TARAS SHEVCHENKO NATIONAL UNIVERSITY OF KYIV
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Department of Complex Systems Modeling

Graduate qualification work for Master's degree

specialty 113 Applied Mathematics

on the topic:

MODELING AND RESEARCH OF CONFLICT SITUATIONS MODEL

Completed by 2nd year student of Master's degree
Fedorova Kseniia Myhajlivna

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(signature)

Supervisor:
Professor, Doctor of Sc.
Khysainov D. Ya.

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(signature)

I assure, there are no derivations from the works of other authors without corresponding references
Student ______________
(signature)

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Head of Department
Cherniy D.I. ______________
(signature)

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ABSTRACT

46 pages, 13 illustrations, 3 tables, 15 reference sources.

THE RICHARDSON MODEL, THE DYNAMIC SYSTEM, THE PHASE PORTRAIT, THE MODEL CONSTRUCTION, THE AFFILIATION, THE STRENGTH OF THE DEVELOPMENT, ARMS RACE, THE MILITARY CONFLICT.

The object of the work is the process of developing a Richardson model based on real statistics. The subject of work is the literature on the course of dynamic systems.

The purpose of this work is to create and apply an algorithm for constructing phase portraits and solution trajectories for dynamic systems with and without delay.

Development methods: computer simulation, methods of finding the system special points, developing an algorithm based on theoretical information. Development Tools: a freely available Matlab R2018b development tool with an academic license.

Results: a large number of literature on dynamic systems was analyzed, algorithms for visualization of dynamic system solutions were constructed, theory and algorithm were applied to construct the model based on statistics, as well as its subsequent analysis.

According to the instrumental data, the work was performed in conjunction with the course work on dynamic systems.

The developed algorithms can be used in the educational process on the course of dynamic systems, as well as political scientists to predict conflicts between states.

CONTENTS

INTRODUCTION

Assessment of the current state of the development object. Today, mathematics is a part of all human life spheres and the method of mathematical modeling of social and economic phenomena is gaining in popularity. The interaction of countries is characterized by a large number of political, economic, geographical, geopolitical and historical factors. Thus, according to these factors, there was a division of countries into groups according to their degree of economic development. But despite this divide, states continue to compete both within the group and among themselves. There is a reason why there is always competition and the possibility of military conflict.

Nowadays many methods for predicting potential conflicts exist. The Richardson model of the arms race is one of them and is a rather attractive target for research. Countries' indicators depend on a large number of parameters, that increases the degree of truthfulness of the model, but at the same time - complicates the process of its analysis. But Richardson model, in theory, allows take into account all reasonable factors. Although, alas, there are not too many works in this field today.

The relevance of the topic and the reasons for its implementation. There are many hot spots in the world today. Some conflicts could be avoided by analyzing the situation of potential enemies in advance and prepare for the arms race. The USA and China are struggling for economic primacy, that's why these countries are a good choice for the current research. Modeling and analyzing can help to anticipate the possibility of a military conflict and, if necessary, to be prepared for it.

The arms race model can be based on a functional balance, expressed by geopolitical status, which include several aspects.

It is extremely interesting to study the system with the delay - this factor brings the system as close as possible to the real conditions and takes into account that any decisions are not made immediately, but take time.

The purpose and tasks of the paper. The purpose of the qualification work is to build an arms race model using real statistics data. To achieve this goal, the following tasks are set.

– Investigate Richardson arms race model without delay.

– Apply theoretical knowledge of dynamic systems to study and build a model with delay.

– Develop algorithms for the phase portraits construction of a dynamic system without delay and with it.

– Gather statistics about arms, imports, GDP level for China and US.

– Build a Richardson model based on statistic data.

– Analyze system behavior after adding and changing a system delay.

Object, methods and means of development. The object is the process of building a Richardson model and analyzing it using the collected theoretical data and the developed visualization algorithms.

Algorithm development and model building were preceded by analysis of literature and statistics, as well as research into the dynamic predator-victim model.

As a tool for model visualization was chosen Matlab programming language, which contains an extremely wide range of mathematical methods and functions, and allows you to do complexity calculations quickly and accurately.

Areas of application are possible. The developed algorithms can be used in the educational process on the course of dynamic systems, as well as by political scientists to predict conflicts between states or, at least, be prepared for it.

Relationships with other works. According to the instrumental data, the work was performed in conjunction with the course work on dynamic systems.

1 MAIN PRINCIPLES OF MODELING

1.1 General principles of dynamic systems modeling

A model is a specially created object of material or abstract nature, which is similar to the original object and reproduces some of the properties, connections and characteristics of the original object that are essential to the task. The process of building, exploring, and using a model for objects' studying is called modeling. The purpose of the simulation is to describe the system behavior, test hypotheses and assumptions, construct theories to clarify the principles of system and predict its future behavior.

Mathematical methods can be divided into three classes:

– Descriptive models - deterministic models that are represented as inequalities or equations that describe the behavior of a system.

– Regulatory models - optimization models containing an expression that must be maximized or minimized under the constraints. These expressions can be represented in algebraic or integral form, or in any other standard form.

– Probability models - models that are expressed in the form of equations and inequalities, but have probabilistic meaning.

When constructing mathematical models, the first difficulty arises when ideas and representations need to be presented in the form of exact expressions, inequalities and equations (to formulate conditions and to set the task correctly). Then the time for mathematical studies of solutions' existence and uniqueness, the determination of their properties, the construction of these solutions and the analysis of the convergence of the chosen algorithm.

Stability and equilibrium are the basic concepts to analyze about the system. The system is considered to have a stable equilibrium state if, after a slight disturbance, it returns to its original position. The system is unstable if after a slight disturbance the distance to the original position of the system decreases.

To be useful and to make sense, a mathematical model must satisfy some requirements.

1) Completeness – by using mathematical methods the model gives a fundamentally new opportunity to reflect the features and characteristics of object that are interesting for researcher.

2) Accuracy - the model provides an acceptable match of the actual parameters and the values found as the model output.

3) *Adequacy* - the ability to accurately quantify and qualitatively describe the characteristics of an object that are important to the task.

4) *Efficiency* - the model should be simple enough and convenient.

5) *Productivity* - the model should contain various parameters of the object components, include functional dependencies, and describe relationships between physical quantities.

Based on the requirements of the model, data is collected and analyzed, hypotheses and a meaningful formulation of the problem are formed. All the data collected as a result of the study is formulated as a technical specification for the design and development of the model.

Further the mathematical formulation of the problem is built - a set of these mathematical relations describing the behavior and properties of the modeling object. Usually, obtained dynamic system refer to ordinary differential equations system with sufficiently smooth right-hand sides that ensure the existence and uniqueness of a solution.

To check the correctness of the obtained system of mathematical relations, you need to perform several checks:

- dimension control;
- order control;
- control the nature of the dependencies;
- control of conservation laws;
- control of boundary conditions;
- control of mathematical closure.

More information about the processes and methods of mathematical modeling can be found in [1].

1.2 Conflict situations modelling

A mathematical approach in conflict situations sphere, about which even people may have different opinions, cannot be a panacea. However, the mathematical formulation of complex problems indicates possible consequences and facilitates the selection of optimal strategies.

The first of the social sciences that was strongly involved into mathematical modeling was not political science, but rather, economic science. It made the transition from verbal expressions to mathematical ones easier because the main subject of its interest was money that related to mathematics.

Political science has been reorganized into the direction of quantitative methods during the 50's and 60's and nowadays it takes second place between social sciences that are using mathematical modeling.

Many politic decisions have a significant economic component. That is why models that are developed within the framework of economic science play a significant role in political science. In [2] is described in more details the processes of mathematical modeling development in political science, but the basic principles will be outlined below.

Money is not the only variable that is interesting for political scientists and can be described mathematically. For example, election results are also given in the form of numbers. Military preparations are usually described in numerical terms (number of missiles, number of tanks, etc.), political opinions are expressed as percentages between different groups of respondents in the survey.

It should also be noted that mathematical modeling is not limited to quantitative characteristics, it can also deal with qualitative characteristics of the political process.

The first reason that drives to political behavior modeling is that the model helps to formalize the phenomena that occur in society. Although much of what happens in politics is usually not unexpected anymore - in fact, having an element of surprise indicates that we have a priori ideas about how events can unfold, and we can grasp the situation. Mathematical models also help to explicate such informal models.

Another reason for applying mathematical modeling is the need to describe the mechanisms that explain our informal predictions explicitly. Although all individuals know what can and cannot be expected from a given political system, they are often unable to determine exactly why and what actually they expect from it. The formal model helps to overcome the too loose formulations of the informal model assumptions and to give an accurate prediction.

Sometimes it seems that mathematical models are just confirming obvious things. However, this is not a weakness, but rather a dignity of modeling, because the assumptions and model forecasts are accurate enough to be verified, as well as to indicate where and how a possible error can occur. That model, which has resisted many attempts to distort it, probably, will make correct predictions in the future. However, the model, which repeatedly gives incorrect forecasts, should probably be removed from consideration.

Therefore, a model is only useful if, in principle, it is possible to demonstrate its falsity. If it is impossible to show that the model is incorrect, then it is also impossible to prove that it is correct, and hence the conclusion about the futility of such model. An informal, intuitive, error-prone model can be a great tactical tool for negotiation, but it is powerless in helping to better understand the mechanism of political behavior.

The third advantage of formal models, compared to intuition or even thoroughly substantiated natural language reasoning, is the ability to systematically handle higher-level entities. Natural languages, like English, emerged as a means of communication, not as a means of inference. Mathematics, on the contrary, was originally conceived as a means of logical inference and systematic operation of concepts. And experience has shown that mathematics in this respect is a very useful tool.

Finally, mathematical modeling allows different scientific disciplines to share their research tools and techniques. Therefore, there are many examples: in the models that used in political science, there are involved not only the basic mathematical tools, but also a lot of techniques borrowed from econometrics, sociology and biology.

Experimental research, which is a complex mathematical model of the distribution of public opinion among different groups of people, is a widespread method used in most social sciences. Borrowing is also going backwards. For example, game theory was first developed by economists and political scientists to analyze the phenomenon of competition and only later became a section of pure mathematics.

So, let's go into more detail about the process of mathematical modeling in politics.

The first step in building a model - inductive - is a selection of the process to be simulated. One possible way of presenting such an initial step is to formulate a problem, that is, to decide what to consider and what to ignore.

Modeling usually involves fewer variables than hypothesis testing: the latter operates with simple processes relating to a large number of variables, whereas models use complex processes associated with a small number of variables.

The second step is to move from problem definition to the actual construction of an informal model. The informal model is a set of such tools that can explain the selected observations, but the degree of their logical interconnection can’t be checked precisely. For example, if the object of the simulation is an arms race then the informal model could look like this: "The arms race is initiated because states are afraid of armament available in other countries; its borders are limited by the cost of armaments." This statement tells us something about the mechanisms that drive the arms race, but it is not specific enough for the final model.

At this stage, the model developer is trying to find different ways to establish a logical match between the model and the real world. This is a critical point in the modeling process.

The next step is searching among existing formal models the one that would be most appropriate for observations.

In formal model, all the assumptions are formulated in mathematical form, but in informal model – no. Existing models are quite specific sets of techniques, and since they have already been studied by someone and the possible conclusions from their original parcels are already known, it provides some direction for further development.

The third step is to translate the informal model into a mathematical model. Such translation involves considering a verbal description of an informal model and finding a suitable mathematical structure that covers the same ideas and processes. This is the most difficult step in the whole modeling process. Many mistakes and ambiguities can be stolen here, because in any translation process the content is both lost and expanded.

The translation stage has two dangers. First, informal models tend to be ambiguous, and there are usually several ways to translate an informal model into a mathematical one, but alternative mathematical models can have completely different meanings.

The second possible danger is adding to the informal model such implicit assumptions that accompany using of specific mathematical methods. This is particularly important in cases where statistical techniques and differential calculus are involved.

For example, if we assume that the reason people participate in the vote [3] is to be able to exert some influence on the election results due to the disruption of potential casual communication, and mathematical analysis shows that the probability of casual communication is very small, we may have underestimated some other reasons for voting, such as a civic responsibility and a desire to express an opinion.

