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MODELING AND HOMOGENIZATION OF PROCESSES IN CYLINDRICAL LAYERED COMPOSITES LIKE CAPACITORS AND TRANSFORMERS

Capacitors and transformers are widely used devices. There is a trend towards minutearization of such devices. Thus, a natural problem of modeling thermal and electromagnetic processes in such devices arises. Understanding the models of such processes allows, for example, to optimize or make cheaper of these devices. But, the geometry of such devices is very specific. For example, a regular capacitor is a long strip with an insulator rolled into a cylinder. Direct modeling of processes in such a device is complicated, since there are very many layers, which leads to equations with highly oscillating coefficients. On the other hand, methods of homogenization theory can be applied to such equations and equations with constant coefficients can be obtained, the solutions of which approximate the solutions of the equations under consideration. The solution of equations with constant coefficients is much simpler both by numerical and analytical methods. As a first step in this direction, one dynamic equation will be considered here, which corresponds to the modeling of wave and thermal processes. Thus, the problem of modeling wave and thermal processes in a cylinder consisting of a large number of alternating layers with different characteristics is considered. Using the asymptotic methods of the homogenization theory, this problem is approximately reduced to modeling processes in a homogeneous (homogenized) cylinder. Asymptotic expansions and homogenized problems are presented, the solutions of which determine the approximate asymptotic of solutions to the problem under consideration. Formulas for calculating the characteristics of the homogeneous cylinder and accuracy estimates of the approximations are given. The presented accuracy estimates of such approximations are essential for numerical calculations and computer modeling of the problems with guaranteed accuracy. In addition, the given approximations are undoubtedly useful for understanding wave and thermal processes in devices similar to capacitors and transformers. This approach can be considered as a basis for considering more general electromagnetic and thermal processes, which are modeled by Maxwell's equations coupled with the heat equation.

Keywords: ***initial boundary value problems, homogenized problems, wave equations, heat equations, asymptotic expansions, heat conductivity coefficients, wave propagation speed coefficients.***

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Introduction

One of the homogenization theory founders, Nikolai S. Bakhvalov, spoke at lectures about the creation of a real rolling mill. This mill rolled out an iron billet into a long thin strip. Then this strip is folded into a roll and moved to cool. The cooling time of the roll was calculated as a metal cylinder. The place of cooling was allocated in accordance with this time. But, the real cooling time turned out to be much longer than the calculated one, which led to overflow of the cooling place. In addition, the increase in cooling time led to the loss of some relevant properties of the roll metal, which had a catastrophic effect on the profitability of the rolling mill. As a result, this real rolling mill had to be closed.

The story, of course, is about the usefulness of calculation formulas for cylindrical layered composite media and materials, since the metal in the roll is not solid metal and can be separated by air layers. For ordinary periodically layered composite media, for example, such formulas for heat conductivity equation are known (Bensoussan, Lions, & Papanicolaou, 1978; Bakhvalov, & Panasenko, 1989). The formulas were obtained by methods of asymptotic expansions for solutions of the wave and heat equation at the beginning of the development of the homogenization theory. But, we are not aware of works where the formulas were obtained for cylindrical layered composites and Bakhvalov's problem was solved. Here the problem will be considered in various layer geometric configurations, in particular for cylindrical layered composites, which are similar to some models for capacitors and transformers.

The usefulness of layered composites for influencing thermal processes is known actually in electrodynamics. Indeed, transformer cores are periodically assembled from thin plates separated by insulating layers. Such a periodic structure of the plates dampens the Foucault currents (Landau, & Lifshitz, 1984), which prevents energy losses for heating the cores. Transformer cores can also be made from long strips with an insulating layer rolled into a toric roll, which simulates a layered structure in cylindrical coordinates and resembles the metal roll. It is known (Bertotti, 1998) that Foucault currents can be modeled by diffusion equations for the magnetic field strengths. Here we consider only one such equation for cylindrical layered composites, which is also used to model thermal processes and is called the heat conduction equation. Note also that diffusion equations model filtration processes. Some filters actually used have a cylindrical layered structure. Therefore, the methods discussed here may be relevant for modeling filtration processes in porous cylindrical layered media with various layer configurations.

Respectively, the transformer cores are the basis for the winding, consisting of turns of conductors, which wrap around part of the magnetic system of the transformer. As a simple model of such a system, one can consider a cylindrical layered composite, the layers of which are formed by a large number of rings or a helix with many turns separated by some material. Often in practice, these can be turns of insulated wires surrounded by air. Such composites will be discussed as well. Layers of such composites are called solenoids in electrodynamics (Vagner, Lembrikov, & Wyder, 2004).

At last, some real capacitors are made from a very thin strip of foil, lined with a thin insulator, which is rolled into a cylinder starting from some core. It is for the modeling of thermal processes in such composites resembling the metal roll that this article was conceived. The problem of describing thermal fields in capacitors was posed by Vladimir V. Mokhor as significant for understanding the distribution of heat, since sometimes capacitors overheat, which leads to irreversible consequences. The use of homogenization methods in solving this problem has led to some applications of these methods to the modeling of layered composites like transformers. Naturally, to fully simulate such processes in composites, it is necessary to consider the complete system of heat conductivity and Maxwell equations (Landau, & Lifshitz, 1984). The approach used here can be considered as a basis for modeling more general processes.

To model processes in cylindrical layered composites, we will use the homogenization theory. The theory arose about fifty years ago with the advent of artificial composite materials. Such materials usually have a periodic structure, which is technologically convenient. Indeed, by fixing the cell of periodicity, it is possible to copy this cell to fill a certain volume, just as city squares are covered with tiles. There are usually a lot of such cells in artificial composites. Therefore, when modeling processes in the composites, it is necessary to choose a very fine mesh to take into account the structure of each cell, which requires very significant computer resources.

Initially, the homogenization theory was developed by N. S. Bakhvalov, I. Babushka, A. Bensoussan, J. L. Lions, and G. Papanicolaou to simplify computer simulations processes in composite materials with a periodic structure; relevant references can be found, for example, in (Bensoussan, Lions, & Papanicolaou, 1978; Bakhvalov, & Panasenko, 1989). According to the theory, for example, modeling of heat processes in composites can be replaced with guaranteed accuracy by modeling the heat process in homogeneous (homogenized) media. In a mathematical sense, this means that the solution of the heat conduction problem for the equation with periodic rapidly oscillating coefficients is close to the solution of the problem for the equation with constant coefficients. The corresponding theorems and accuracy estimates were proved, for example, in (Bensoussan, Lions, & Papanicolaou, 1978; Bakhvalov, & Panasenko, 1989) for sufficiently regular data.

Later, it turned out that it is much easier to prove the convergence or multiscale convergence of solutions to solutions of some homogenized multiscale problems, starting essentially from (Allaire, 1992). This approach, usually, does not require additional data regularity that is essential for nonlinear problems as, for example, in (Piat, & Sandrakov, 2004; Sandrakov, 2004a; 2005) for variational inequalities whose solutions are not regular as a rule. However, the relevant convergence theorems are based entirely on a priori solution estimates, and for problems with additional parameters such estimates may not be exact, which can lead to incorrect homogenized multiscale problems. Thus, inferring homogenized problems from solution estimates requires high mathematical skills

In contrast to the method of asymptotic expansions, since the construction of asymptotics leads to a small residual by definition and, therefore, a priori estimates provide guaranteed accuracy of the asymptotic approximation, which implies convergence of solutions. This approach allows to separate fast and slow variables, similar methods are used, for example, in (Diaz et al., 2019a; 2019b). But, for homogenized multiscale problems such variables are not separated and the type of the corresponding problems is not obvious. In addition, the accuracy of the approximations was not clarified and proved in the multiscale case, further details and relevant references can be found, for example, in (Mielke, Reichelt, & Thomas, 2014; Jager, & Woukeng, 2021; Garttnerat, et al., 2022; Ray, & Schulz, 2023; Garttnerat, Knabner, & Ray, 2024). An approach leading also to homogenized multiscale problems based on the periodic unfolding method is presented, for example, in (Cioranescu, Damlamian, & Griso, 2018; 2023). It is obvious that such multiscale convergence methods are convenient justification methods for results of the homogenization theory.

