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ON THE SEMILATTICE OF ALL FINITELY GENERATED SUBSEMIGROUPS OF A SEMIGROUP

This article is dedicated to the upper semilattice $FSub(S)$ of all finitely generated subsemigroups of a semigroup S. We prove some necessary and sufficient conditions for this semilattice to form a lattice or to be a complete semilattice and describe some of its atoms and dual atoms. We also studied finitely generated semigroups, all of whose proper finitely generated subsemigroups are finite, and lattices of their finitely generated subsemigroups. A sufficient condition for such a semigroup to be finite was found.

***Keywords:** semigroup, finitely generated semigroup, lattice of subsemigroups, semilattice of finitely generated subsemigroups.*

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Introduction

In the study of semigroups, certain combinatorial structures related to semigroups, in particular, subsemigroup lattices, play a crucial role. There are many works devoted to the connection between the structure of a semigroup and the structure of its subsemigroup lattice. The results obtained by the middle of the 1980s are summarized in the monograph (Shevrin, & Ovsyannikov, 1996); see also (Sit, & Sue, 1975; Jones, 1981; Jones, 2010; Pasku, 2006; Thron, & Koppitz, 2003).

Let S be a semigroup. It is well known that the set $Sub(S)$ of all its subsemigroups forms a complete lattice with operations $T_1 \wedge T_2 = T_1 \cap T_2$ and $T_1 \vee T_2 = \langle T_1 \cup T_2 \rangle$, $T_1, T_2 \in \mathrm{Sub}(S)$.

In this article, we consider a subsemilattice of this lattice–the semilattice of all finitely generated subsemigroups of S. At first we state some auxiliary results.

In what follows, for a semigroup S, the notation S^1 means the semigroup S with an adjoined unit if S does not have one, and the semigroup S itself, otherwise.

Proposition 1. *Suppose a finitely generated semigroup S has an indecomposable element a. Let A be a finite generating system of S and $A_1 = A \setminus a$. Then $S \setminus a$ is finitely generated; moreover,*

$$S \setminus \{a\} = \langle \{a^2, a^3\} \cup A_1 \cup aA_1 \cup A_1 a \cup aA_1 a \rangle.$$

Proof. From indecomposability of a we obtain that the set $S \setminus a$ is closed with respect to multiplication, and therefore it forms a subsemigroup of S.

Each word over the alphabet A, different from a, breaks down into subwords that either do not contain the letter a, or are of the form ba^kc or ada, where $b, c \in \langle A_1 \rangle^1$, $k \geq 1$, $d \in \langle A_1 \rangle$. Thus, we obtain the desired equality.

□

A subset B of a set A is called cofinite if $|A \setminus B| < \infty$. Each infinite cyclic semigroup is isomorphic to the semigroup $(N; +)$. Hence, theorem 1 from (Sit, & Sue, 1975) can easily be reformulated in terms of arbitrary cyclic semigroups.

Proposition 2. *Each subsemigroup of a cyclic semigroup is finitely generated and is a cofinite subset of one of its cyclic subsemigroups.*

Proposition 3. (Shevrin, & Ovsyannikov, 1996; Lemma I.3.1 and Proposition I.3.2)

1. *An element $T \in \mathrm{Sub}(S)$ is an atom if and only if $T = e$, where e is an idempotent.*

2. $\mathrm{Sub}(S)$ *contains a unique atom* (*which is also the zero in* $\mathrm{FSub}(S) \setminus \emptyset$) *if and only if the semigroup S is periodic and contains a unique idempotent.*

1. Main results

Proposition 4. *The set $\mathrm{FSub}(S)$ of all finitely generated subsemigroups of a semigroup S forms an upper subsemilattice of the lattice $\mathrm{Sub}(S)$.*

Proof. Let $T_1, T_2 \in \mathrm{FSub}(S)$. If $T_i = \langle A_i \rangle$, $|A_i| < \infty$, $i = 1,2$, then $T_1 \vee T_2 = \langle T_1 \cup T_2 \rangle = \langle A_1 \cup A_2 \rangle$ is a finitely generated semigroup.