A third danger is that some research methods may be reactive. In other words, there is a danger that the study procedure itself may change the behavior of those to be investigated so that the results will prove to be incorrect.

We can list several major sources of measurement error:

– Differences in the distribution between the constant characteristics. For example, the influence of intelligence and the influence of political ideology can be mixed, and we will not be able to identify differences in estimates due to ideological factors and distinguish them from differences caused by intellectual factors.

– Differences in the distribution of temporal characteristics that are reflected in measurements. For example, the nature of the answers to the questionnaire may be determined by the respondent's mood or health. The only way to protect yourself from this kind of impact is to pay attention to the signals that warn that individual objects are affected by such factors and perform procedures designed to test the reliability of the measurements.

– Differences that occur when interpreting a measuring instrument by different people. This problem arises only when people answer the questions directly, not when the researcher makes measurements while observing behavior. If the questions are ambiguous, then the respondents will also give them different interpretations that may lead to differences in the measurement estimates.

– Differences in measurement conditions. This source of measurement error is also found mainly in research, the results of which are measured by answering questions. For example, in a sample survey, it is well established that the nature of the answers may be influenced by the interviewer's race, gender, and age. Answers may vary from interview to interview depending on the characteristics of the interviewer himself. One way to avoid this kind of measurement error is to make every effort standardize the measurement conditions.

– Differences in the application of measuring instruments. Objects attributes' estimations may differ as a result of various errors that arise when collecting and recording information. Interviewers may misunderstand the instructions and ask questions in a different way than the researcher intended. Poor lighting may cause the respondent to incorrectly mark the items in the questionnaire. At the most crucial moment, the pencil may break and the ink will run out of ink, so that observers will not be able to capture key events for group interaction. All these types of deviations in the measurement instruments application lead to differences in estimates, regardless of any differences in the actual values of the studied variable. The primary way to combat measurement errors is to pre-check the measurement tools.

– Differences in data processing and analysis. Before information can be analyzed, it must be carefully processed. For example, interviewers can record the answers to a question by writing down every word spoken to the interlocutors. Subsequently, when encoding the answers, these written snippets can be reduced to a single digit. This number can be fixed in bits of the magnetic disk or magnetic tape. At each of these stages, data analysis becomes easier, but with each successive step, the likelihood of errors as a result of which the objects begin to differ by some variable increases, while in reality, they do not differ. The possibility of such errors leads to double and triple-check the data at each conversion and save the original form for further verification.

– Differences in the response of individuals to the shape of measuring instruments. This problem becomes important when the objects of analysis are people. The best way to avoid this type of measurement error is to use more than one form of measurement to operationalize each concept.

All these factors can lead to the appearing of measurement errors. The various errors listed above are usually divided into systematic and random. Systematic errors are errors that arise from the confusion of variables in the real world or from the features of the instrument itself. They appear every time this tool is used, and they are constantly accompanying objects and studies that use the same measurement. Constant mistakes make the results invalid. Random errors are appeared in different ways and caused be the passing characteristics of the objects, situational differences in the use of the tool, errors in the measurement, data processing and other factors. They make measurements invalid the same as systematic errors. Besides, random errors make measurements unreliable in the sense that the occurrence of random errors does not make it possible to obtain the same results all the time when using the same measurement.

Therefore, it is important to check the validity of measurements, but this is only possible after the data has been collected.

Once the data is validated and, if necessary, corrected, you can move on to the next step - mathematical processing of the formal model. Here the entire arsenal of mathematical methods - logical, algebraic, geometric, differential, probabilistic - is applied to formally derive nontrivial consequences from the initial assumptions of the model. At the stage of mathematical processing, regardless of the task nature, we usually deal with pure abstractions and use the same mathematical tools, regardless of the original problem. This stage is the deductive core of the simulation, which is to find nontrivial and unpredictable conclusions from plausible assumptions.

The lessons learned to go through another translation process - this time from the math language back to the natural language. The precautions mentioned in connection with the translation into the formal model language remain relevant here because translation inevitably entails the loss and addition of some information and some assumptions. This final translation may be the most difficult step in the modeling process.

Although the model developer is generally interested in getting a very definite result that has quite a certain real meaning, but modeling often produces unexpected results that may be even more interesting than originally expected.

Then you need to go back to the initial stages of modeling to make some changes to the model. It is possible to return to the formal comparison and refinement of the model many times before empirical validation becomes possible, which acts as the final stage of modeling necessary to establish the degree of validity of the model.

Empirical verification is not always necessary: in some cases, the initial assumptions describe the process exhaustively (this applies, for example, to the rules of the electoral procedure), and the conclusions of the model need not be validated. But usually, the initial assumptions contain factors that are not fully specified in the theoretical development of the model and require estimates based on evidence.

Since, in reality, all models of social processes assume a considerable element of chance, empirical tests also help to establish the predicted power of the model.

Therefore, mathematical models have advantages over natural-language models. First, they sort out the mental models we normally use. Second, they are devoid of inaccuracy and ambiguity. Third, mathematical writing, unlike natural expressions, allows us to operate at a very high level of deductive complexity. Finally, mathematical models help to find common solutions to problems that appear to be heterogeneous.

2 DYNAMIC MODELS OF CONFLICT SITUATIONS

2.1 Systems without delay

Arms control is defined as country's attempt to minimize conflict consequences and limit the weapon quantity used. Has to be mentioned, such definition doesn't provide minimizing conflict consequences, because in this case the likelihood of an armed conflict and losses should to be minimized as well. But their compatible probability cannot be qualitatively estimated.

The immediate aim is to reduce arms in order to reduce the likelihood of a conflict, as well as its intensity, if it does start.

To show effectively in quantitative terms probabilistic nature of certain actions and degree of objectives achievement, firstly it is necessary to develop a measuring scale for these values.

The most popular arms racing model is Richardson model [3]. Here is a list of assumptions:

1) In arms race between two countries each will seek to increase arms proportionally to the size of the other arms [3,4].

Put $x_1(t),\ x_2(t)$ – arms levels of two countries, $k > 0$ and $l > 0$ - parameters of the reaction of each country to the arms of the other. Then dynamic of arms increasing can be shown as the system of two differential equations:

$$\dot{x_1}(t) = kx_2(t),\ \dot{x}_2(t) = lx_1(t).$$

At this step, the equilibrium position of the system is a saddle. If initial conditions don't lie on stable separatrix, partial solutions grow by exponential law, and that's why arms can grows to the infinity.

2) Economic is a restriction for arms, that seeks to reduce the arms growth rate in proportion to the size of the existing armed forces [3,4].

Let's add "fatigue" or costs coefficients - a and b, then arms growing dynamic model has a look:

$$\dot{x_1}(t) = kx_2(t) - ax_1(t),\ \dot{x}_2(t) = lx_1(t) - bx_2(t). \tag{2.1}$$

Eigenvalues of the system are:

$$\lambda_{1,2} = -\frac{a + b \pm \sqrt{(a-b)^2 + 4kl}}{2}.$$

The roots of a characteristic equation are real numbers, that's why equilibrium position could be [4]:

- stable node, if $ab > lk$;
- line of a critical points, if $ab = lk$;
- saddle, if $ab < lk$.

In such a way, if "fatigue" coefficients more, than reaction of each country to the arms of the other, arms coincides to zero.

3) The country will increase arms guided by their national aspirations and also by hostility to other countries even if any country does not constitute a threat to the existence of it [3,4].

Mark the coefficients g and h as economic possibilities of arms of each of the countries, or in other words, as claim coefficients if they are positive, and goodwill coefficients if they are negative. Then model is increased to linear non-homogeneous system and looks like (2.2):

$$\dot{x_1}(t) = kx_2(t) - ax_1(t) + g,\ \dot{x_2}(t) = lx_1(t) - bx_2(t) + h. \qquad (2.2)$$

In this case, with the condition $ab > lk$ a stable equilibrium position is non-zero, but stationary one, that is determined as the solution of the system:

$$kx_2 - ax_1 + g = 0, lx_1 - bx_2 + h = 0$$

and have a look:

$$\overline{x_1} = \frac{bg + kh}{ab - kb}, \overline{x_2} = \frac{lg + ah}{ab - kb}.$$

As for equation (2.1), we obtain or node, or line of critical points, or saddle. Both countries are trying to maintain a constant armament that is allowed by their economics.

So, three options of model behavior when $t \to \infty$ exist:

- Infinite arms race, when $x_1 \to \infty, x_2 \to \infty$;
- Mutual disarmament, when $x_1 \to 0, x_2 \to 0$;
- Equilibrium of arms, when $x_1 \to x_1^*, x_2 \to x_2^*$.

It is understandable that each country is going to obtain level of arms expenses where costs are stable and don't depend on time.

Based on the analysis of the model coefficients, we can say the following:

- When $ab > kl, g > 0, h > 0$ then equilibrium point exists;
- When $ab < kl, g > 0, h > 0$ model leads to unlimited race escalation;
- When $ab < kl, g < 0, h < 0$ full mutual disarmament will be achieved;
- When $ab < kl,$ та $g < 0$ or $h < 0$ the result will depend on the initial conditions.

But in real life all decisions are making not instantly, the time for solution development and implementation is needed. Also, technical delays during the realization can occur. Therefore, it makes sense to use a differential equation model with delay.

2.2 Systems with delay

Given that in Richardson model $k > 0$ and $l > 0$ - parameters of the reaction of each country to the arms of the other, and in real life any decisions are not taken immediately, especially decisions like the reactions to actions by other countries (at least, because the time to spread the news to the world is needed), that's why it would be logical to try to add a delay into the system for parameters mentioned above.

1) Consider the case when the countries don't have an economic restriction and that's why they can increase their strength up to infinity. Therefore, we have the system with pure delay:

$$x'(t) = Bx(t-\tau) + f(t), \tag{2.2}$$

where $x(t) = \binom{x_1(t)}{x_2(t)}, B = \begin{bmatrix} 0 & k \\ l & 0 \end{bmatrix}, f(t) = \binom{g}{h}$.

As a solution of Cauchy problem of the system

$$x'(t) = Bx(t-\tau) \tag{2.3}$$

we will mean a certain vector function $x(t)$ for $t \geq -\tau,$ that is continuously differentiated when $t \geq 0$ and it satisfies the system (2.3) and conditions $x(t) = \varphi(t),$

$-\tau \leq t \leq 0$. Namely, solution of the Richardson model is the size arms that the country has under certain constraints.

In the general case, solution of Cauchy problem of homogeneous system with a pure delay has the form:

$$x(t) = \exp_\tau\{B, t\}\varphi(-\tau) + \int_{-\tau}^{0} \exp_\tau\{B, t - \tau - s\}\varphi'(s)ds$$

with the initial conditions $x(t) = \varphi(t), -\tau \leq t \leq 0$,

where $\exp_\tau\{B, t\} = \begin{cases} 0, -\infty < t < -\tau \\ I, -\tau \leq t \leq 0 \\ I + B\frac{t}{1!}, 0 \leq t \leq \tau \\ \cdots \\ I + B\frac{t}{1!} + \cdots + B^k \frac{[t-(k-1)\tau]^k}{k!}, (k-1)\tau \leq t < k\tau \end{cases}$,

and when $\tau \to 0$ on the interval $t \geq 0$ there is a convergence $\lim_{\tau \to 0}\{\exp_\tau\{B, t\}\} = e^{Bt}$.

But just for the Richardson model, that is studied in this paper, we predict that initial conditions don't depend on t, and that's why the solution shown above is not satisfy us.