The homogenization method was originally based on asymptotic solution expansions for second order divergent equations according to (Bensoussan, Lions, & Papanicolaou, 1978; Bakhvalov, & Panasenko, 1989). Here we will use a method of asymptotic solution expansions for elliptic equations of arbitrary order, the use of which does not require high skills as detailed in (Sandrakov, 1991). This approach makes it possible to consider various problems for modeling of processes in cylindrical layered composites like capacitors and transformers. This will allow us to consider in sufficient detail the actual problems for such modeling under various geometric assumptions. The exact formulation of such problems is presented in the next section. Various variants of geometric structures for cylindrical layered composites are discussed in sections 3–5. Composites consisting of periodic layers, rolled strips, and layers of periodic rings will be considered sequentially in the sections. Finally, section 6 deals with composites with helix layers that model some transformers.

1. Models for processes in cylindrical layered composites

Models for wave and heat processes in composite materials will be considered. Precise definitions will be needed for the exact formulation of the models. Let

$$\Omega = B \times (0, l\,) = \{(x, y) \in \mathbb{R}^2 :\ \kappa < x^2 + y^2 < R^2\} \times (0, l\,)$$

be a cylinder of radius R and length l with a removed core of radius κ. Here B denotes an open circle of radius R without origin with some neighborhood of the positive radius κ. The origin is removed here to correctly use cylindrical coordinates, and some neighborhood is removed to correctly define the boundary conditions, otherwise such conditions would be quite specific.

The coordinates $(r, \varphi, z) \in (0, R) \times [0,2\pi) \times (0, l)$ will be applied on the domain Ω. We will also use functional spaces, which are defined, for example, in (Bensoussan, Lions, & Papanicolaou, 1978; Bakhvalov, & Panasenko, 1989; Duvaut, & Lions, 1972). Such spaces are necessary for the correct setting of the initial boundary value problems considered here, as is customary in mathematical and numerical physics.

Let the functions $u_0, u_1 \in C_0^\infty(\Omega)$ and $f \in C_0^\infty((0, T) \times \Omega)$ be given, where the positive T is fixed. We define the function $u = u(t, x, y, z) \in C^0([0, T]; H_0^1(\Omega))$ as a solution to the initial boundary value problem for the wave equation

$$\begin{aligned} &m^\varepsilon u''_{tt} + s^\varepsilon u'_t - div\,(A^\varepsilon \nabla u) = g^\varepsilon f \quad in \quad \Omega \times (0, T), \\ &u|_{t=0} = u_0,\ \ u'_t|_{t=0} = u_1 \ \ in\ \ \Omega, \ \ u = 0 \ \ on \ \ \partial\,\Omega \times (0, T), \end{aligned} \tag{1}$$

where the equation coefficients of the problem characterize the component properties of the composite materials and ε^{-1} is a large parameter specified below as the number of layers. The equation of this problem is understood in the distribution sense (Duvaut, & Lions, 1972).

In the case when $m^\varepsilon = 0$ and the initial condition $u'_t|_{t=0} = u_1$ is removed, the coefficient s^ε characterizes the density (normalized to the specific heat capacity) of the components and similarly for $g^\varepsilon = s^\varepsilon$, the matrix A^ε determines the heat conductivity of the components in the directions of coordinate axes, and this problem models the heat process in the domain Ω as it is generally accepted. However, the case of non-zero m^ε may be used to model heat processes at extreme temperatures, as suggested, for example, in (Saedodin, & Barforoush, 2017) with reference to the work of J. C. Maxwell 1867. This case is also called the telegraph equation or the relativistic heat equation.

In general, the coefficient m^ε characterizes the density of the components and similarly for $g^\varepsilon = m^\varepsilon$, the matrix A^ε determines the permeability of the components in the directions of coordinate axes, and the problem models the wave process with damping in the domain Ω, where s^ε is the characteristic of such damping. Sometimes, Maxwell's equations can be reduced to wave equations with damping. In this case, the matrix A^ε is usually diagonal and can characterize the electrical or magnetic permeability of the components, and the coefficient s^ε determines the specific conductivity of the components when Ohm's law is fulfilled. But, for models of electromagnetic processes, it is necessary to consider a system of equations for electric and magnetic fields. Here we are only studying one equation as a model for wave or heat processes, which can be considered as basic for the general case.

For thermal processes, for example, the condition $u = 0$ on the boundary is called the homogeneous Dirichlet condition and means that the temperature on the boundary of the cylinder is zero, which is convenient in the mathematical sense. If for some process the ambient temperature is equal to $\eta \neq 0$, then the boundary condition should be replaced by the condition $u = \eta$. But, one can replace the unknown function $u = u(t, x, y, z)$ in problem (1) with a new unknown function $v = u - \eta$ for which the homogeneous boundary condition is fulfilled. It is possible to consider other types of boundary conditions in (1), for example, the Neumann or mixed boundary conditions. In thermal physics and electrodynamics, it is also customary to reduce the equations of the problem to a dimensionless form; for simplicity, this is not done here.

Let us now define the coefficients in (1), which are specific to layered composites. Using cylindrical coordinates, we rewrite the equation of the problem in the form

$$m^\varepsilon u''_{tt} + s^\varepsilon u'_t - \partial_r(a_1^\varepsilon\, \partial_r u) - r^{-1} a_1^\varepsilon\, \partial_r u - r^{-2}\, \partial_\varphi(a_2^\varepsilon\, \partial_\varphi u) - \partial_z(a_3^\varepsilon\, \partial_z u) = g^\varepsilon f \tag{2}$$

in $\Omega \times (0, T)$, where ∂_r, ∂_φ, ∂_z denote partial derivatives with respect to cylindrical coordinates. It is assumed that the matrix $A^\varepsilon = A^\varepsilon(r)$ is diagonal and depends only on the radial variable r. The last assumption will also be made for these other coefficients $m^\varepsilon = m^\varepsilon(r)$, $s^\varepsilon = s^\varepsilon(r)$ and $g^\varepsilon = g^\varepsilon(r)$.

Fixing a sufficiently large integer N, we will consider $\varepsilon = R/N$ as a small parameter and represent the cylinder as the union of tubes

$$\Omega = \cup_{k=1}^N B_k \times (0, l) = \cup_{k=1}^N \{ (x, y) \in \mathbb{R}^2 : \ \kappa + \varepsilon\,(k-1) \le \sqrt{x^2 + y^2} < \kappa + \varepsilon\, k \} \times (0, l),$$

where B_k is a ring of thickness ε and suitable radii for $k = 1, \dots, N$.

Let us define the coefficients of equation (2) by the equalities

$$m^\varepsilon = m(r/\varepsilon), \quad s^\varepsilon = s(r/\varepsilon), \quad a_i^\varepsilon = a_i(r/\varepsilon), \quad g^\varepsilon = g(r/\varepsilon), \tag{3}$$

where the functions $m = m(\tau)$, $s = s(\tau)$, $a_i = a_i(\tau)$ and $g = g(\tau)$ for $i = 1,2,3$ are 1-periodic (periodic with period 1 relative to the variable $\tau \in \mathbb{R}$) and are defined on the period by the following equalities

$$\begin{aligned} m(\tau) = m_0 \ \ \mathit{if} \ \ \tau \in [0, \sigma] \cup [1-\sigma, 1] \ \ \mathit{and} \ \ m(\tau) = m_1 \ \ \mathit{if} \ \ \tau \in (\sigma, 1-\sigma), \\ a_i(\tau) = a_0^i \ \ \mathit{if} \ \ \tau \in [0, \sigma] \cup [1-\sigma, 1] \ \ \mathit{and} \ \ a_i(\tau) = a_1^i \ \ \mathit{if} \ \ \tau \in (\sigma, 1-\sigma) \end{aligned} \tag{4}$$

for $i = 1,2,3$. Similarly, $s(\tau)$ is equal to s_0 or s_1 and $g(\tau)$ is equal to g_0 or g_1, where all constants m_0, m_1, a_0^i, a_1^i, g_0, g_1 and σ are positive and fixed. In addition $\sigma \in (0, 0.5)$. Further, all 1-periodic functions defined on the period are continued through the periodicity. Schematically, the coefficient model is presented in Fig. 1.