□

From the proof of this proposition we obtain that the mapping $\varphi: \mathfrak{B}_{fin}(S) \to \mathrm{FSub}(S), A \mapsto \langle A \rangle$, where $\mathfrak{B}_{fin}(S)$ is the set of all finite subsets of S, ordered by inclusion, is an endomorphism of semilattices.

Each element of the semilattice $\mathrm{FSub}(S)$ can be written as the join of a finite family of cyclic semigroups:

$$T = \langle a_1, a_2, \dots, a_n \rangle = \langle a_1 \rangle \vee \langle a_2 \rangle \vee \dots \vee \langle a_n \rangle.$$

Proposition 5. *If for elements $T_1, T_2 \in \mathrm{FSub}(S)$ their least upper bound in $\mathrm{FSub}(S)$ exists, then it is equal to $T_1 \cap T_2$.*

Proof. It is obvious that $T_1 \wedge T_2 \subseteq T_1 \cap T_2$. Let $s \in T_1 \cap T_2$. Then $\langle s \rangle \subseteq T_1$ and $\langle s \rangle \subseteq T_2$. From the definition of the greatest lower bound, we obtain that $\langle s \rangle \subseteq T_1 \wedge T_2$, and therefore $s \in T_1 \wedge T_2$. Hence, each element of the set $T_1 \cap T_2$ lies in $T_1 \wedge T_2$, so $T_1 \cap T_2 \subseteq T_1 \wedge T_2$. Thus, $T_1 \wedge T_2 = T_1 \cap T_2$.

□

Corollary 1. *The partially ordered set* $\mathrm{FSub}(S)$ *is a lattice if and only if an intersection of two arbitrary finitely generated subsemigroups of* S *is finitely generated. In this case,* $\mathrm{FSub}(S)$ *is a sublattice of* $\mathrm{Sub}(S)$.

Since in a nilpotent semigroup the lengths of nonzero products are bounded from above, every finitely generated subsemigroup of such a semigroup is finite. From this and proposition 5 we obtain the following.

Corollary 2. *If a semigroup* S *is nilpotent, then* $\mathrm{FSub}(S)$ *is a lattice*.

Proposition 6. *Let* $T_1, T_2 \in \mathrm{FSub}(S)$ *be such that* $T_1 \cap T_2 \in \mathrm{FSub}(S)$. *For any finite generating system* A *of the semigroup* $T_1 \cap T_2$, *there exist finite generating systems* B *and* C *of* T_1 *and* T_2, *respectively, such that* $B \cap C = A$.

Proof. Let A_1 and A_2 be some finite generating systems of the semigroups T_1 and T_2, respectively. Then $A \cup A_1$ and $A \cup A_2$ also generate these semigroups. Since $A \cup A_1$ and $A \cup A_2$ are finite, they contain some minimal generating systems, B and C, that contain A. Suppose that $B \setminus A \nsubseteq T_1 \setminus (T_1 \cap T_2)$. Then there exists an element $c \in T_1 \cap T_2$ such that $c \in B \setminus A$. Since c is generated by A, the set $B \setminus c$ is a generating system of $T_1 \cap T_2$ and contains A, which contradicts the minimality of B. Hence, $B \setminus A \subseteq T_1 \setminus (T_1 \cap T_2)$. Analogously, it can be shown that $C \setminus A \subseteq T_2 \setminus (T_1 \cap T_2)$. Thus, $B \cap C = A$.

□

Corollary 3. *For any* $T_1, T_2 \in \mathrm{FSub}(S)$, *such that* $T_1 \cap T_2 \in \mathrm{FSub}(S)$, *there exist finite generating systems,* A_1 *and* A_2, *such that* $T_1 \cap T_2 = \langle A_1 \cap A_2 \rangle$.

Proposition 7. *Let* $\varphi : S \to S_1$ *be a homomorphism of semigroups. A subsemigroup of* $\varphi(S)$ *is finitely generated if and only if it is a homomorphic image of some finitely generated subsemigroup of* S.