Requiring $\varphi(t)$ to be only continuous for the system (2.3), obtain:

$$x(t) = \exp_\tau\{B, t - \tau\}\varphi(0) + B\int_{-\tau}^{0} \exp_\tau\{B, t - 2\tau - s\}\varphi(s)ds$$

The solution of the non-homogeneous system (2.2) is much more complicated and has the form:

$$x(t) = \exp_\tau\{B, t - \tau\}\varphi(0) + B\int_{-\tau}^{0} \exp_\tau\{B, t - 2\tau - s\}\varphi(s)ds$$

$$+ \int_{0}^{t} \exp_\tau\{B, t - \tau - s\}f(s)ds.$$

In arms race model, that is studied, $f(t) = f_0, t \geq 0$, initial conditions are constant, $x(t) = \varphi_0$, so we can write the solution as:

$$x(t) = \exp_\tau\{B, t-\tau\}\varphi_0 +$$

$$+B[I\tau + \sum_{i=1}^{n-1} \frac{B^i}{(i+1)!}\left[(t-i\tau)^{i+1} - (t-(i+1)\tau)^{i+1}\right] + \frac{B^n}{(n+1)!}[(t-n\tau)^{n+1} -$$

$$-(t-(n+1)\tau)^{n+1}]]\,\varphi_0 + \{I(n+1)\tau +$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{i} \frac{B^j}{(j+1)!}\left[\left((i-j+1)\tau\right)^{j+1} - \left((i-j)\tau\right)^{j+1}\right] + I(t-n\tau) +$$

$$+\sum_{i=1}^{n} \frac{B^i}{(i+1)!}[(t-(i+1)\tau)^{i+1} - (n\tau-(i-1)\tau)^{i+1}]\}f_0.$$

2) Model, when countries have the same arms claims, is represented as a system:

$$x'(t) = Ax(t) + Bx(t-\tau) + f(t),$$

where matrixes A, B – commute, or other words, $AB = BA$. In our case, in is true when $a = b$ and so $x(t) = \binom{x_1(t)}{x_2(t)}, A = \begin{bmatrix} -a & 0 \\ 0 & -a \end{bmatrix}, B = \begin{bmatrix} 0 & k \\ l & 0 \end{bmatrix}, f(t) = \binom{g}{h}$.

The matrix

$$X_0(t) = \exp(At)\exp_\tau\{B_1, t\}, B_1 = \exp\{-A\tau\}, -\tau \le t \le 0$$

is the solution of a homogeneous system. If a homogeneous system satisfies the initial conditions $x(t) = \varphi(t), -\tau \le t \le 0$, then the solution is next:

$$x(t) = X_0(t-\tau)\varphi(-\tau) + \int_{-\tau}^{0} X_0(t-\tau-s)[\varphi'(s) - A\varphi(s)]ds.$$

3) In the general case, we have:

$$x'(t) = Ax(t) + Bx(t-\tau) + f(t),$$

or, in terms of arms race between two countries:

$$\begin{cases} x_1'(t) = kx_2(t-\tau) - ax_1(t) + g \\ x_2'(t) = lx_1(t-\tau) - bx_2(t) + h \end{cases}. \qquad (2.3)$$

The initial conditions are written as follows:

$$x(t) = \varphi(t), -\tau \le t \le 0, x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \varphi(t) = \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix}.$$

The system above (2.3) can be interpreted as a system of differential equations in infinite dimensional Banach space, the points of which are segments of trajectories $x(t-s), -\tau \leq s \leq 0$ [5].

In this case, to show its solution in the same simple form as for the equations without delay, a kind of problematic.

The solution of the non-homogeneous system (2.3) can be written as

$$x(t) = x_0(t) + \bar{x}(t),$$

where $x_0(t)$ - the solution of the homogeneous system, that satisfies the initial conditions $x(t) = \varphi(t), -\tau \leq t \leq 0$, and $\bar{x}(t)$ - the solution of the non-homogeneous system, that satisfies zero initial conditions.

To find a solution of the Cauchy problem of a homogeneous system, we are going to use Rise theorem and possibility to present linear operator in the integral form [6].

Theorem 2.1: The solution of the system

$$x'(t) = Ax(t) + Bx(t-\tau), \tag{2.4}$$

that satisfies the initial conditions $x(t) = \varphi(t), -\tau \leq t \leq 0$ can be written in integral form:

$$X(t) = X_0(t)\varphi(-\tau) + \int_{-\tau}^{0} X_0(t-\tau-s)\varphi'(s)ds. \tag{2.5}$$

The Theorem 2.1 above requires a function φ(t) to be continuously differentiated. This condition can be weakened by requiring from φ(t) to be only continuous. For this, take the integral by parts and get:

$$X(t) = X_0(t-\tau)\varphi(0) + \int_{-\tau}^{0}[AX_0(t-\tau-s) + BX_0(t-2\tau-s)]\varphi(s)ds \tag{2.6}$$

Integral view of the solution of (2.4) depend on "special" solution $X_0(t)$, that come from the solution of equation (2.5) with singular initial conditions $X_0(t) = I, -\tau \leq t \leq 0, X_0(t) = \Theta, t \leq \tau$.

To find it, we will use Laplace transform

$$\overline{X_0}(p) = \int_0^{\infty} e^{-pt} X_0(t)dt.$$

After expanding image of a matrix function $X_0(t)$ into a series and computing commenced inverse transformation, we will obtain the original.

Solution (2.6) can be implemented for arms race model.

Let denote $X(t)$ – the matrix function that is a solution of matrix differential equation (2.7):

$$\frac{dX(t)}{dt} = AX(t) + BX(t-\tau) \tag{2.7}$$

with singular initial conditions $X_0(t) = I, -\tau \le t \le 0, X_0(t) = \Theta, t \le \tau$.

Theorem 2.2: Solution of the equation (2.7) with initial conditions $X_0(t) = I, -\tau \le t \le 0, X_0(t) = \Theta, t \le \tau$ has a form:

$$X_0(t) = I + \sum_{k=0}^{n-1} \varphi_k(t), (n-1)\tau < t \le n\tau,$$

$$\varphi_k(t) = \frac{(t-k\tau)^{k+1}}{k!} \sum_{i=0}^{\infty} A^i \frac{(t-k\tau)^i}{(k+i+1)!} B^k (A+B).$$

Consider the special case when $\det A \neq 0$. In this case, the solution can be written not as a series, but as a finite sum.

Theorem 2.3: Assume that the matrix A is non-degenerate. In this case, the solution of equation (2.7) with the initial conditions $X_0(t) = I, -\tau \le t \le 0, X_0(t) = \Theta, t \le \tau$ has a form:

$$X_0(t) = I + \sum_{k=0}^{n-1} \varphi_k(t), (n-1)\tau < t \le n\tau,$$

$$\varphi_k(t) = \sum_{i=0}^{\infty} (-1)^i \frac{(t-k\tau)^{k-i}}{(k-i)!} A^{-(i+1)} e^{A(t-k\tau)} B^k (A+B) + (-1)^{k+1} A^{-(k+1)} B^k (A+B).$$

For arms race,

$$\varphi_K(t) = \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} (-1)^i \frac{(t-K\tau)^{m+K-i}}{m!\,(K-i)!} \begin{bmatrix} (-a)^{m-(i+1)} k^K l & (-a)^{m-(i+1)} k^K b \\ -a(-b)^{m-(i+1)} l^K & (-b)^{m-(i+1)} l^K k \end{bmatrix} + (-1)^i \begin{bmatrix} (-a)^{-(K+1)} k^K l & (-a)^{-(K+1)} k^K b \\ -a(-b)^{-(K+1)} l^K & (-b)^{-(K+1)} l^K k \end{bmatrix},$$

where a, b, l, k – the corresponding coefficients from the Richardson model.

Obtained solutions of differential equations systems with delay have a rather complex structure and it is really hard to solve them analytically. But the Matlab environment could be used for it, where the function should be defined once and then can be used for calculations with different incoming data.

But we are more interested not so much in solutions, as in system stability. Because in terms of the arms race, it is important for us to understand if the system is in equilibrium, because it is significant in terms of political relations between countries. But purely solution of the system provides only a quantitative understanding of arms value that are not too substantial.

Definition 2.1: Solution $x(t) = \varphi(t)$ of the system (2.3) is called stable by Lyapunov if for $\forall\varepsilon > 0$ exists $\delta(\varepsilon) > 0$, such that for any solution $x = x(t)$ of the system, from the proximity of the initial conditions $\|\varphi(t_0) - y(t_0)\|_\tau < \delta$ implies proximity when $t \geq t_0$, scilicet $|\varphi(t) - y(t)| < \varepsilon$.

Definition 2.2: a linear non-homogeneous system (2.3) is called stable (asymptotically stable) if all its solutions are Lyapunov stable (asymptotically stable).

Theorem 2.4: For the stability of linear non-homogeneous system (2.3) for any free term $f(t)$ necessary and sufficient for zero solution $x(t) = 0$ of a corresponding homogeneous system (2.4) to be stable [6].

Proof:

Let $y(t) = \varphi(t)$ fixed solution of a homogeneous system. Replace $y(t) = \varphi(t) + x(t)$:

$$\varphi'(t) + x'(t) = A(t)(\varphi(t) + x(t) + B(t)\big(\varphi(t-\tau) + x(t-\tau)\big) + f(t).$$

As $\varphi(t)$ – the solution of the non-homogeneous system, then

$$\varphi'(t) = A(t)\varphi(t) + B(t)x(t-\tau) + f(t),$$

and after reduction the system remains:

$$x'(t) = A(t)x(t) + B(t)x(t-\tau).$$

Thus, the stability of an arbitrary solution $y(t) = \varphi(t)$ of non-homogeneous system doesn't dependent on function $f(t)$, but it is determined be the matrix $A(t)$.

Thus, the necessary and sufficient conditions for stability of linear stationary systems:

1) For a zero solution $x(t) = 0$ of the linear system with constant coefficients to be asymptotically stable, necessary and sufficient, for all roots of the characteristic equation to have a negative real part, i.e. $Re\ \lambda_i(A) < 0$.

2) If at least one root has a positive real part, i.e. λ_s, such that $Re\ \lambda_s(A) > 0$, then the solution is unstable.

The characteristic equation of a homogeneous system with a delaying argument (2.3) is written in the form:

$$\beta(\lambda) = \det\left(\lambda I - A - Be^{-\tau\lambda}\right) = 0 \qquad (2.5)$$

Theorem 2.5:

– If all the roots of a characteristic quasi-polynomial (2.5)

$$\det\left(\lambda I - A - Be^{-\tau\lambda}\right) = 0$$

lie in the left half plane $C_- = \{\lambda \in C: Re\ \lambda < 0\}$, then the zero solution of system (2.3) is asymptotically stable.

– If there is a root of a quasi-polynomial (2.5) lying in the right half plane $C_+ = \{\lambda \in C: Re\ \lambda > 0\}$, then the zero solution of system (2.3) is unstable [7].

Given the dynamic model of conflict situations, we can say, that when all the roots of a characteristic equation have a negative real part, we can talk about the stability of such a system, such that the parts of the conflict have reached a comfortable level of arms and balance of power, taking into account the economic component of their country and its opponent that studied in model. From now on, costs for armament are constant and unchanged, until one of the model parameters will be changed.

Let's apply the Theorem 2.5 above to our system and coefficients. Firstly, consider the zero-equilibrium position $\left(x_1(t), x_2(t)\right) = (0,0)$. The system of first approximation has the form [8]:

$$\begin{cases} x_1'(t) = -ax_1(t) \\ x_2'(t) = -bx_2(t) \end{cases}.$$

Since $a > 0$ and $b > 0$, obviously, the solution is stable.

Now consider the roots of the characteristic equation in the general case for a homogeneous system:

$$\det\left(\lambda I - A - Be^{-\tau\lambda}\right) = 0$$

or, for the model that is studied:

$$\beta(\lambda) = \det\left(\lambda I - A - Be^{-\tau\lambda}\right) = det\begin{bmatrix} \lambda + a & -ke^{-\lambda\tau} \\ -le^{-\lambda\tau} & \lambda + b \end{bmatrix} =$$

$$= (\lambda + a)(\lambda + b) - kle^{-2\lambda\tau} = 0$$

It is quite difficult to find the roots analytically, but it can be done using the computer. The system will be stable if the roots of the characteristic equation satisfy the Theorem 2.5 [9].