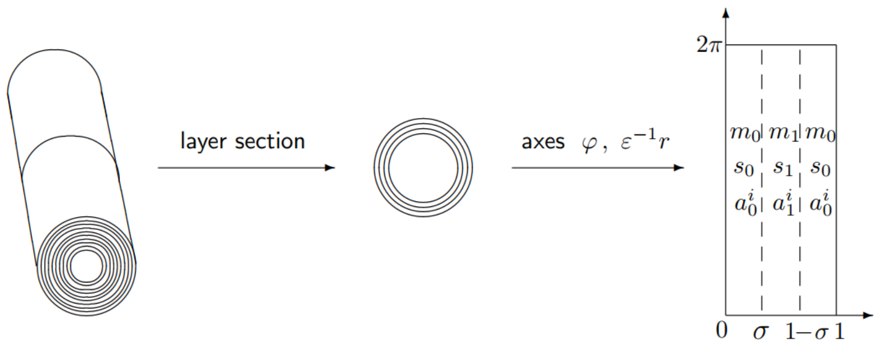

Fig. 1. Model of coefficients of cylindrical layered composites

2. Asymptotic expansions and homogenization

Under the assumptions made, problem (1) has a unique solution, for example, in accordance with (Duvaut, & Lions, 1972). Thus, one can try to construct an approximation of this solution for sufficiently small ε. Following (Sandrakov, 1991), one may choose an asymptotic approximation for the solution of problem (1) in the form

$$u_a = v(t, r, \varphi, z) \ + \ \varepsilon\, v_1(t, r, \varphi, z, r/\varepsilon) \ + \ \varepsilon^2 v_2(t, r, \varphi, z, r/\varepsilon), \tag{5}$$

where by definition the functions $v_1(t, r, \varphi, z, \tau)$ and $v_2(t, r, \varphi, z, \tau)$ are 1-periodic with respect to the variable $\tau \in \mathbb{R}$.

To determine the terms of asymptotic expansion (5), the following differentiation formula for functions of a special form $w = w(t, r, \varphi, z, r/\varepsilon)$ will be used

$$\partial_r w(t, r, \varphi, z, r/\varepsilon) = (\varepsilon^{-1} \partial_\tau w(t, r, \varphi, z, \tau) + \partial_r w(t, r, \varphi, z, \tau))|_{\tau=r/\varepsilon}. \quad (6)$$

The main idea of constructing asymptotic expansions is to observe that the variables t, r, φ, z, τ are independent inside the brackets of the formula. In the sense that if $\partial_\tau w(t, r, \varphi, z, \tau) = 0$, for example, in (6) for all variables, then this is also true for $\tau = r/\varepsilon$. Note that the first term in expansion (5) does not depend on r/ε, which is determined by the kernel of the relevant operator on 1-periodic functions according to (Sandrakov, 1991). For divergent operators of the second order, the kernel is generated by 1, which is well known in homogenization theory (Bensoussan, Lions, & Papanicolaou, 1978).

Let's substitute asymptotic approximation (5) into equation (2) and take formula (6) into account. Then, retaining only the essential terms, we obtain

$$\begin{gathered} m\, v''_{tt} + s\, v'_t - \varepsilon^{-1} \partial_\tau (a_1\, \partial_\tau v_1) - \varepsilon^{-1} \partial_\tau (a_1\, \partial_r v) - \\ - \partial_\tau (a_1\, \partial_\tau v_2) - \partial_r (a_1\, \partial_r v) - \partial_\tau (a_1\, \partial_r v_1) - \partial_r (a_1\, \partial_\tau v_1) - \\ - r^{-1} a_1 \partial_\tau v_1 - r^{-1} a_1\, \partial_r v - r^{-2} a_2\, \partial_\varphi^2 v - a_3\, \partial_z^2 v - \varepsilon\, r^{-1} a_1\, \partial_r v_1 - \\ - \varepsilon\, \partial_\tau (a_1\, \partial_r v_2) - \varepsilon\, \partial_r (a_1\, \partial_r v_1) - \varepsilon\, \partial_r (a_1\, \partial_\tau v_2) = g\, f \end{gathered} \quad (7)$$

for $\tau = r/\varepsilon$, where $m = m(\tau)$, $s = s(\tau)$, $a_i = a_i(\tau)$ and $g = g(\tau)$ for $i = 1,2,3$ in accordance with definition (3).

Equating the coefficients at ε^{-1} in asymptotic equality (7), we have

$$- \partial_\tau (a_1\, \partial_\tau v_1) = \partial_\tau (a_1\, \partial_r v). \quad (8)$$

This equality will be fulfilled if we choose $v_1 = \Phi(\tau)\, \partial_r v\, (t, r, \varphi, z)$ and define the function $\Phi = \Phi(\tau)$ as a solution of the following 1-periodic problem

$$- \partial_\tau (a_1\, \partial_\tau \Phi) = \partial_\tau (a_1) \quad in \quad (0,1). \quad (9)$$

It is known (Bensoussan, Lions, & Papanicolaou, 1978) that such a solution to problem (9) exists and is defined up to a constant, which will be fixed by the condition $\langle \Phi \rangle = \int_0^1 \Phi(\tau)\, d\tau = 0$. Indeed, we have $a_1 \partial_\tau (\Phi + \tau) = C$ and $\partial_\tau \Phi = a_1^{-1} C - 1$. Therefore, we obtain $C = \langle a_1^{-1} \rangle^{-1}$ and

$$\Phi = \int_0^\tau (a_1^{-1}(\eta)\, \langle a_1^{-1} \rangle^{-1} - 1) d\eta + \langle \Phi \rangle.$$

Equating the coefficients at ε^0 in asymptotic equality (7) and using the definition already made that $v_1 = \Phi\, \partial_r v$, we get

$$\begin{gathered} - \partial_\tau (a_1\, \partial_\tau v_2) = \partial_\tau (a_1 \Phi)\, \partial_r^2 v + (a_1 + a_1 \partial_\tau \Phi)\, \partial_r^2 v + \\ + r^{-1} (a_1 + a_1 \partial_\tau \Phi)\, \partial_r v + r^{-2} a_2\, \partial_\varphi^2 v + a_3\, \partial_z^2 v + g\, f - m\, v''_{tt} - s\, v'_t. \end{gathered} \quad (10)$$

By analogy with (8), consider the equality as a 1-periodic problem for $v_2 = v_2(\tau)$ with fixed variables t, r, φ, z. It is known (Bensoussan et al., 1978) that the problem is correct if and only if the right-hand side is orthogonal to 1 in the space $L^2(0,1)$ and therefore we obtain

$$\hat{m}\, v''_{tt} + \hat{s}\, v'_t - \hat{a}_1\, \partial_r^2 v - r^{-1} \hat{a}_1\, \partial_r v - r^{-2} \hat{a}_2\, \partial_\varphi^2 v - \hat{a}_3\, \partial_z^2 v = \hat{g}\, f \quad in \quad \Omega \times (0, T), \quad (11)$$

where, using that $a_1 + a_1 \partial_\tau \Phi = \langle a_1^{-1} \rangle^{-1}$, the constants are defined by the equalities

$$\begin{gathered} \hat{m} = \langle m \rangle = \int_0^1 m(\tau)\, d\tau, \quad \hat{s} = \langle s \rangle, \quad \hat{g} = \langle g \rangle, \\ \hat{a}_1 = \langle a_1^{-1} \rangle^{-1}, \qquad \hat{a}_2 = \langle a_2 \rangle, \qquad \hat{a}_3 = \langle a_3 \rangle. \end{gathered} \quad (12)$$

Thus, starting from equation (2) with complex rapidly oscillating coefficients, we have obtained an equation with constant coefficients, which is called *homogenized*. Adding initial and boundary conditions to this equation, as in problem (1), we get a homogenized problem of the same type, but with constant coefficients.