Proof. If $T = \langle a_1, a_2, \dots, a_n \rangle \in \mathrm{FSub}(S)$, then

$$\varphi(T) = \langle \varphi(a_1), \varphi(a_2), \dots, \varphi(a_n) \rangle \in \big(\varphi \mathrm{FSub}(S)\big).$$

Conversely, suppose $U = \langle b_1, b_2, \dots, b_n \rangle \in \big(\varphi \mathrm{FSub}(S)\big)$. Let $a_1, a_2, \dots, a_n \in S$ be some preimages of the generators $b_1, b_2, \dots, b_n$, respectively. Then, for the semigroup $T = \langle a_1, a_2, \dots, a_n \rangle \in \mathrm{FSub}(S)$, we have:

$$U = \langle \varphi(a_1), \varphi(a_2), \dots, \varphi(a_n) \rangle = \varphi(\langle a_1, a_2, \dots, a_n \rangle) = \varphi(T).$$

The claim follows.

□

Proposition 8. *If the lattice* $\mathrm{Sub}(S)$ *is distributive, then* $\mathrm{FSub}(S)$ *is its sublattice. In particular,* $\mathrm{FSub}(S)$ *is also distributive.*

Proof. Suppose $T_1 \in \mathrm{Sub}(S)$, $T_2 \in \mathrm{FSub}(S)$ and $T_2 = \langle a_1, a_2, \dots, a_n \rangle$. Applying distributivity, we obtain:

$$T_1 \cap T_2 = T_1 \cap (\langle a_1 \rangle \vee \dots \vee \langle a_n \rangle) = (T_1 \cap \langle a_1 \rangle) \vee \dots \vee (T_1 \cap \langle a_n \rangle).$$

Proposition 2 and the fact that $\mathrm{FSub}(S)$ is an upper semilattice imply that $T_1 \cap T_2 \in \mathrm{FSub}(S)$. Thus, by corollary 1, $\mathrm{FSub}(S)$ is a sublattice of $\mathrm{Sub}(S)$. But then it is itself distributive.

□

Let L be a lower semilattice. By $a \wedge b$ we denote the greatest lower bound of the elements a and b. Because of the associativity, commutativity and idempotency of the operation $\wedge$, the semilattice L can be viewed as a commutative band with respect to $\wedge$. The same holds for upper semilattices with respect to the operation of taking the least upper bound.

From the proof of proposition 8 we obtain

Corollary 4. *If the lattice* $\mathrm{Sub}(S)$ *is distributive, then the lower semilattice* $(\mathrm{FSub}(S); \cap)$ *is an ideal in* $(\mathrm{Sub}(S); \cap)$ *(in the semigroup-theoretic sense).*

The distributivity criteria of the lattice of all subsemigroups of a semigroup are stated in (Shevrin, 1961). Proposition 3 immediately implies the following.

Proposition 9:

1. *An element* $T \in \mathrm{FSub}(S)$ *is an atom if and only if* $T = e$, *where* e *is an idempotent.*
2. $\mathrm{FSub}(S)$ *contains a unique atom (which is also the zero in* $(S) \setminus \emptyset$*) if and only if the semigroup* S *is periodic and contains a unique idempotent.*

Proposition 1 immediately implies the following.

Proposition 10. *If a finitely generated semigroup* S *contains an indecomposable element* a, *then* $S \setminus a$ *is a dual atom in* $\mathrm{FSub}(S)$.

Note that if a semigroup S is finite, then $\mathrm{FSub}(S)$ always contains dual atoms. For infinite semigroups, $\mathrm{FSub}(S)$ can have dual atoms even if S does not contain indecomposable elements. For example, in $\mathrm{FSub}(Z)$ of the semigroup $(Z; +)$, each of the subsemigroups N_0, $-N_0$ and pZ for prime p is a dual atom.

Proposition 11. *If the upper semilattice* $\mathrm{FSub}(S)$ *is complete, then* S *is finitely generated*.