Another popular method of studying the stability of delayed systems is the second Lyapunov method. As follows from the Definition 2.1 of the stability of the zero solution and the basic Lyapunov theorems, the initial perturbation is inside the surface of the Lyapunov function level. Analytically, this condition looks like:

$$V(x(s), s) < V(x(t), t), s < t.$$

The advantage of this approach is the ability to use a well-designed apparatus of finite-dimensional Lyapunov functions (especially for nonlinear systems). It should be noted, that in the practical study of systems with delay, both approaches give (mainly) only sufficient stability conditions and majoritarian estimates of solutions convergence. The main theorem of asymptotic stability is as follows:

Theorem 2.6: Suppose there is a function $V(x, t)$ that continuously differentiated, is defined in some region of the zero-equilibrium position and satisfy the conditions:

1) Function $V(x, t)$ positively defined, that is $V(0, t) = 0$ and $V(x, t) \geq W(|x|) > 0$ when $x \neq 0$.
2) The full derivative of the function $V(x, t)$ through a system of curves satisfy the Razumihin condition $V(x(s), s) < V(x(t), t), s < t$.

Then the zero solution of the system will be asymptotically stable.

Or, Krasovsky was formulated an equivalent theorem:

Theorem 2.7: Suppose we have a system

$$x'(t) = F(x(t), x(t-\tau)), t \geq 0, x(t) \epsilon R^n,$$

and there is a functional $V[x(t)]$, that satisfies bilateral estimates

$$a(||x(t)||) \leq V[x(t)] \leq b(||x(t)||),$$

a derivative of which along the solutions $x(t)$:

$$\frac{d}{dt}V[x(t)] = \lim_{h \to +0} \frac{1}{h}\{V[x(t) + hF(x(t), x(t-\tau))] - V[x(t)]\}$$

is a functional, that satisfies

$$\frac{d}{dt}V[x(t) \leq -c(||x(t)||)], a(\cdot), b[\cdot], c(\cdot), a(0) = b(0) = c(0) = 0,$$

and $||x(t)||$ - some norm in functional space $C^1_{[-\tau,0]}$. Then zero solution $x(t) = 0$ is asymptotically stable.

As rule, for linear dynamic systems, the quadratic Lyapunov function is used. If we denote the surfaces of the Lyapunov function level

$$\partial V^\alpha = \{x \in R^n: V(x) = \alpha, V^\alpha = \{x \in R^n: V(x) < \alpha\},$$

then, using the inequalities of quadratic forms, the Razumihin condition can be written as follows:

$$\lambda_{min}(H)|x(s)|^2 \leq V(x(s)) \leq V(x(t)) \leq \lambda_{max}(H)|x(t)|^2.$$

From here we get:

$$|x(s)| < \sqrt{\varphi(H)}|x(t)|, \varphi(H) = \frac{\lambda_{max}(H)}{\lambda_{min}(H)}.$$

Theorem 2.8: Let a linear stationary system without delay

$$x'(t) = (A + B)x(t)$$

be asymptotically stable. If there is positively defined matrix H, which is the solution of the matrix equation

$$(A + B)^T H + H(A + B) = -C, \forall C > 0,$$

and for which inequality is fulfilled

$$\lambda_{min}(-A^T H - HA) - 2|HB|\left(1 + \sqrt{\varphi(H)}\right) > 0, \varphi(H) = \frac{\lambda_{max}(H)}{\lambda_{min}(H)},$$

then the delayed system (2.4) is asymptotically stable for any delay $\tau > 0$.

For a dynamic model of conflict situations, stability means, that the parties to the conflict have reached a comfortable level of arms and balance of power, taking into account the economic component of its side and the side of its opponent that is studied in model.

From now on, armaments costs become constant and unchanged until one of the model's parameters will be changed [4].

2.3 Modeling algorithms

2.3.1. Behavior of the systems without delay

Consider an ordinary model without delay, which has the form (2.2):

$$\begin{cases} x_1'(t) = kx_2(t) - ax_1(t) + g \\ x_2'(t) = lx_1(t) - bx_2(t) + h \end{cases}.$$

To analyze the system, let’s apply an algorithm for constructing phase portraits and solution trajectories, which will later be applied to numerical model. In Matlab, the algorithm is defined as follows:

1) Determine the set of points where x' and y' will be calculated. The set will be in the form of a grid at interval (a, b) with the step $\frac{b-a}{n}$:

$x1 = linspace(a, b, n);$

$x2 = linspace(a, b, n);$

$[x, y] = meshgrid(x1, x2);$

2) Values x_1' and x_2' of the differential equations system $myode$ are calculated for each point of the set mentioned above:

$u = zeros(size(x));$

$v = zeros(size(y));$

$for\ i = 1{:}numel(x)$

$\quad Yprime = myode(t, [x(i), y(i)]);$

$\quad u(i) = Yprime(1);$

$\quad v(i) = Yprime(2);$

$end;$

3) Normalize obtained values and construct a set of vectors on the plane. For this purpose a special standard function $quiver$ is used.

4) Construct multiple solutions' trajectories under different initial conditions – this can be performed by using Matlab function $ode45(f, [t0; T])$ $[x0, y0])$, that solves the system of ordinary differential equations f with initial conditions $(x0, y0)$ on a time interval $[t0; T]$.

5) Using $plot$ function, visualize the results [10].

2.3.2. Behavior of the systems with delay

Definitely, if delay is appeared, the phase portrait and system trajectories change. But how much and whether it affects the stability of the system depends on the magnitude of the delay, the type of singular point, and so on.

As was described above, in general, to build a phase portrait, we need to know the derivatives values. But in case of delay, the problem is somehow complicated by the fact that to find $x_1'(t)$ and $x_2'(t)$ we need to know $x_1(t- \tau)$ and $x_2(t- \tau)$, but these values are usually unknown to us, and in general, the term 'phase portrait' for a systems with delay doesn't exist. But we can plot multiple trajectories and normals at each of the selected points, and by its nature, this plot will have sense just like a normal phase portrait of a system without delay.

An algorithm in the Matlab programming language was written, which greatly simplifies calculations.

Suppose we have a system of two differential equations in the general form (2.3):

$$\begin{cases} x_1'(t) = kx_2(t- \tau) - ax_1(t) + g \\ x_2'(t) = lx_1(t- \tau) - bx_2(t) + h \end{cases}$$

1) The set of values $x_1(t), x_2(t)$ of delay differential equations system $ddefun$ is determinated for $\forall t \in [t0, T]$ with fixed delay lag and initial conditions $history$ by using the standart Matlab function $dde23(@ddefun, [lag], [history], [t0, T]$.

2) Values $x_1{}'(t)$ and $x_2'(t)$ are calculated for $\forall t \in [t0, T]$ the points of the set obtained in the first step:

$x1dot\ =\ zeros(1, length(x1));$

$x2dot\ =\ zeros(1, length(x2));$

$tmp\ =\ sol.x;$

$for\ i\ =\ 1{:}\,10{:}\, length(x1dot)$

$t\ =\ tmp(i)\ -\ lag;$

$if\ t\ <=0$

$yint\ =\ [y01; y02];$

$else$

$yint\ =\ deval(sol, t);$

$end;$

$[x1dot(i), x2dot(i)]\ =\ ddefun(t, x1(i), x2(i), yint);$

$end;$

3) With obtained values the set of normilized vectors are going to be constructed through the function $quiver$.

4) Multiple solution trajectories are constructed under the given initial conditions using a standard function $plot$.

5) To check the stability of the system according to Lyapunov conditions, the eigenvalues are found through the ordinary function $solve$ [10].

3 NUMERICAL SOLUTIONS

3.1 Model building

To build Richardson model, we need to collect statistic data, handle it in some way and provide model's coefficients. Today, RAND corporation model of geopolitical status is very popular and we are going to use it [11].

By this model, geopolitical power of a country is a dimensionless convolution of private parameters, that names geopolitical potential. The common formula of country power has a form [12,13]:

$$S(t) = F_A(t) * G(t)$$

$$G(t) = 0.5 * (1 + X_M^{0.43}) * X_E^{0.27}$$

$$F_A = \left(1 - \frac{J}{Y}\right)^{0.27} \left(1 - \frac{W_a}{W_G + W_a}\right)^{0.43},$$

– X_E – the ability of the country to save economic and technical position or goes up. In model, we will use GDP as this parameter;

– X_M – the ability of the country to protect its positioning in the world. In model, this parameter will be calculated as the total cost of armaments, number of regular troops, number of military equipment units, nuclear weapons;

– $\frac{W_a}{W_G+W_a}$ - the military - human factor, as the ability of the country to defend its position in the world. It is characterized by the ratio of the number of regular troops to the total number of troops;

– $\frac{J}{Y}$ - is a financial factor and represents as the ratio of imports to GDP;

– $G(t)$ – the geopolitical potential of the country;

– $F_A(t)$ – an influence function, which is a set of all factors that are not explicitly related to geopolitical potential;

– $S(t)$ – status at the moment of time.

Consider the model on the example of two countries – USA and China, using statistics for 2001, 2006, 2011 and 2016 years (Table 1 and Table 2) [14,15].

Table 1 – Statistics of China

Year	Military budget (mln $)	Regular troops	Regular army + reserve	Number of military equipment units	Nuclear weapons	Import	GDP
2001	122.19	3210000	4503000	15298	410	271300	5081000
2006	126	3755000	4503000	15298	130	632000	6593000
2011	123.33	2945999	3503000	27504	240	1664000	13810000
2016	126	2993000	3503000	27320	250	1950000	17961000

Table 2 – Statistics of USA

Year	Military budget (mln $)	Regular troops	Regular army + reserve	Number of military equipment units	Nuclear weapons	Import	GDP
2001	301.69	1420700	2349950	22553	8776	1398700	10621000
2006	527.6	1546000	2349950	22553	5521	1727000	13093000
2011	698.18	1520100	2349950	76962	7150	2314000	15517000
2016	612.5	1381250	2349950	68333	7506	2273000	19348000

Let's calculate coefficients $G(t), F_A, S(t)$ for each country in each of the listed years. We obtain the following data (table 3):

Table 3 – Parameters of geopolitical potential

Country	Year	$G(t)$	F_A	$S(t)$
China	2001	11094	0.439	48799
	2006	23368	0.4497	10509
	2011	25767	0.4381	11288
	2016	27848	0.4233	11789
USA	2001	17648	0.6459	11399
	2006	19333	0.6069	11733
	2011	20406	0.6119	12487
	2016	20783	0.6605	13727

Consider a six-equation system to find the coefficients of the Richardson arms race model, where $x_i, y_i, i = 1,2,3,4$ – the status of China and USA at 2001, 2006, 2011 and 2016 years respectively:

$$\begin{cases} x_2 = ky_1 - ax_1 + g \\ y_2 = lx_1 - by_1 + h \\ x_3 = ky_2 - ax_2 + g \\ y_3 = lx_2 - by_2 + h \\ x_4 = ky_3 - ax_3 + g \\ y_4 = lx_3 - by_3 + h \end{cases},$$

Dividing it into two systems, we find the coefficients:

$$k = 0.6, a = 0.4, g = 3665.1, l = 0.39, b = 1.6, h = 6752.8.$$

The Richardson model will look like:

$$\begin{cases} x_1' = -0.4x_1 + 0.6x_2 + 3665.1 \\ x_2' = 0.39x_1 - 1.6x_2 + 6752.8 \end{cases}.$$

3.2 Computer calculations

3.2.1. Systems without delay

Consider the example. Let's have a system

$$\begin{cases} x_1' = -0.4x_1 + 0.6x_2 + 3665.1 \\ x_2' = 0.39x_1 - 1.6x_2 + 6752.8 \end{cases}. \tag{3.1}$$

By equating the right parts of the system to zero, we find that the critical point is in (24423.3, 10173.7). Do a substitution:

$$\begin{cases} x_1 = 24423.3 + \chi_1 \\ x_2 = 10173.7 + \chi_2 \end{cases},$$

and return back to (x_1, x_2) variables, we obtain a homogeneous system:

$$\begin{cases} x_1' = -0.4x_1 + 0.6x_2 \\ x_2' = 0.39x_1 - 1.6x_2 \end{cases}.$$

We see, that $ab = 0.64, lk = 0.234, ab > lk,$ that's why critical point should to be a stable node.

