІ МОДАЛЬНІ ЛОГІКИ КВАЗІАРНИХ ПРЕДИКАТІВ З РІВНІСТЮ

Досліджено транзиційні модальні логіки (ТМЛ) квазіарних предикатів з рівністю. Модальні логіки успішно використовують для опису й моделювання різноманітних предметних областей. Епістемічні логіки застосовують для опису систем штучного інтелекту, інформаційних та експертних систем. З допомогою темпоральних логік описують динамічні системи, специфікації та верифікації програм. Традиційні модальні логіки базуються на класичній логіці предикатів. Проте ця логіка недостатньо враховує неповноту, частковість, структурованість інформації про предметну область. Такі обмеження ведуть до необхідності побудови нових, програмно-орієнтованих логік, що зумовлює актуальність пропонованої роботи. Ми розглядаємо чисті першопорядкові ТМЛ квазіарних предикатів, не обмежених умовою монотонності та збагачених предикатами рівності. Виділено два різновиди таких логік: $TML^{Q\equiv}$ з предикатами строгої рівності $\equiv_{xy}$ та $TML^{Q=}$ з предикатами слабкої рівності $=_{xy}$. Характерною особливістю цих логік є використання композиції розширеної реномінації. Для елімінації кванторів у $TML^{Q\equiv}$ використано спеціальні тотальні предикати-індикатори Ez, а в $TML^{Q=}$ у ролі часткових предикатів-індикаторів постають $=_{zz}$. Семантичною базою ТМЛ є транзиційні модальні системи (ТМС). Для $TML^{Q\equiv}$ та $TML^{Q=}$ виділено найважливіші різновиди цих систем: загальні ($GMS^{Q\equiv}$ і $GMS^{Q=}$), темпоральні ($TmMS^{Q\equiv}$ і $TmMS^{Q=}$) та мультимодальні ($MMS^{Q\equiv}$ і $MMS^{Q=}$). Описано семантичні моделі та мови цих різновидів ТМС, особливу увагу приділено властивостям, пов'язаним із предикатами рівності. Розглянуто чотири типи відношень логічного наслідку для множин, специфікованих станами формул у цих TMS, за неспростовністю (IR), істинністю (T), хибністю (F) та сильний (TF). Описано особливості заміни рівних у $TML^{Q\equiv}$ та $TML^{Q=}$. Властивості відношень логічного наслідку для множин специфікованих станами формул є семантичною основою подальшої побудови для досліджених логік відповідних числень секвенційного типу.

Ключові слова: *модальна логіка, транзиційна модальна система, предикат, рівність, логічний наслідок.*

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