Proof. Suppose the semilattice $\mathrm{FSub}(S)$ is complete. Then the set $\{\langle s \rangle\}_{s \in S}$ must have the least upper bound in $\mathrm{FSub}(S)$. From the equality

$$\bigcup_{s \in S} \langle s \rangle = S,$$

it follows that the least upper bound of this set has to be equal to S. Thus, $S \in \mathrm{FSub}(S)$, which means that S is finitely generated.

□

Proposition 12. *Let* $T \in \mathrm{Sub}(S) \setminus \mathrm{FSub}(S)$. *If, in the semilattice* $\mathrm{FSub}(S)$, *the least upper bound,* U, *of the set* $\{\langle t \rangle\}_{t \in T}$ *exists, then* T *contains all the indecomposable elements of* U. *In particular, no generating system of* U *can consist purely of indecomposable elements.*

Proof. By the definition of the least upper bound, no proper finitely generated subsemigroup of U can contain T. Let a be an indecomposable element of U. By proposition 1, $U \setminus \{a\} \in \mathrm{FSub}(U) \setminus U$. Hence, $T \nsubseteq U \setminus a$, which means $a \in T$. If U were generated by its indecomposable elements, T would contain a generating system of U, which would imply $T = U$, which is impossible, since $T \notin \mathrm{FSub}(S)$.

□

Proposition 13. *If all proper finitely generated subsemigroups of a finitely generated semigroup S are finite, then $\mathrm{FSub}(S)$ is a complete lattice*.

Proof. Since S is finitely generated, then the upper semilattice $\mathrm{FSub}(S)$ contains a unit. An intersection of an arbitrary family of finitely generated subsemigroups of S either equals S or is finite; hence, it is always finitely generated. Therefore, $\mathrm{FSub}(S)$ is a complete lower semilattice. By lemma 34 from (Grätzer, 2011), $\mathrm{FSub}(S)$ is a complete lattice.

□

Proposition 14. *Suppose S is a finitely generated semigroup, all of whose proper finitely generated subsemigroups are finite. Let A an irreducible generating system of S and $a \in A$. Then:*

1. *The set $S \setminus S^1aS^1$ is finite.*
2. *If T is an infinitely generated subsemigroup of S, then the set $T \cap S^1aS^1$ is nonempty and is a two-sided ideal of T.*
3. *If T is an infinitely generated subsemigroup of S, then, in the lattice $\mathrm{FSub}(S)$, the following equality holds:*

$$\bigvee_{t\in T} \langle t \rangle = S.$$

Proof. 1. Each element of the set $S \setminus S^1aS^1$ is generated by the elements of the set $A \setminus a$. Hence, $S \setminus S^1aS^1 \subseteq \langle A \setminus \{a\} \rangle \in \mathrm{FSub}(S)$. Since $\langle A \setminus \{a\} \rangle$ is a proper finitely generated subsemigroup of S, it is finite, which means $S \setminus S^1aS^1$ is finite as well.

2. Suppose $T \cap S^1aS^1 = \emptyset$. Then statement 1 implies that $|T| < \infty$, which contradicts the infinite generatedness of T. The second part of this statement is obvious.

3. In the lattice $\mathrm{Sub}(S)$, the following equality holds:

$$\bigvee_{t\in T} \langle t \rangle = T.$$

Since $\mathrm{FSub}(S)$ is a sublattice of $\mathrm{Sub}(S)$ and is complete, the least upper bound of the set $\{\langle t \rangle\}_{t\in T}$ exists in it and contains T. Since T is infinite, the only element of $\mathrm{FSub}(S)$ that can contain it is S itself.

□

Let S be the semigroup from Proposition 14. The ideal S^1aS^1 is the set of all elements of S that are generated by a (in combination with other generators). Hence, the following equality holds:

$$S = \bigcup_{a\in A} S^1aS^1.$$

If S is infinite, then at least one of these ideals is infinite.