Also, from theoretical knowledge, for arms race when $ab > kl, g > 0, h > 0$ stable equilibrium position should exist. This means, both countries will uniformly increase their strength with a slight separation from each other.

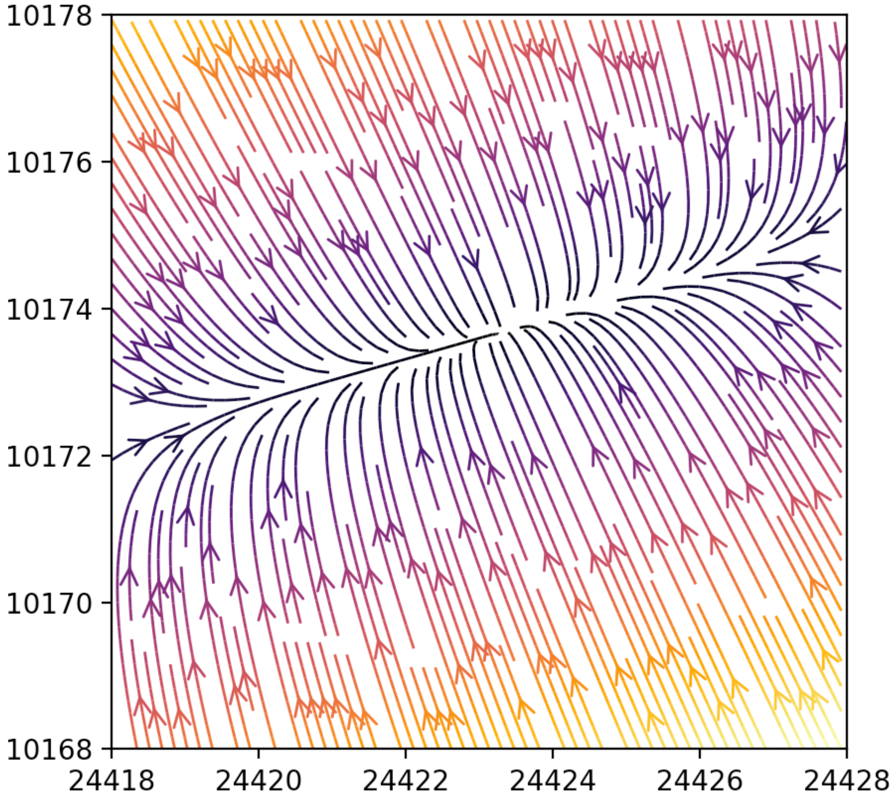
Let's check this by finding the eigenvalues in analytical way:

$$\beta(\lambda) = \det\begin{pmatrix} -0.4 - \lambda & 0.6 \\ 0.39 & -1.6 - \lambda \end{pmatrix} = (-0.4 - \lambda) * (-1.6 - \lambda) - 0.234 =$$

$$= 0.406 + 2\lambda + \lambda^2 = 0,$$

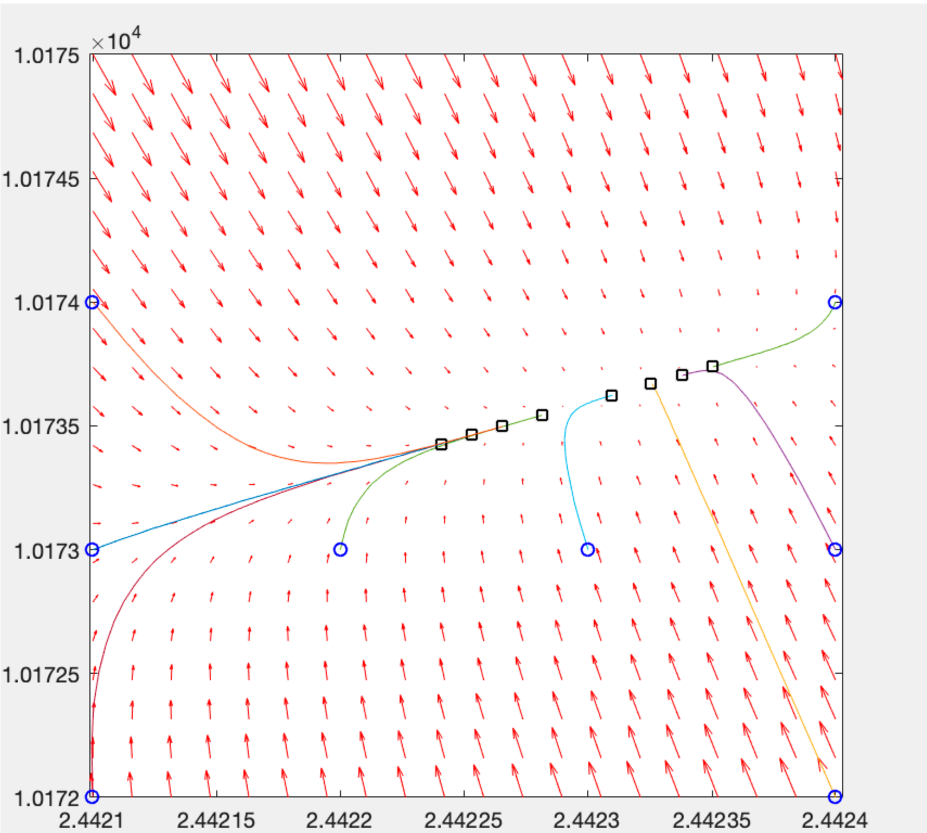
wherefrom $\lambda_1 = -0.23, \lambda_2 = -1.77$, that indicating critical point is a stable node.

For the differential equations system (3.1) below, phase portrait is shown on the Pic.1:



Picture 1 – Phase portrait for the system without delay

Solutions of the system with different initial conditions (Pic.2):



Picture 2 – Solutions of the system without delay with different initial conditions

On Pic.1 and Pic.2 we see, that the system is stable. That's why we can assume, that having a very small delay although system behavior will change but it will not influence on stability. Therefore both countries all the time are in the arms race state, but will not start the war [4].

3.2.2. Systems with delay

Let's apply algorithm for the same Richardson model as in previous paragraph, where the model without delay was investigated:

$$\begin{cases} x_1'(t) = -0.4x_1(t) + 0.6x_2(t-\tau) + 3665.1 \\ x_2'(t) = 0.39x_1(t-\tau) - 1.6x_2(t) + 6752.8 \end{cases} \quad (3.2)$$

For a delayed system, we cannot construct a phase portrait in its usual sense. And therefore, consider trajectories with different constant initial conditions, as described in the algorithm in the previous paragraph.

Equation for finding the system eigenvalues:

$$\beta(\lambda) = \det(\lambda I - A - Be^{-\tau\lambda}) = det\begin{bmatrix} 0.4+\lambda & -0.6e^{-\lambda\tau} \\ -0.39e^{-\lambda\tau} & 1.6+\lambda \end{bmatrix} =$$

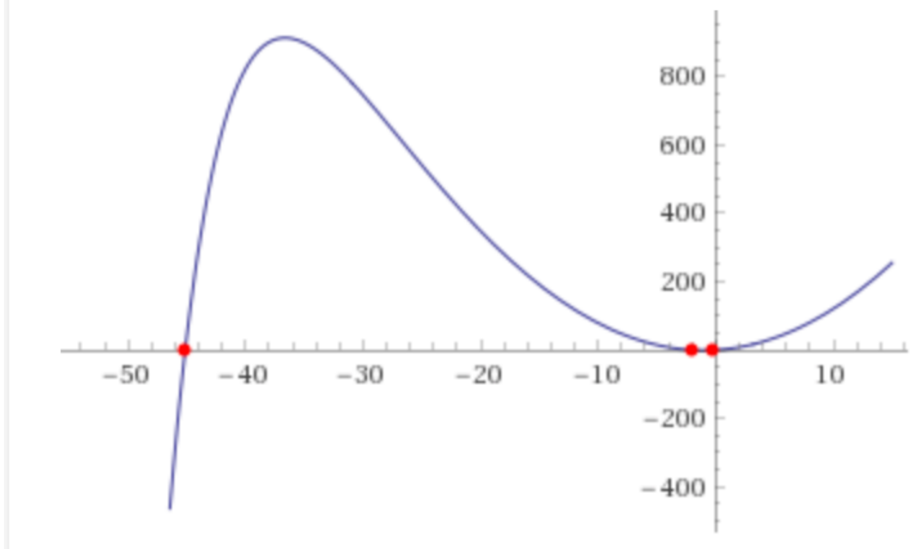
$$= (0.4+\lambda)(1.6+\lambda) - 0.234e^{-2\lambda\tau} = 0$$

Consider the trajectories and find eigenvalues for different values of τ.

We will start from small delay and step by step will increace it. From theoretical knowledges, system with a small delay by its begaviour should be very similar to initial one, and with increacing the delay, begavior is going to change. Let's check it.