It is known (Duvaut, & Lions, 1972) that the initial boundary value problem for the homogenized equation has a unique regular solution. Similarly, the unique regular solvability holds for the heat case (when formally $m^\varepsilon = 0$ and the condition $u'_t|_{t=0} = u_1$ is removed). By the method of separation of variables, the solution can be constructed as a convergent series, which cannot be done for problem (1). The series can be used for a computer representation. Moreover, each term of the series is a particular solution of the homogenized problem with suitable initial conditions. Also, the homogenized problem can be solved numerically by standard methods.

Consequently, two terms in the asymptotic approximation (5) are defined. In order to determine the third term $v_2 = v_2(t, r, \varphi, z, z/\varepsilon)$, we subtract equation (11) from equality (10). Then, we obtain

$$- \partial_\tau (a_1\, \partial_\tau v_2) = \tilde{a}_1 \partial_r^2 v + r^{-2}\, \tilde{a}_2 \partial_\varphi^2 v + \tilde{a}_3 \partial_z^2 v + \tilde{g}\, f - \tilde{m}\, v''_{tt} - \tilde{s}\, v'_t, \quad (13)$$

where the determined coefficients $\tilde{a}_1 = \partial_\tau (a_1 \Phi)$, $\tilde{a}_2 = a_2 - \hat{a}_2$, $\tilde{a}_3 = a_3 - \hat{a}_3$, $\tilde{g} = g - \hat{g}$, $\tilde{m} = m - \hat{m}$ and $\tilde{s} = s - \hat{s}$ are 1-periodic functions and such that they are orthogonal to 1 in the space $L^2(0,1)$. Note, for example, that the equalities $\langle \partial_\tau (a_1 \Phi) \rangle = -\langle (a_1 \Phi)\, \partial_\tau 1 \rangle = 0$ are valid, which means that $\partial_\tau (a_1 \Phi)$ and 1 are orthogonal. Here we have used the integration by parts formula and the 1-periodicity of the function $a_1 \Phi$.

Hence, equality (13) and therefore equality (10) are fulfilled if we choose

$$v_2 = \Phi_1 \partial_r^2 v + r^{-2}\, \Phi_2\, \partial_\varphi^2 v + \Phi_3\, \partial_z^2 v + \Phi_g f - \Phi_m\, v''_{tt} - \Phi_s\, v'_t \quad (14)$$

and define the functions Φ_m and Φ_s as 1-periodic solutions of the problems

$$\partial_\tau (a_1\, \partial_\tau \Phi_m) = \tilde{m}, \quad \partial_\tau (a_1\, \partial_\tau \Phi_s) = \tilde{s} \quad in \quad (0,1), \quad (15)$$

for example. It is known (Bensoussan, Lions, & Papanicolaou, 1978) that the solutions to problems (15) exist and are determined up to constants, which will be fixed by the conditions $\langle\Phi_m\rangle = \langle\Phi_g\rangle = 0$. So, one can define the 1-periodic functions Φ_1, Φ_2, Φ_3 and Φ_g in (14). With this choice of the function $v_2 = v_2(t, r, \varphi, z, z/\varepsilon)$, all coefficients at ε^0 in asymptotic equality (10) are equal to zero. Likewise, one can construct additional terms $v_3 = v_3(t, r, \varphi, z, z/\varepsilon)$, $v_4 = v_4(t, r, \varphi, z, z/\varepsilon)$ in expansion (5) according to (Sandrakov, 1991). But, the available terms will be enough for approximating the solution of problem (1).

Thus, the asymptotic equality (7) is fulfilled with accuracy up to $O(\varepsilon)$, since the solution to the homogenized problem is regular due to the assumptions on the initial data and so the terms in (5) are regular. So, the asymptotic approximation $u_a = u_a(t, r, \varphi, z, z/\varepsilon)$ also approaches the equation (2) with accuracy up to $O(\varepsilon)$. Hence, following (Sandrakov, 1998), this can be verified by energetic methods that the last statement implies, for example, the following accuracy estimate

$$\begin{aligned} &\left\| r^{1/2}\left(\partial_r u - \theta^\varepsilon \partial_r v + r^{-1}\partial_\varphi u - r^{-1}\partial_\varphi v + \partial_z u - \partial_z v \right)\right\|^2_{C^0([0,T];L^2(\Omega))} \\ &+ \| u'_t - v'_t \|^2_{C^0([0,T];L^2(\Omega))} + \| u - v \|^2_{C^0([0,T];L^2(\Omega))} \le C\,\varepsilon, \end{aligned} \tag{16}$$

with $\theta^\varepsilon = \theta(r/\varepsilon)$, where $\theta(\tau) = 1 + \partial_\tau\Phi(\tau)$, $\Phi = \Phi(\tau)$ is the solution of problem (9), u is the solution of problem (1) with coefficients from (3), v is the solution of homogenized problem with coefficients from (12) and the constant C does not depend on ε when $0 < \varepsilon \le \varepsilon_0$ for the corresponding ε_0.

The specified accuracy estimate may be proved as follows. It is known (Duvaut, & Lions, 1972) and can be directly verified that the solution of problem (1) satisfies the following energy estimate

$$\begin{aligned} &\| u'_t \|^2_{C^0([0,T];L^2(\Omega))} + \| u \|^2_{C^0([0,T];L^2(\Omega))} + \| \nabla u \|^2_{C^0([0,T];L^2(\Omega))} \le \\ &\le C\left(\| f \|^2_{L^1(0,T;L^2(\Omega))} + \| u_0 \|^2_{H^1_0(\Omega)} + \| u_1 \|^2_{L^2(\Omega)} \right), \end{aligned}$$

where C does not depend on ε. This energy estimate is also valid for a solution of the homogenized problem for equation (11). Indeed, this homogenized problem can be written in the form (1), but for an equation with constant coefficients, for which the energy estimate is well known. Further, from the equation of this problem, we can express $v''_{tt}|_{t=0}$ as an element of the space $C_0^\infty(\Omega)$. Hence, v'_t satisfies the similar problem, but with the right side $f'_t \in C_0^\infty((0,T)\times\Omega)$. Thus, the energy estimate is carried out for v'_t with a constant C independent of ε. Continuing by induction, we conclude that for the n-th derivative the inclusion $v_t^{(n)} \in C^0([0,T]; H^1_0(\Omega))$ holds for the required n.

Now, using elliptic regularity (according to (Kinderlehrer, & Stampacchia, 1980), for example) and writing the homogenized equation in the following form

$$div\,(\hat{a}\nabla v) = \hat{m}\, v''_{tt} + \hat{s}\, v'_t - \hat{g}\, f$$

we get that $v_t^{(n-2)} \in C^0([0,T]; H^3(\Omega) \cap H^1_0(\Omega))$, where $\hat{a}$ is a diagonal matrix with constant elements from (12). Accordingly, we also have that $v_t^{(n-4)} \in C^0([0,T]; H^5(\Omega) \cap H^1_0(\Omega))$. Continuing by induction and using embedding theorems, we conclude that $v \in C^n([0,T]\times\bar{\Omega})$ with the required n and norm estimates that do not depend on ε. (Similar inclusions are fulfilled for thermal processes when $m^e = 0$.) Such inclusions imply the smoothness of the solution of the homogenized problem in cylindrical coordinates, which completely describes the structure of the terms of the asymptotic expansion (5). Namely, each such term is a smooth function multiplied by a bounded one, which follows from the properties of the solutions of problem (9) and similar problems, for example, (15).