Proposition 15. *Suppose S is a finitely generated semigroup, all of whose proper finitely generated subsemigroups are finite. Let T be a proper subsemigroup of S. Then:*

1. $\mathrm{Sub}(T)$ *is a complete sublattice of* $\mathrm{Sub}(S)$.
2. $\mathrm{FSub}(T)$ *is a sublattice of* $\mathrm{FSub}(S)$*; moreover,* $\mathrm{FSub}(T) \cup T$ *is a complete lattice.*
3. $(\mathrm{Sub}(T);\cap)$ *is an ideal (in the semigroup-theoretic sense) in* $(\mathrm{Sub}(S)\,;\cap)$*, and* $(\mathrm{FSub}(T);\cap)$ *is an ideal in both* $(\mathrm{FSub}(S);\cap)$ *and* $(\mathrm{Sub}(S);\cap)$.

Proof. 1. This is obvious.

2. This follows from corollary 1 and the fact that all proper finitely generated subsemigroups of T are finite.

3. If F_1 and F_2 are some subsemigroups of S and T, respectively, then $F_1 \cap F_2$ is a subsemigroup of T. Moreover, if F_2 is finite, so is $F_1 \cap F_2$, which means $F_1 \cap F_2 \in \mathrm{FSub}(T)$.

□

Proposition 16. *Suppose S is a finitely generated semigroup, all of whose proper finitely generated subsemigroups are finite. If S contains an indecomposable element, then it is finite*.

Proof. Let a be an indecomposable element of S. By proposition 1, $S \setminus a$ is a proper finitely generated subsemigroup of S. Hence, it is finite. But then S is itself finite.

□

Remark. In (Pasku, 2006), there is a criterion of finiteness of a finitely generated semigroup. However, it is not suitable for our purpose.

Proposition 17. *Let $\langle a \rangle$ and $\langle b \rangle$ be infinite cyclic semigroups and have non-empty intersection. Let k be the smallest integer such that $a^k = b^n$ for some n, $gcd\, gcd\, (k,n) = d$ and $k = k_0 d$. Then*

$$\langle a \rangle \cap \langle b \rangle = \left\langle a^k, a^{k+k_0}, \dots, a^{k+(d-1)k_0} \right\rangle.$$

Moreover, the above generating system is a unique irreducible generating system of this semigroup.

Proof. Consider arbitrary l and m such that $a^l = b^m$. We have:

$$b^{nl} = (b^n)^l = (a^k)^l = (a^l)^k = (b^m)^k = b^{mk}.$$

Since the semigroup $\langle b \rangle$ is infinite, the above equality implies that $km = nl$. For a positive integer l, there exists a positive integer m such that $km = nl$ if and only if $k_0 | l$. Hence, $a^l \in \langle b \rangle$ if and only if $l = k + k_0 t$ for some $t \in N_0$.

Now we will show that $\langle a\rangle \cap \langle b\rangle = \langle a_1, a_2, \ldots, a_d\rangle$, where $a_i = a^{k+(i-1)k_0}, i = 1, \ldots, d$. For any $t = qd + r$, $0 \le r < d$, we have $k + tk_0 = k + qdk_0 + rk_0 = k + rk_0 + qk$, which means $a^{k+tk_0} = a_{r+1}a_1^q$. Hence, $\langle a\rangle \cap \langle b\rangle \subseteq \langle a_1, a_2, \ldots, a_d\rangle$. The converse inclusion is obvious. Since each of the elements $a_1, a_2, \ldots, a_d$ is indecomposable, this is a unique irreducible generating system for this semigroup.

□

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ПРО НАПІВҐРАТКУ ВСІХ СКІНЧЕННО ПОРОДЖЕНИХ ПІДНАПІВГРУП НАПІВГРУПИ

Присвячено дослідженню верхньої напівґратки $FSub(S)$ всіх скінченно породжених піднапівгруп напівгрупи S. Знайдено деякі необхідні та достатні умови для того, щоб ця напівґратка утворювала ґратку або була повною; розглянуто її атоми та коатоми. Проаналізовано також скінченно породжені напівгрупи, у яких усі власні скінченно породжені піднапівгрупи скінченні, та ґратки їхніх скінченно породжених піднапівгруп. Визначено достатню умову скінченності такої напівгрупи.

Ключові слова: ***напівгрупа, скінченно породжена напівгрупа, ґратка піднапівгруп, напівґратка скінченно породжених піднапівгруп.***

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