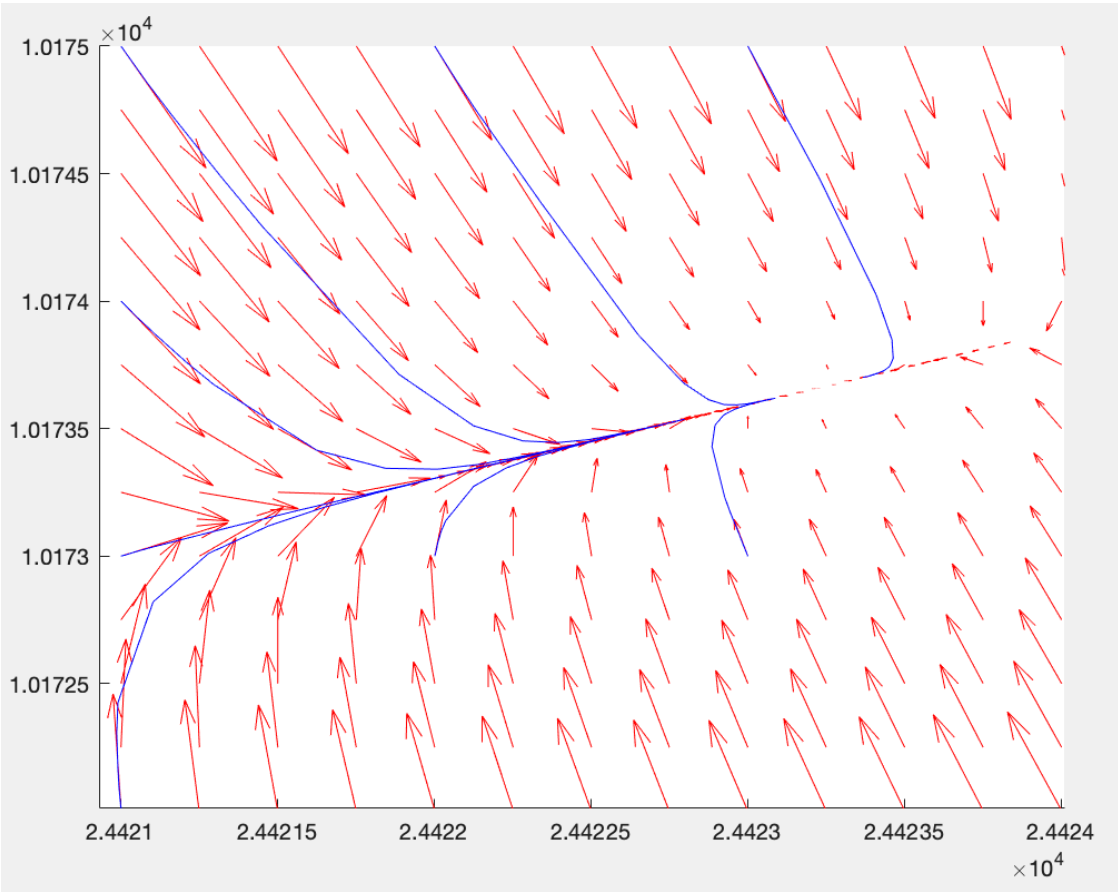
a) $\boldsymbol{\tau = 0.1}$**:**

$$\beta(\lambda) = (0.4+\lambda)(1.6+\lambda) - 0.234e^{-0.2\lambda} = 0.$$

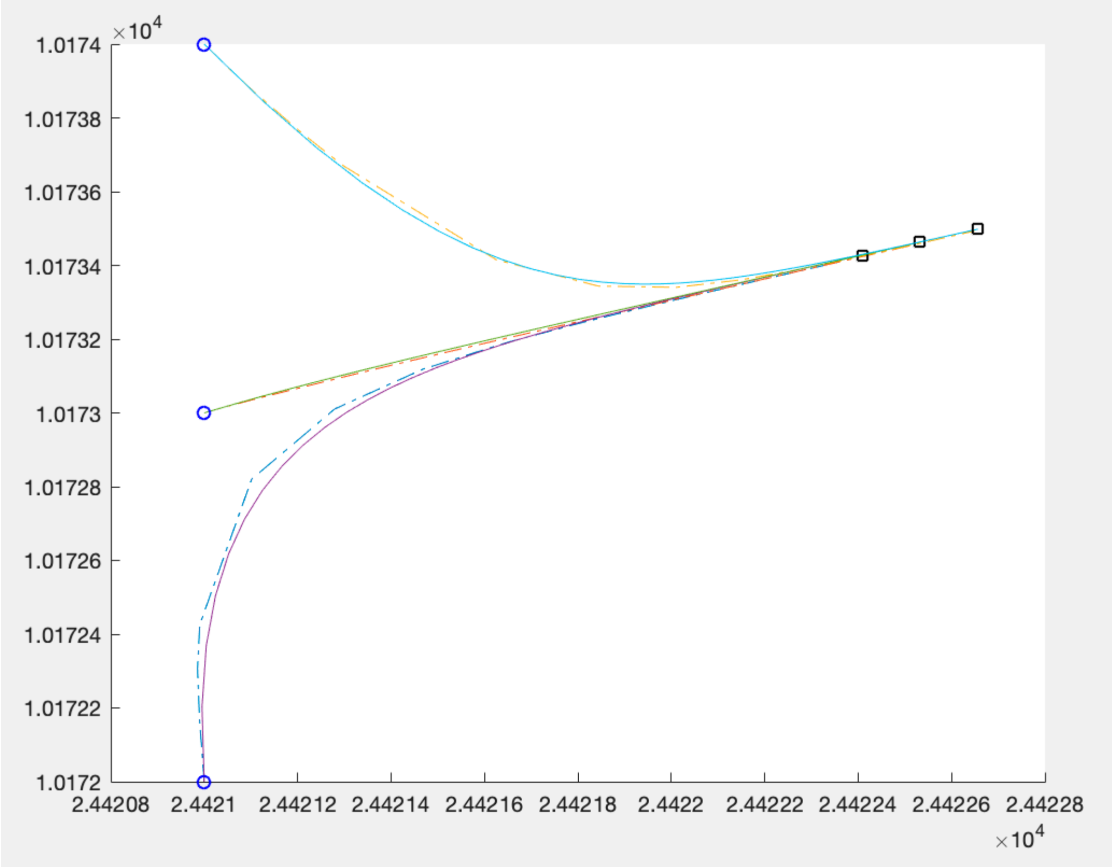


Picture 3 – Roots of characteristic polynomial

Using Matlab, we found, that $\lambda = \{-45.13, -1.84, -0.22\}$ - the eigenvalues is real and negative (Pic.3), that means system stability. Also, Pic. 4 and Pic. 5 are proving the system stability.



Picture 4 – Phase portrait and solutions of the system with delay $\tau = 0.1$



Picture 5 – Trajectories of the system with delay τ $= 0.1$ and without it

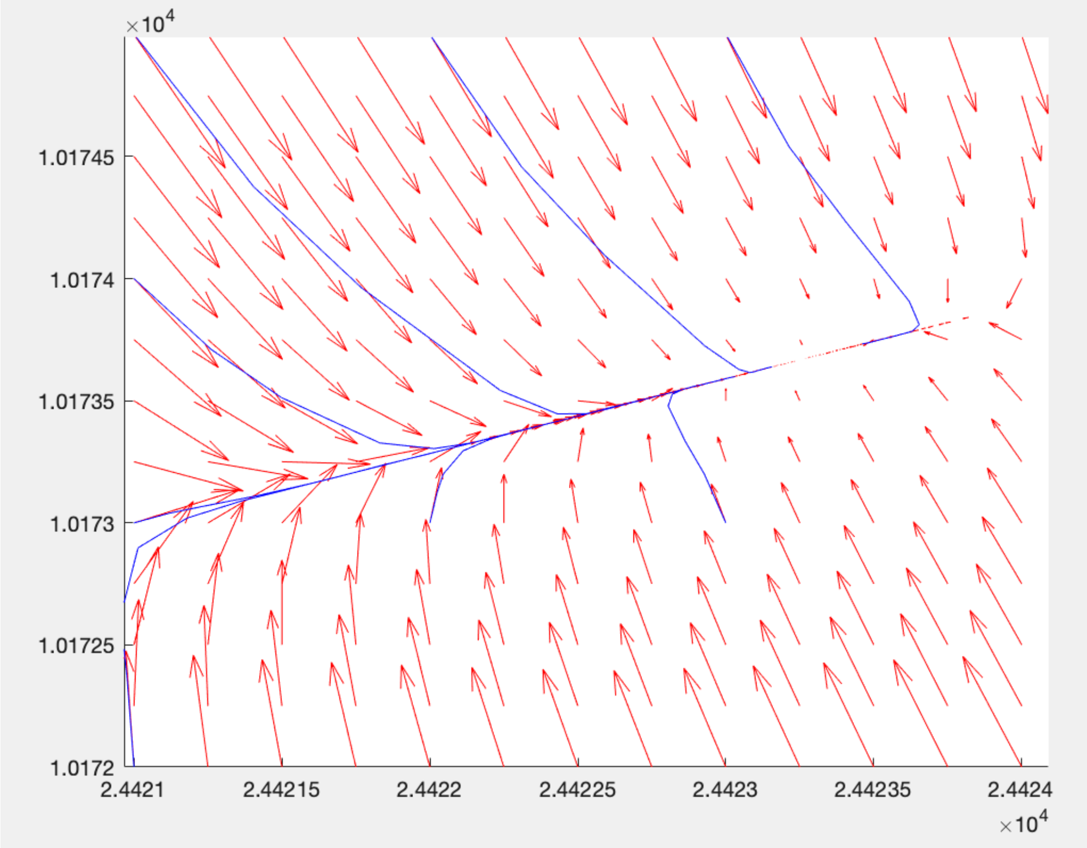
Indeed, due to the fact that argument deviation is not significant, phase trajectories and trajectories of the system with a deviating argument almost are not different from the system without deviating of argument. You can see it more clearly in the Pic.5 above, where trajectories of systems with delay $\tau = 0.1$ (solid line) and without (dotted line) with the same initial conditions are shown.

Increase delay:

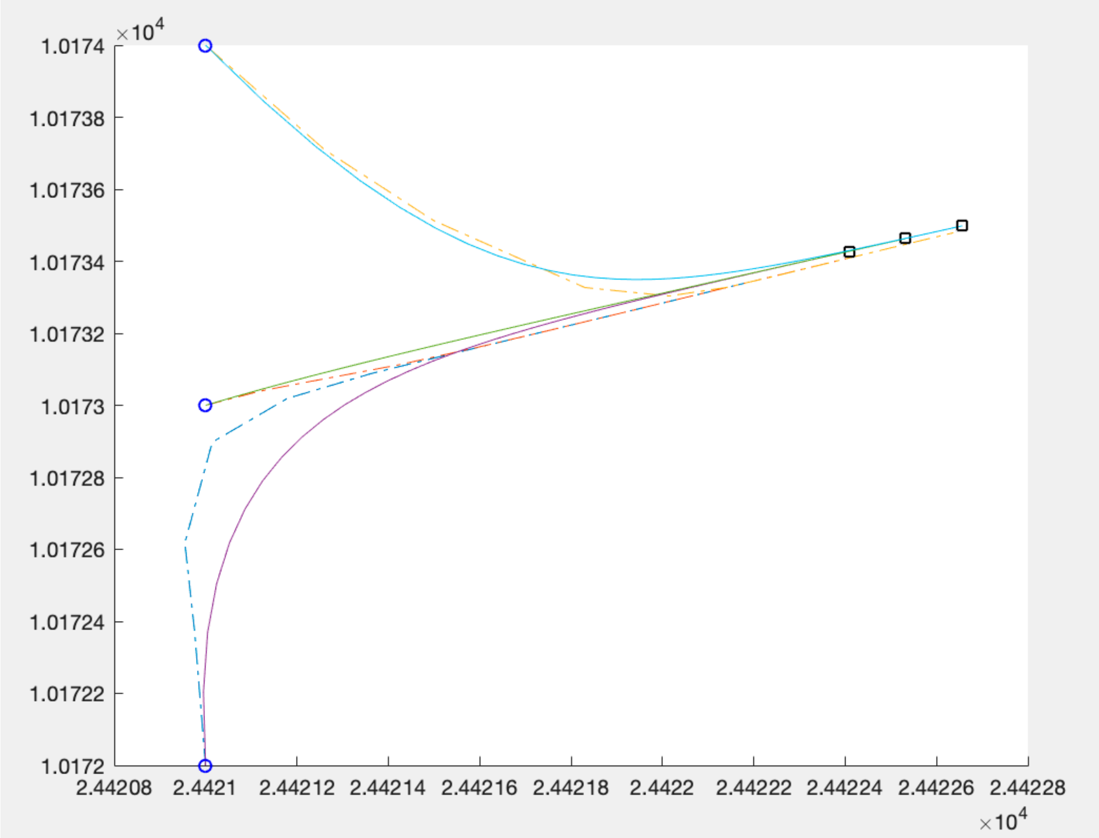
b) $\boldsymbol{\tau = 0.5}$

$$\beta(\lambda) = (0.4 + \lambda)(1.6 + \lambda) - 0.234e^{-\lambda} = 0.$$

Characteristic equation has a finite number of roots, eogenvalue $\lambda = -0.197$ - real and negative, and all other roots are complex, but with negative real part. That's why system stays to be stable (Pic. 6 and Pic. 7).