According to the construction of u_a, the right side of the equation for $w = u - u_a$ is equal to εF, where the function F is the sum of terms, each of which is a smooth (and so bounded) function multiplied by a bounded one, which follows from definitions and the asymptotic equality (7), based on formula (6). This yields the estimate (16), since the gradient in cylindrical coordinates is $(\partial_r w, r^{-1}\partial_\varphi w, \partial_z w)$ and the relevant Jacobian is r. It is also used here that $\partial_r u_a = \theta^\varepsilon \partial_r v + O(\varepsilon)$ according to (5) and (6), since $v_1 = \Phi(r/\varepsilon)\,\partial_r v$ by definition. More precisely and correctly, the terms v_1 and v_2 in the asymptotic expansion (5) should be multiplied by a smooth cutoff function (equal to zero at the boundary and to one for $(x,y,z) \in \Omega$ distant from the boundary by more than ε). But, such a multiplication does not change the estimate (16), as is well known (Bensoussan, Lions, & Papanicolaou, 1978; Bakhvalov, & Panasenko, 1989).

Note that the accuracy estimate (16), proved for smooth and compactly supported data, easily implies convergence for irregular energy data by density. Indeed, suppose that $f \in L^1(0,T; L^2(\Omega))$ in (1), and the other conditions remain unchanged. It is known (Duvaut, & Lions, 1972) that there exists a sequence of $\{f_i\}_{i=1}^\infty \subset C_0^\infty((0,T)\times\Omega)$ such that

$$\| f - f_i \|_{L^1(0,T;L^2(\Omega))} \le \delta_i \to 0 \quad for \quad i \to \infty.$$

For the solution of problem (1) with the right-hand side f_i, estimate (16) holds with a constant C_i that does not depend on ε, but may depend on i.

Hence, using the energy estimate for these problems (with f, f_i and $f - f_i$ as right-hand sides of the equation) and the triangle inequality, we obtain the following estimate

$$\begin{aligned} &\left\| r^{1/2}\left(\partial_r u - \theta^\varepsilon \partial_r v + r^{-1}\partial_\varphi u - r^{-1}\partial_\varphi v + \partial_z u - \partial_z v \right)\right\|^2_{C^0([0,T];L^2(\Omega))} \\ &+ \| u'_t - v'_t \|^2_{C^0([0,T];L^2(\Omega))} + \| u - v \|^2_{C^0([0,T];L^2(\Omega))} \le C_i\,\varepsilon + \delta_i. \end{aligned}$$

(We use here, for example, that $\| u - v \| \le \| u - u_i \| + \| v - v_i \| + \| u_i - v_i \|$ in natural notations, for example, v_i is the solution to problem (11) with the right-hand side f_i.)

For every positive δ one can take and fix i such that $\delta_i < \delta/2$. Now, there is ε_0 such that $C_i\,\varepsilon < \delta/2$ for $0 < \varepsilon < \varepsilon_0$. By definition, this means that the right hand side in the last highlighted inequality tends to zero as $\varepsilon \to 0$. This completes the proof of the convergence of solutions of problem (1) to the solution of the homogenized problem as $\varepsilon \to 0$ in the case of irregular energy data.

Thus, we have a fairly good accuracy estimate for solutions and gradients for sufficiently small ε. In this case, for modeling wave processes (respectively, heat processes when $m^\varepsilon = 0$) in cylindrical layered composites, instead of solving problem (1) (for example, numerically through computer simulations), it is possible to solve the homogenized problem with guaranteed accuracy. Naturally, the numerical solution of problem (1) is much more difficult than the solution of the homogenized problem, since a very fine calculation grid is required to take into account the geometry of a very large number of layers in Ω, since the coefficients in (3) are rapidly oscillating.

Remark that the estimate (16) is fulfilled when all constants m_0, m_1, a_0^i, a_1^i, g_0 and g_1 in (4) are positive and fixed. If one of these constants is small compared to ε, for example, the case $a_0^1 = C\,\varepsilon^2$ is interesting. In this case, composites with highly variable properties should be considered. The homogenization problem in Cartesian coordinates for this case was considered, for example, in (Sandrakov, 1998; 2004b). Interestingly, the composites discussed in (Sandrakov, 2004b) have the properties of semiconductors for waves, which transmit or not waves depending on their frequencies. This is a consequence of the fact that the homogenization problem in Cartesian coordinates is hyperbolic (parabolic) problem for integro-differential equations. Optimization issues for such equations were considered, for example, in (Anikushyn, Lyashko, & Samosonok, 2024; Anikushyn, & Hranishak, 2025) and for the parabolic case in (Kapustyan et al., 2025). Using the approaches of these works, for example, it is possible to optimize the properties of the homogenized equation coefficients.

All coefficients of the homogenized equation in (12) are calculated as averages, except for $\hat{a}_1$. The coefficient is calculated using the so-called harmonic mean. As an example, for air the heat conductivity is $a_0^1 = a_0^2 = a_0^3 = 0.02$, and for iron we have $a_1^1 = a_1^2 = a_1^3 = 100$ according to reference books in suitable dimensional values.

Hence, $\langle\, a_1^{-1}\rangle = 50 \cdot 2\,\sigma + 0.01 \cdot (\,1 - 2\,\sigma)$ and, for example, when $\sigma = 0.25$, we obtain

$$\hat{a}_1 = \langle\, a_1^{-1}\rangle^{-1} = (25.005)^{-1} \approx 0.04, \quad \hat{a}_2 = \hat{a}_3 \approx 50.01.$$

In addition, we get

$$\hat{a}_1 \approx 0.42, \quad \hat{a}_2 = \hat{a}_3 \approx 99.95 \quad for \quad \sigma = 0.025,$$
$$\hat{a}_1 \approx 0.02, \quad \hat{a}_2 = \hat{a}_3 \approx 5.01 \quad for \quad \sigma = 0.475.$$

For comparison, for glass $a_1^1 = a_1^2 = a_1^3 = 1$. Let $a_0^1 = a_0^2 = a_0^3 = 0.02$ also. Then, we have

$$\hat{a}_1 \approx 0.04, \quad \hat{a}_2 = \hat{a}_3 \approx 0.51 \quad for \quad \sigma = 0.25,$$
$$\hat{a}_1 \approx 0.29, \quad \hat{a}_2 = \hat{a}_3 \approx 0.95 \quad for \quad \sigma = 0.025.$$
$$\hat{a}_1 \approx 0.02, \quad \hat{a}_2 = \hat{a}_3 \approx 0.07 \quad for \quad \sigma = 0.475.$$

This explains why air layer materials are good heat insulators in a direction perpendicular to the layers. In reality, this fact for multiple layers has long been used in thermoses. There was a report by a specialist in composite mechanics, at which homogenization methods for conventional composites with periodic structure were stated to be in good agreement with experimental data even at $\varepsilon = 0.1$. But, we cannot give an exact reference. As a naive example of thermoses, it can be seen that thermal insulation is well achieved even for two or three layers of thermal insulation. Naturally, this will be even better for composites with many layers considered here. In conventional layered geometries, this fact is actually used in multiple glazing for good heat insulation. Of course, all these conclusions are applicable to wave processes, since $\hat{a}_1$ determines the speed of wave propagation in the direction perpendicular to the layers. It is known, for example, that multiple glazing is a good sound insulator.

Comparing these two cases with the predominance of air (for $\sigma = 0.475$), it can be seen that the heat insulation properties of composites with iron and glass are not very different, at least for the direction perpendicular to the layers with a large number of such alternating layers. But, this is not entirely true for other directions. Nevertheless, glass flask thermoses have long been used for a long time, and iron flask thermoses are more recently available. Accordingly, in the case of wave processes, a system of repeating iron sheets can be used for sound insulation.

3. Models of processes in cylindrical rolled layered composites

We have considered composites which are the union of thin cylindrical tubes. Some capacitors are made from a very thin strip of foil, lined with a thin insulator, which is rolled into a cylinder. This case can be considered by the same homogenization method. Indeed, consider a new coordinate

$$\rho \;=\; r \;+\; \varepsilon\,(\,\varphi/2\pi).$$

It would be possible to use this coordinate as new instead of r in equation (2). But, we will use the coordinate only in coefficients. Note that fixing $r = r_0 > \kappa$, we have $\rho = r_0 + \varepsilon\,k$ for $\varphi = 2\pi k$, where $k = 1, \dots, N$. Thus, the curve $\rho = r_0 + \varepsilon\,(\varphi/2\pi)$ describes a spiral with step ε for positive φ, which is called *Archimedean*, and models the cylindrical rolled layered composite if we add the coordinate $z \in (0, l)$ and constrain the value of the variable φ.

Now we define the coefficients of equation (2) by the equalities

$$m^\varepsilon = m(\rho/\varepsilon), \quad s^\varepsilon = s(\rho/\varepsilon), \quad a_i^\varepsilon = a_i(\rho/\varepsilon), \quad g^\varepsilon = g(\rho/\varepsilon), \tag{17}$$

where the functions $m = m(\tau)$, $s = s(\tau)$, $a_i = a_i(\tau)$ and $g = g(\tau)$ for $i = 1,2,3$ are 1-periodic and are defined on the period by equalities (4). We remark, for example, that if $m^\varepsilon = m(\rho/\varepsilon) = m_0$ for $r = r_0$ and $\varphi = 0$, then $m^\varepsilon = m(\rho/\varepsilon) = m(1 + r_0/\varepsilon) = m_0$ for $r = r_0$ and $\varphi = 2\pi$ by 1-periodicity, and we can assume that $\varphi \in [0,2\pi)$ in (17). Schematically, such a coefficient model is presented in Fig. 2.

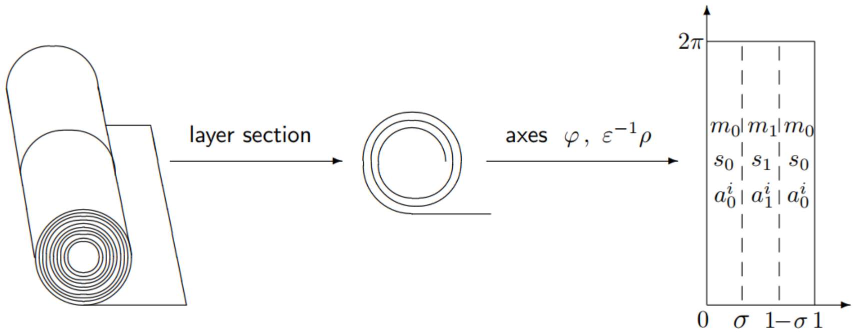

Fig. 2. Model of coefficients of cylindrical rolled layered composites

Further, one may choose an asymptotic approximation for the solution of problem (1) with coefficients (17) in the form

$$u_a = v(t,r,\varphi,z) + \varepsilon\, v_1(t,r,\varphi,z,\rho/\varepsilon) + \varepsilon^2\, v_2(t,r,\varphi,z,\rho/\varepsilon), \tag{18}$$

where, as before, the functions $v_1(t,r,\varphi,z,\tau)$ and $v_2(t,r,\varphi,z,\tau)$ are 1-periodic relative to the variable $\tau\in\mathbb{R}$ by definition.

Hence, the differentiation formula (6) should now be replaced by the formulas

$$\partial_r\, w(t,r,\varphi,z,\rho/\varepsilon) = (\varepsilon^{-1}\partial_\tau\, w(t,r,\varphi,z,\tau) + \partial_r\, w(t,r,\varphi,z,\tau))|_{\tau=\rho/\varepsilon},$$
$$\partial_\varphi\, w(t,r,\varphi,z,\rho/\varepsilon) = \left((\partial_\varphi + (2\pi)^{-1}\partial_\tau)\, w(t,r,\varphi,z,\tau)\right)\Big|_{\tau=\rho/\varepsilon} =$$
$$= \left(\partial_\varphi\, w(t,r,\varphi,z,\tau) + (2\pi)^{-1}\partial_\tau\, w(t,r,\varphi,z,\tau)\right)\Big|_{\tau=\rho/\varepsilon},$$

because $\tau=\rho/\varepsilon=(r/\varepsilon)+(2\pi)^{-1}\varphi$. Taking the last formula into account, we obtain

$$\partial_\varphi(a_2^\varepsilon\,\partial_\varphi\, v) = (\partial_\varphi + (2\pi)^{-1}\partial_\tau)(a_2(\tau)\,\partial_\varphi\, v) = a_2\,\partial_\varphi^2\, v + (2\pi)^{-1}(\partial_\tau a_2)\,\partial_\varphi\, v, \tag{19}$$

since v does not depend on τ. Therefore, retaining the essential terms, instead of (7) we have

$$\begin{gathered} m\, v''_{tt} + s\, v'_t - \varepsilon^{-1}\partial_\tau(a_1\,\partial_\tau\, v_1) - \varepsilon^{-1}\partial_\tau(a_1\,\partial_r\, v) - \\ - \partial_\tau(a_1\,\partial_\tau\, v_2) - \partial_r(a_1\,\partial_r\, v) - \partial_\tau(a_1\,\partial_r\, v_1) - \partial_r(a_1\,\partial_\tau\, v_1) - \\ - r^{-1}a_1\,\partial_\tau\, v_1 - r^{-1}a_1\,\partial_r v - r^{-2}a_2\,\partial_\varphi^2\, v - r^{-2}(2\pi)^{-1}(\partial_\tau a_2)\,\partial_\varphi\, v - \\ - a_3\,\partial_z^2 v - \varepsilon\, r^{-1}a_1\,\partial_r v_1 - \varepsilon\,\partial_\tau(a_1\,\partial_r v_2) - \varepsilon\,\partial_r(a_1\,\partial_r v_1) - \varepsilon\,\partial_r(a_1\,\partial_\tau v_2) = g\, f \end{gathered} \tag{20}$$

for $\tau=r/\varepsilon$, where $m=m(\tau)$, $s=s(\tau)$, $a_i=a_i(\tau)$ and $g=g(\tau)$ for $i=1,2,3$ in accordance with definition (17).

Further, we get the 1-periodic problem (9) and equality (10), in which the term $r^{-2}a_2\,\partial_\varphi^2\, v$ should be replaced by

$$r^{-2}\, a_2\,\partial_\varphi^2\, v + r^{-2}(2\pi)^{-1}(\partial_\tau\, a_2)\,\partial_\varphi\, v$$

according to (20). Applying the averaging to the new equality (10), we have exactly the homogenized equation (11), since $\langle\partial_\tau(a_2)\rangle = -\langle(a_2)\,\partial_\tau 1\rangle = 0$ through using the integration by parts formula and the 1-periodicity of the function $a_2=a_2(\tau)$. Therefore, one more term $r^{-2}\Phi_2^1\,\partial_\varphi\, v$ should be added to definition (14), where (in addition to problems (15) and similar problems for Φ_1, Φ_2, Φ_3, Φ_g) the function Φ_2^1 is a 1-periodic solution of the problem

$$-\partial_\tau(a_1\,\partial_\tau\,\Phi_2^1) = (2\pi)^{-1}\,\partial_\tau\, a_2 \quad in \quad (0,1).$$

As before, the regular solution to this problem exists and is uniquely determined up to a constant. Consequently, this allows us to completely determine by definition (14) the function $v_2=v_2(t,r,\varphi,z,\rho/\varepsilon)$ and achieve the fulfillment of equation (2) up to $O(\varepsilon)$. Therefore, following (Sandrakov, 2004b), this can be proven by energetic methods that the last statement implies the accuracy estimate (16). Similar results are valid for the case of heat processes (when formally $m^\varepsilon=0$ and the condition $u'_t|_{t=0}=u_1$ is removed). The results are natural, since at the micro level cylindrical layered composites and cylindrical rolled layered composites are indistinguishable, at least for wave and heat flows. This may not be true for electric and magnetic flows, since Maxwell's equations form a system of equations.

4. Models of processes in cylindrical layered composites, the layers of which are formed by rings

We have considered cylindrical layered composites which are the union of thin cylindrical tubes. Now we will assume that each tube is a union of rings surrounded by some material. Let, as before, the cylindrical domain without the core is such union. In addition, we will assume that $L=l/\varepsilon$ is an integer for simplicity.