Picture 6 – Phase portrait and solutions of the system with delay $\tau = 0.5$



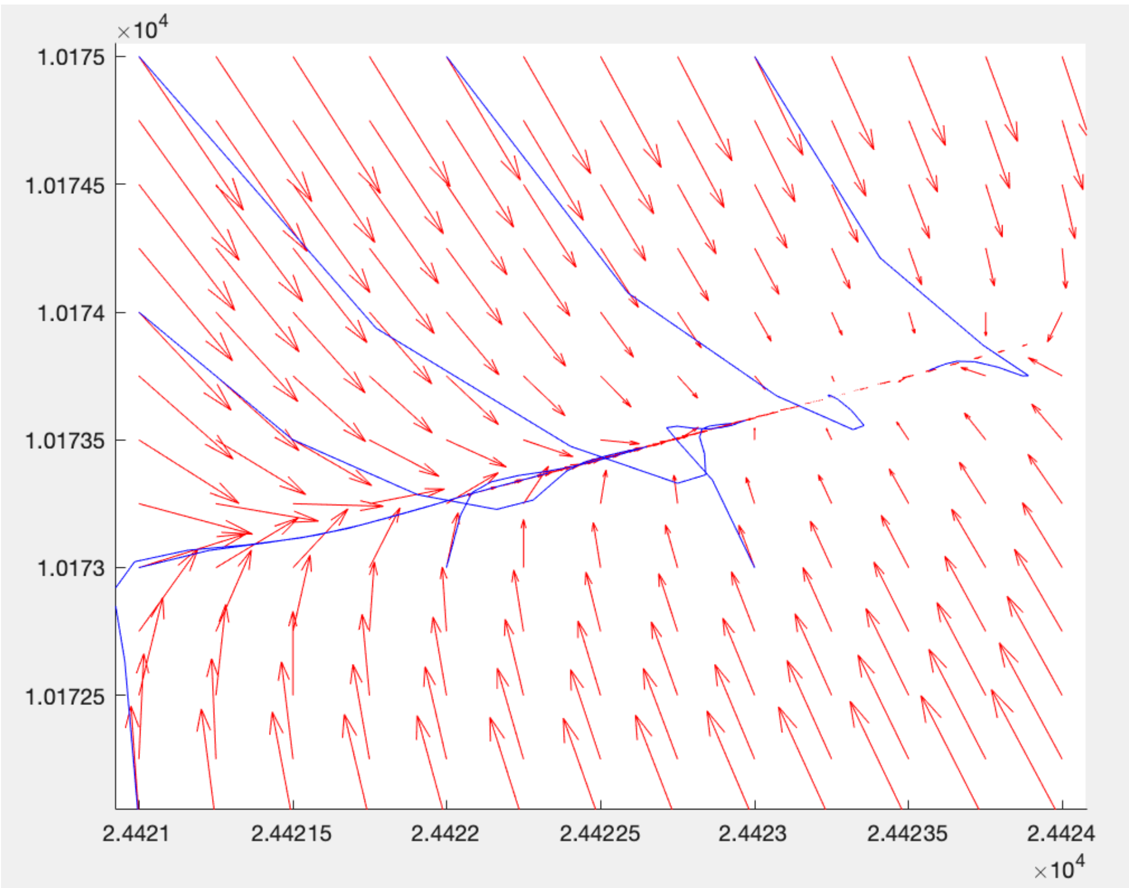
Picture 7 – Trajectories of the system with delay $\tau = 0.5$ and without it

Obviously, we need to increase delay parameter more, to see the difference in trajectories more clearly. In Richardson model terminology, we want to see how is delay in reaction on changes in one country influence the relationship between countries:

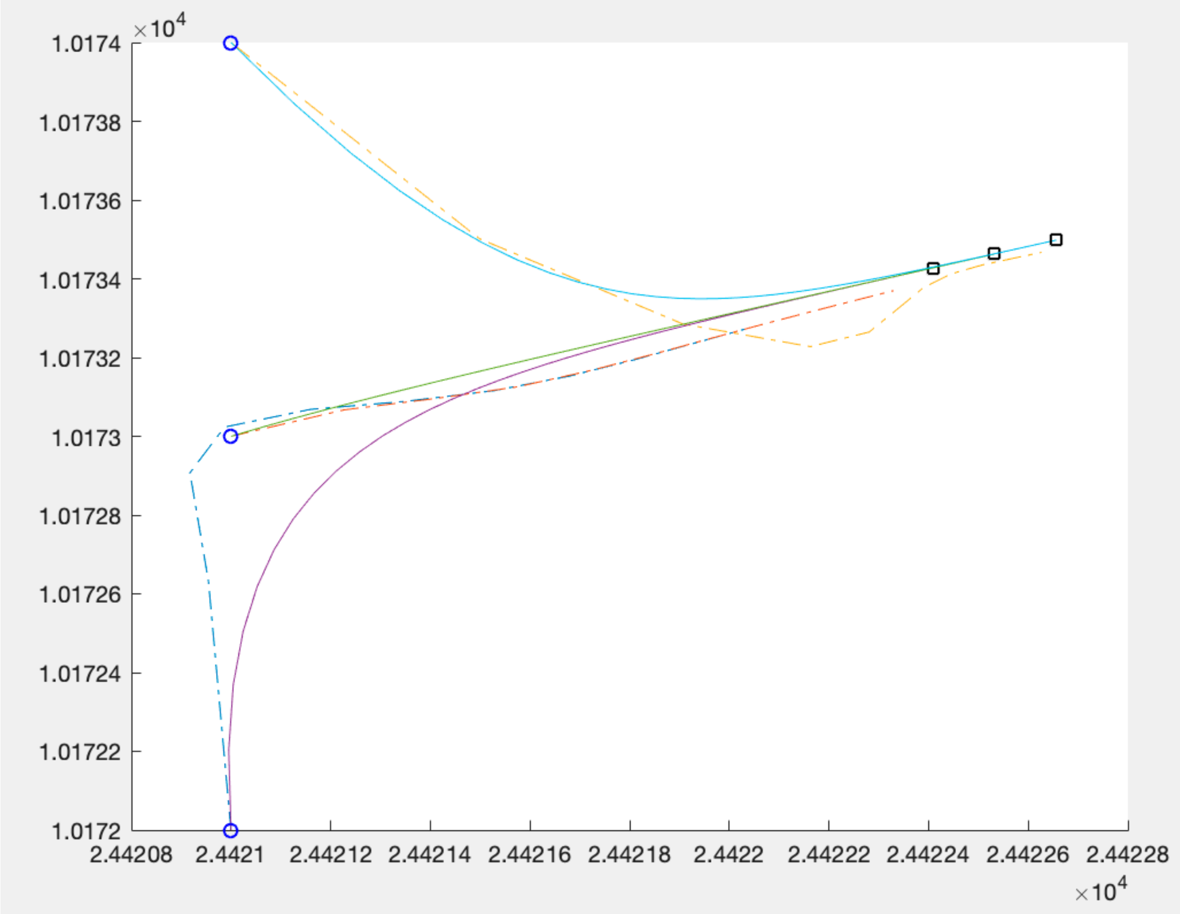
c) $\tau = \mathbf{1}$:

$$\beta(\lambda) = (0.4 + \lambda)(1.6 + \lambda) - 0.234e^{-2\lambda} = 0.$$

With a delay equal to one, root with the biggest real part - $\lambda = -0.17$ is negative, that's why the system is still stable. But now it is possible to see that trajectories are not longer like linear and by approaching the critical point system behaves somewhat unpredictably (Pic. 8 and Pic. 9).



Picture 8 – Phase portrait and solutions of the system with delay $\tau = 1$



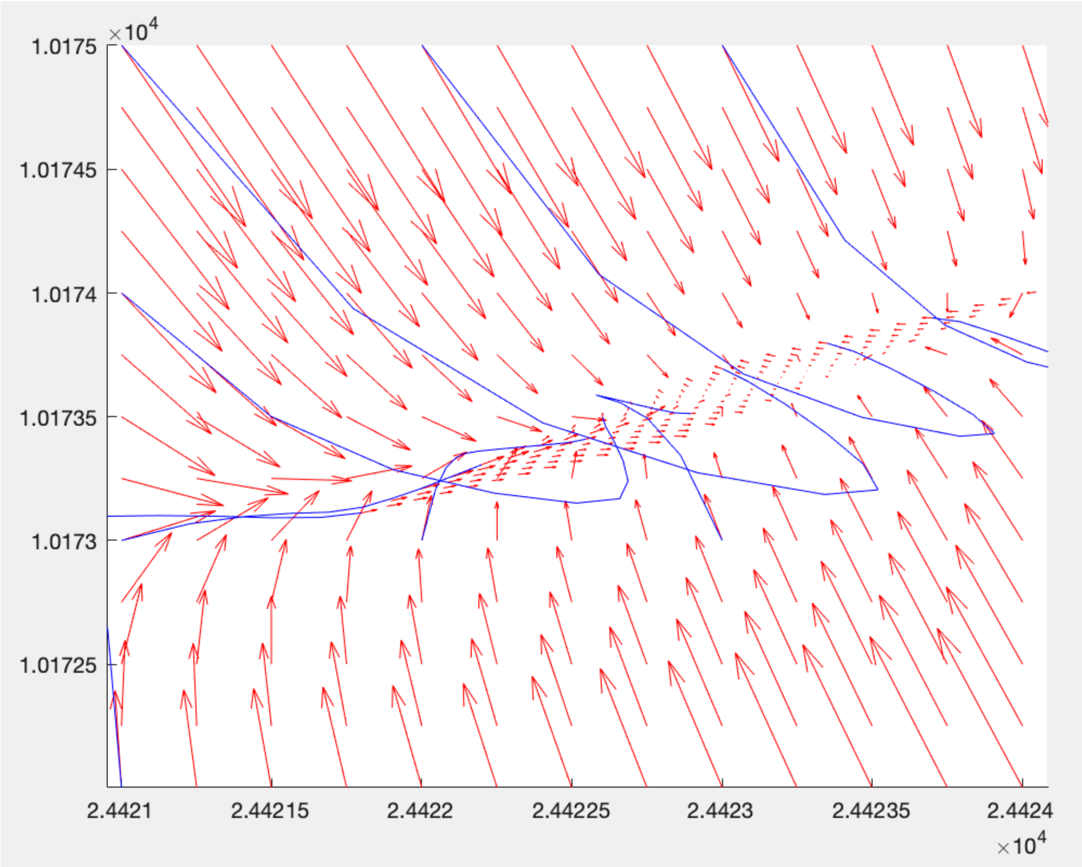
Picture 9 – Trajectories of the system with delay $\tau = 1$ and without it

Thinking logically, with even greater delay, the system though may still remain stable, but the trajectories behavior will become even more unpredictable, let's check it out:

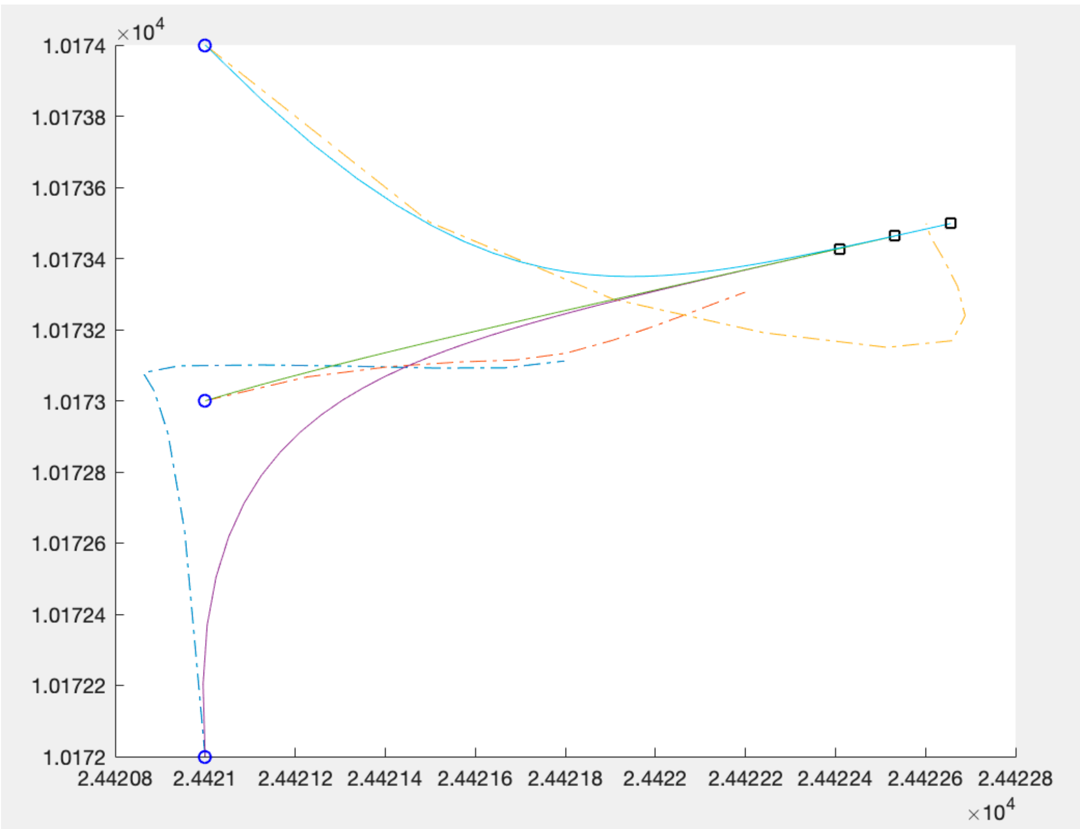
d) $\tau = 2$:

$$\beta(\lambda) = (0.4 + \lambda)(1.6 + \lambda) - 0.234e^{-4\lambda} = 0.$$

Found that $\lambda = -0.13$ – root with the biggest real part, all other roots lie to the left, that means that the system still remains stable (Pic. 10 and Pic. 11).



Picture 10 – Phase portrait and solutions of the system with delay $\tau = 2$



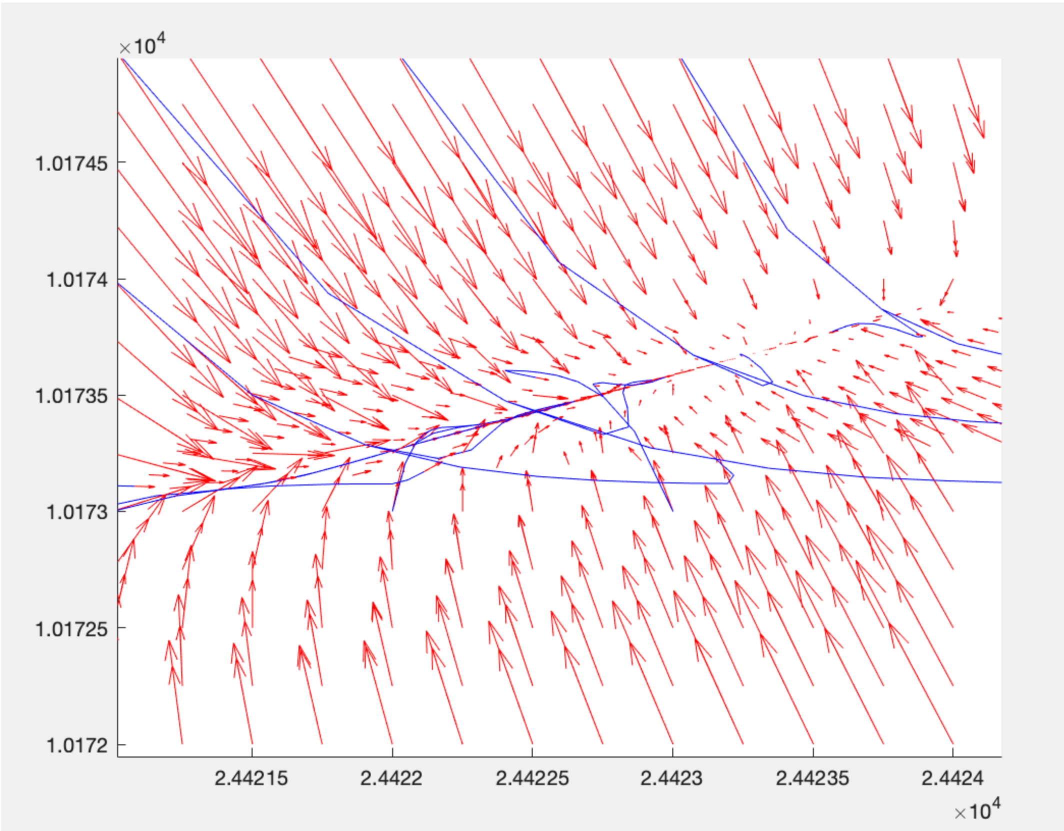
Picture 11 – Trajectories of the system with delay $\tau = 2$ and without it

We see, that system unpredictiable behavior was increased, trajectories looks more complicated, eigenvalues' real part is approaching zero. Let's try to break system stability by increasing the delay parameter twice:

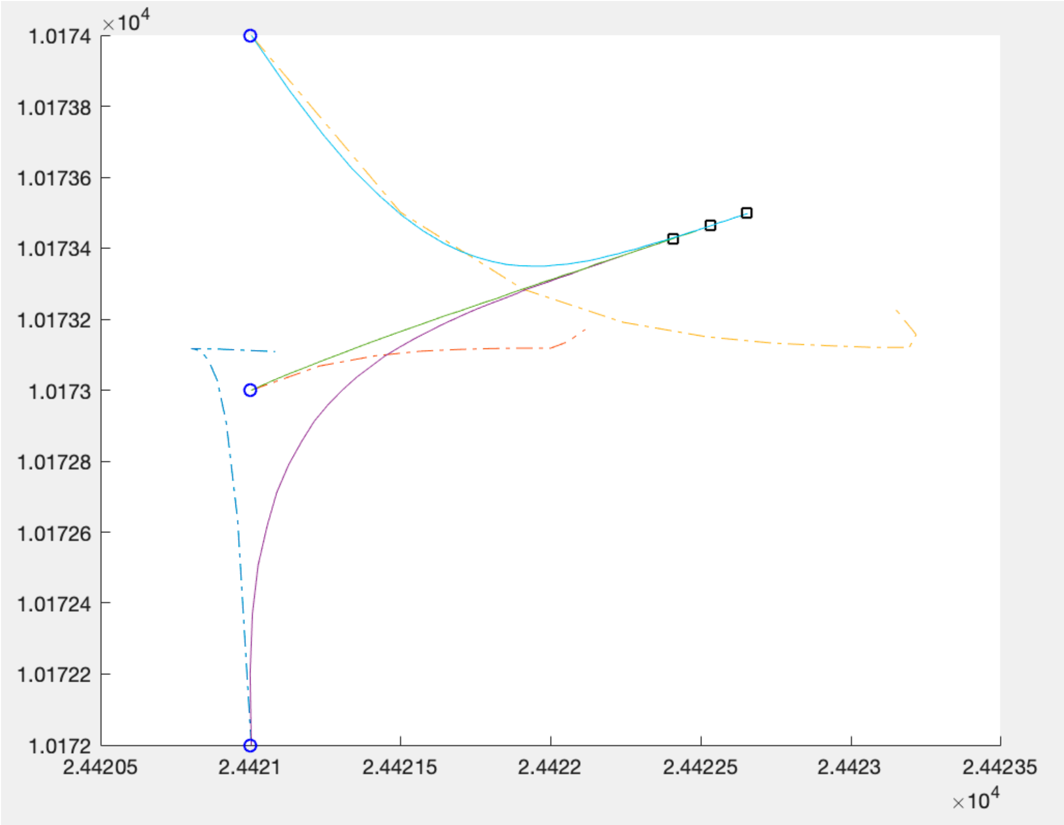
e) $\tau = \mathbf{4}$:

$$\beta(\lambda) = (0.4 + \lambda)(1.6 + \lambda) - 0.234e^{-8\lambda} = 0.$$

Eigenvalues' real parts is still negative, the biggest one for real eigenvalue $\lambda = -0.08$. But on the picture of phase trajectories (Pic. 12) and trajectories of the solution (Pic. 13) although visible that all of them approximately seek to the critical point, but readability of the plot is completely lost and mathematical attractiveness of this plot is missed, trajectories and directions look confusing and unclear.



Picture 12 – Phase portrait and solutions of the system with delay $\tau = 4$



Picture 13 – Trajectories of the system with delay $\tau = 4$ and without it

Also, the bigger delay was studied for this paper. In that case, the system stays stable, but behaviour – unclear and unpredictible. From mathematical point of view, plots becomes unreadable, and that's why they are not added to this work.

I want to assume, that system stays stable exactly because the critical point is node and it is really hard to broke its stability.

Returning to the arms race model (3.2), if we add a delay, economics' reaction doesn't change and it is seeking to reducing the rate of arms growth and the cost of arms. But this does not happen immediately, it happen after receiving information about the armaments and costs of another country. Therefore, trajectories become more confusing and unpredictable, because the time for analysis for both countries is needed. But nevertheless, at given system parameters, despite the delay, system stay stable. In other words, both countries all the time will be in arms race state trying to overtake opponent, but they will never start the war.

CONCLUSIONS

For this paper, Richardson model was selected. As modeling I understood representation of the social process as a differential equations system, selection of system parameters based on statistics, as well as further study of the model to obtain certain conclusions.

Based on the analysis of a large number of literature and sources, theoretical information about the model, the value of its parameters and their effect on the stability of the system are presented in the paper.

It has been suggested that a country's response to changes from another country may not be immediate, and thus, delay parameter was added to the system and conducted an appropriate analysis of the obtained results.

Also, based on the sources about GDP, the volume of imports, the cost of weapons and army a Richardson model was built. Data for 2001, 2006, 2011 and 2016 were taken for analysis.

The model is in the process of a steady buildup of forces by both sides. The character of the model is an arms race where the constant rivalry present. As soon as one of the states breaks into the lead, the other seeks to increase its geopolitical status by varying model parameters. This depends on the speed of response to changes in one country by another.

Indeed, analyzing the results obtained and analyzing China-US relations in 2017-2020 (years that were not included to the model), we see that the results of the model are really adequate, and these countries are in a state of steady arms increase.

So, in the general case, the arms race is when the arms escalation rates of both countries are equal or approximately equal. Here, from any starting point, the solution eventually comes to the point of equilibrium, "balance of power" achieved, regardless of the initial level of arms. In this case, the race does not lead to military conflict.

In an unstable arms race, there are disparities in the dynamics of the country's level of arms, and such race must end with a war.

However, the model has unforeseen contingency risks, which can lead to dramatic changes in model parameters, such as weather disasters, nuclear war, epidemics, and more.

Today, as never before, obtained results may be useful for world political scientists and economists. After the end of the global pandemic, redistribution of the world economics and forces will definitely take place. Already, amid a pandemic, the level of hostility between the US and China is increasing, and by adding a few new parameters to the resulting model, you can predict how the confrontation will end. With the obtained information, possible military conflicts can be avoided, or, at least, people can be prepared for them.

LIST OF REFERENCE SOURCES

1. Маценко В.Г. Математичне моделювання. Навчальний посібник / В.Г. Маценко. – Чернівці: Чернівецький національний університет, 2014. – 519 с. - ISBN 978-966-423-294-1

2. Mathematical models of arms control and disarmament / T.L. Saaty. – New York: John Wiley & Sons Inc., 1968. – 304 c.

3. Об управляемых вариантах модели Л. Ричардсона в политологии / М.С. Никольский // Труды ИМП УрО РАН. - 2011. – Т.17 с.121-128

4. Lewis Fry Richardson: His Intellectual Legacy and Influence in the Social Sciences / N. P. Gleditsh. – Cham: Springer Nature Switzerland AG, 2019. – 148 c.

5. Comparative analysis and new field of application of Lanchester's combat models / A.V. Shatyrko, B. Puzha, V. Novotna. – Brno: Conference MITAV, 2018. – 118 c.

6. Введення в моделювання динамічних систем / Д.Я. Хусаінов, А.В. Шатирко, І.І. Харченко. – К.: Київський національний університет імені Тараса Шевченка, 2010. – 130 с.

7. Абсолютная устойчивость автоматических систем з запаздыванием / В. Резван. – М.: Наука, Главная редакция физико-математической науки, 1983. – 339 с. (Вс 21707)

8. Линейные динамические системы с последействием. Представление решений, устойчивость, управление, стабилизация / Д.Я. Хусаинов, Й. Диблик, М. Ружичкова – К.: ГП Информ.-аналит. агентство, 2015. – 252 с. (Ва 792456)

9. Введение в теорию дифференциальных уравнений с отклоняющимся аргументом / Л.Э. Эльсгольц, С.Б. Норкин. – М.: Наука, Главная редакция фізико-математической науки, 1971. – 296 с. (Ва 168029).

10. Get Started with MatLab [Electronic resource] https://it.mathworks.com/help/matlab/getting-started-with-matlab.html

11. Опыт предварительного определения состава геополитического потенциала / А.Б. Волинчук, В.Б. Шведов. – Евразийский журнал региональных

и политических исследований, 2008. – 50 с.

12. Politics Among Nations: The Struggle for Power and Peace / H.J. Morgenthau. – New York: Aftred A. Knopf, 1986. – 356 с.

13. Геополитический статус государства: сущностные характеристики / Фартушняк Л.Л.- [Electronic resource] http://www.jurnal.org/articles/2008/polit80.html

14. Militaty Strength Ranking [Electronic resource] https://www.globalfirepower.com/countries-listing.asp

15. Статистика стран мира [Electronic resource] http://svspb.net/danmark/vvp-stran.php