Let us denote by $Y=(0,1)^2$ the cell, which will be the periodicity cell in what follows. Further, we assume that $Y=Y_0\cup Y_1\cup\partial Y_1$ is the union of two open sets with the regular boundary $\partial Y_1=\partial Y_0$, where Y_0 and Y_1 have non-zero Lebesgue measures. Let's also define the following functions

$$\begin{gathered} m(\tau,\xi)=m_0 \;\; if \;\; (\tau,\xi)\in Y_0 \;\; and \;\; m(\tau,\xi)=m_1 \;\; if \;\; (\tau,\xi)\in Y_1, \\ a_i(\tau,\xi)=a_0^i if(\tau,\xi)\in Y_0 \; and \; a_i(\tau,\xi)=a_1^i if(\tau,\xi)\in Y_1 \end{gathered} \tag{21}$$

for $i=1,2,3$. Similarly, $s(\tau,\xi)$ is equal to s_0 or s_1 and $g(\tau,\xi)$ is equal to g_0 or g_1, where all constants m_0, m_1, a_0^i, a_1^i, g_0 and g_1 are positive and fixed. Further, the functions are continued by 1-periodicity. But, now 1-periodicity means periodicity with period 1 with respect to the two variables $\tau\in\mathbb{R}$ and $\xi\in\mathbb{R}$.

Next, we define the coefficients of equation (2) by the equalities

$$m^\varepsilon = m(r/\varepsilon, z/\varepsilon),\quad s^\varepsilon = s(r/\varepsilon, z/\varepsilon),\quad a_i^\varepsilon = a_i(r/\varepsilon, z/\varepsilon),\quad g^\varepsilon = g(r/\varepsilon, z/\varepsilon) \tag{22}$$

for $i=1,2,3$. Schematically, such a coefficient model is presented in Fig. 3.

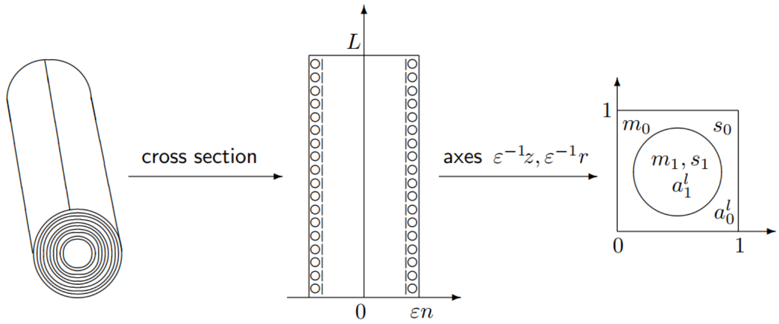

Fig. 3. Model of coefficients of cylindrical composites with layers from rings

Note that the further constructions of asymptotic solution expansions and results do not change for the more general case $Y = Y_0 \cup Y_1 \cup Y_2 \cup \partial Y_1 \cup \partial Y_2$, for example, it can be an insulated wire surrounded by air. For simplicity, we restrict ourselves to the case. It would also be possible to choose a relevant non-degenerate parallelogram as the periodicity cell. But, using a non-degenerate change of variables, the case can always be reduced to the one under consideration.

In this case, to construct an asymptotic expansion, following (Sandrakov, 1991), one may choose an asymptotic approximation for the solution of (1) in the form

$$u_a = v(t,r,\varphi,z) + \varepsilon\, v_1(t,r,\varphi,z,r/\varepsilon,z/\varepsilon) + \varepsilon^2\, v_2(t,r,\varphi,z,r/\varepsilon,z/\varepsilon), \tag{23}$$

where the functions $v_1(t,r,\varphi,z,\tau,\xi)$ and $v_2(t,r,\varphi,z,\tau,\varphi)$ are 1-periodic relative to $(\tau,\xi) \in \mathbb{R}^2$ and the first term does not depend on $\tau = r/\varepsilon$ and $\xi = z/\varepsilon$, since the kernel of the relevant operator on 1-periodic functions is generated by 1.

Now, to define the terms of asymptotic expansion (23), the following differentiation formulas for functions of a special form $w = w(t,r,\varphi,z,r/\varepsilon,z/\varepsilon)$ will be used

$$\begin{aligned} \partial_r\, w(t,r,\varphi,z,r/\varepsilon,z/\varepsilon) &= (\varepsilon^{-1}\partial_\tau\, w(t,r,\varphi,z,\tau,\xi) + \partial_r\, w(t,r,\varphi,z,\tau,\xi))|_{\tau=r/\varepsilon,\xi=z/\varepsilon}, \\ \partial_z\, w(t,r,\varphi,z,r/\varepsilon,z/\varepsilon) &= \left(\varepsilon^{-1}\partial_\xi\, w(t,r,\varphi,z,\tau,\xi) + \partial_z\, w(t,r,\varphi,z,\tau,\xi)\right)\Big|_{\tau=r/\varepsilon,\xi=z/\varepsilon}. \end{aligned} \tag{24}$$

Let's substitute asymptotic approximation (23) into equation (2) and take formulas (24) into account. Then, retaining only the essential terms, we obtain

$$\begin{aligned} &-\varepsilon^{-1}\partial_\tau(a_1\,\partial_\tau v_1) - \varepsilon^{-1}\partial_\xi\big(a_3\,\partial_\xi v_1\big) - \varepsilon^{-1}\partial_\tau(a_1\,\partial_r v) - \varepsilon^{-1}\partial_\xi(a_3\,\partial_z v) - \\ &- \partial_\tau(a_1\,\partial_\tau v_2) - \partial_\xi\big(a_3\,\partial_\xi v_2\big) - \partial_r(a_1\,\partial_r v) - \partial_z(a_3\,\partial_z v) + m\,v''_{tt} - \\ &- \partial_\tau(a_1\,\partial_r v_1) - \partial_r(a_1\,\partial_\tau v_1) - \partial_\xi(a_3\,\partial_z\, v_1) - \partial_z\big(a_3\,\partial_\xi v_1\big) + s\,v'_t - \\ &- r^{-1}a_1\,\partial_\tau\, v_1 - r^{-1}a_1\,\partial_r\, v - r^{-2}a_2\,\partial_\varphi^2\, v - \varepsilon\,\partial_r(\,a_1\,\partial_r v_1) = g\,f \end{aligned} \tag{25}$$

for $\tau = r/\varepsilon$ and $\xi = z/\varepsilon$, where $m = m(\tau,\xi)$, $s = s(\tau,\xi)$, $a_i = a_i(\tau,\xi)$, $g = g(\tau,\xi)$ according to definition (22).

Equating the coefficients at ε^{-1} in asymptotic equality (25), we have

$$-\partial_\tau(\,a_1\,\partial_\tau v_1) - \partial_\xi(\,a_3\,\partial_\xi v_1) = \partial_\tau(\,a_1\,\partial_r\, v) + \partial_\xi(\,a_3\,\partial_z\, v). \tag{26}$$

This equality will always be satisfied if we choose

$$v_1 = \Phi^1(\tau,\xi)\;\partial_r\, v(t,r,\varphi,z) + \Phi^3(\tau,\xi)\;\partial_z\, v(t,r,\varphi,z)$$

and define the functions $\Phi^1 = \Phi^1(\tau,\xi)$ and $\Phi^3 = \Phi^3(\tau,\xi)$ as solutions of the following 1-periodic problems choose

$$\begin{aligned} -\partial_\tau(\,a_1\;\partial_\tau\,\Phi^1\,) - \partial_\xi(\,a_3\;\partial_\xi\,\Phi^1\,) = \partial_\tau(\,a_1) \quad in \quad Y = (0,1)^2, \\ -\partial_\tau(\,a_1\;\partial_\tau\,\Phi^3\,) - \partial_\xi(\,a_3\;\partial_\xi\,\Phi^3\,) = \partial_\xi(\,a_3) \quad in \quad Y = (0,1)^2. \end{aligned} \tag{27}$$

It is known (Bensoussan, Lions, & Papanicolaou, 1978) that the solutions to problems (27) exist and are determined up to constants, which will be fixed by the condition

$$\langle\langle\,\Phi^i\,\rangle\rangle = \int_0^1\!\!\int_0^1 \Phi^i(\tau,\xi)\,d\tau\,d\xi = 0 \quad for \quad i = 1{,}3.$$

Note that the solvability follows from the equalities $\langle\langle\,\partial_\tau(a_1)\,\rangle\rangle = -\langle\langle\,(a_1)\;\partial_\tau\,1\,\rangle\rangle = 0$. Problems (27) can be considered as problems for an elliptic operator on the 2-torus as a manifold without boundary. It is known (Kinderlehrer, & Stampacchia, 1980) that the solutions are Holder functions, since $a_i \in L^\infty(Y)$ for $i = 1{,}3$. Moreover, if the boundary ∂Y_1 is quite regular, then $\Phi^i \in W^{1,\infty}_{per}(Y)$, where the index per means 1-periodicity with respect to $(\tau,\xi) \in \mathbb{R}^2$, since the coefficients of these operator are constants and Φ^i are smooth outside ∂Y_1. So the solutions to the problems are quite regular. In the previous cases, the regularity follows from explicit formulas, for example, for solutions of problem (9).

Equating the coefficients at ε^0 in asymptotic equality (25) and using the choice already made, that $v_1 = \Phi^1\,\partial_r v + \Phi^3\,\partial_z v$, we obtain

$$\begin{aligned} &-\partial_\tau(\,a_1\;\partial_\tau\, v_2) - \partial_\xi\big(\,a_1\;\partial_\xi\, v_2\big) = g\,f - m\,v''_{tt} - s\,v'_t + \\ &+(a_1 + a_1\,\partial_\tau\Phi^1)\,\partial_r^2 v + (a_1\,\partial_\tau\Phi^3)\,\partial_r\,\partial_z v + \big(a_3 + a_3\,\partial_\xi\Phi^3\big)\,\partial_z^2 v + \big(a_3\,\partial_\xi\Phi^1\big)\,\partial_z\,\partial_r v + \\ &+ r^{-1}(a_1 + a_1\,\partial_\tau\Phi^1)\,\partial_r\, v + r^{-1}(a_1\,\partial_\tau\Phi^3)\,\partial_z\, v + r^{-2}a_2\;\partial_\varphi^2\, v \\ &+ \partial_\tau(a_1\Phi^1)\,\partial_r^2\, v + \partial_\tau(a_1\Phi^3)\,\partial_r\,\partial_z\, v + \partial_\xi(a_3\Phi^3)\,\partial_z^2\, v + \partial_\xi(a_3\Phi^1)\,\partial_z\,\partial_r\, v. \end{aligned} \tag{28}$$

By analogy with (26), consider the equality as a 1-periodic problem for the function $v_2 = v_2(\tau,\xi)$ with fixed t, r, φ, z. It is known (Bensoussan, Lions, & Papanicolaou, 1978) that the problem is correct if and only if the right-hand side is orthogonal to 1 in the space $L^2(Y)$. Consequently, we get

$$\hat{m}\, v''_{tt} + \hat{s}\, v'_t - \hat{a}_1^1\, \partial_r^2 v - r^{-1}\hat{a}_1^1\, \partial_r v - \hat{a}_1^3\, \partial_r \partial_z v - r^{-1}\hat{a}_1^3\, \partial_r v - \\ - \hat{a}_3^3\, \partial_z^2 v - \hat{a}_3^1\, \partial_z \partial_r v - r^{-2}\hat{a}_2\, \partial_\varphi^2 v = \hat{g}\, f \quad in \quad \Omega\times(0,T), \tag{29}$$

where the following notation was introduced for the constants

$$\hat{m} = \langle\langle\, m\,\rangle\rangle = \int_0^1\int_0^1 m(\tau,\xi)\, d\tau d\xi,\ \ \hat{s} = \langle\langle\, s\,\rangle\rangle,\ \ \hat{g} = \langle\langle\, g\,\rangle\rangle, \\ \hat{a}_2 = \langle\langle\, a_2\,\rangle\rangle,\ \ \hat{a}_1^1 = \langle\langle a_1 + a_1\,\partial_\tau\Phi^1\rangle\rangle,\ \ \hat{a}_1^3 = \langle\langle a_1\,\partial_\tau\Phi^3\rangle\rangle, \\ \hat{a}_3^3 = \langle\langle a_3 + a_3\,\partial_\xi\Phi^3\rangle\rangle, \qquad \hat{a}_3^1 = \langle\langle a_3\,\partial_\xi\Phi^1\rangle\rangle. \tag{30}$$

Thus, starting from equation (2) with complex oscillating coefficients, we have obtained an equation with constant coefficients, which is homogenized. Adding boundary and initial conditions to this equation, as in problem (1), we obtain a homogenized problem. Note that, in the general case, the matrix $\hat{a} = \{\,\hat{a}_i^j\,\}_{i,j=1,3}$ is not diagonal and, therefore, homogenized media may be anisotropic. However, it is known (Bensoussan et al., 1978) that the matrix with constant components is both symmetric and elliptic. It is also known that the following inequalities hold in the matrix sense

$$\langle\langle\, a^{-1}\,\rangle\rangle^{-1} \le \hat{a} \le \langle\langle\, a\,\rangle\rangle,$$

where a is the diagonal matrix with diagonal entries $a_1 = a_1(\tau,\xi)$ and $a_3 = a_3(\tau,\xi)$.

To explicitly calculate the elements of the homogenized matrix â, it is necessary to solve problems (27). In the general case, this is done numerically through computer simulations. Note also that if the coefficients in (21) do not depend on $\xi \in [0,1]$, that is $m = m(\tau)$, $s = s(\tau)$, $a_i(\tau)$, $g = g(\tau)$. Then the matrix $\hat{a}$ is diagonal and the homogenized problem (29) coincides with problem (11), where the coefficients are defined in (12), which coincide with (30). Indeed, in the case, the second equation in (27) is trivial and Φ^3=0. In addition, the first equation coincides with equation (9).

In turn, if the coefficients in (21) do not depend on $\tau \in [0,1]$, that is, the coefficients have the form $m = m(\xi)$, $s = s(\xi)$, $a_i = a_i(\xi)$, $g = g(\xi)$. Then the matrix $\hat{a}$ is also diagonal and the homogenized problem (29) coincides with problem (11), but the coefficients of this problem should be determined by the following equalities

$$\hat{m} = \langle m\rangle = \int_0^1 m(\xi)\, d\xi,\ \ \hat{s} = \langle s\rangle,\ \ \hat{g} = \langle g\rangle, \\ \hat{a}_1 = \langle a_1\rangle, \qquad \hat{a}_2 = \langle a_2\rangle, \qquad \hat{a}_3 = \langle a_3^{-1}\rangle^{-1}.$$

Therefore, in comparison with (12), good heat (wave) insulation will be in the direction of the axis of the cylinder, which is layered composed of pancakes.

In the general case, the matrix $\hat{a}$ is diagonal if the coefficients $a_1 = a_1(\tau,\xi)$ and $a_3 = a_3(\tau,\xi)$. in definition (21) are symmetric with respect to the coordinate axes passing through the center $(0.5{,}0.5)$ of the periodicity cell Y according to (Bakhvalov, & Panasenko, 1989). For example, if Y_1 is a square or circle centered at this point and of suitable size, as in Fig. 2, it is known (Duvaut, & Lions, 1972) that the homogenized problem (29) (with initial and boundary conditions from (1)) has a unique regular solution. Similar results are valid for the case of heat processes (when formally $m^\varepsilon = 0$ and the condition $u'_t|_{t=0} = u_1$ is removed). The solution can be found numerically, which is a standard problem in computational mathematics. In the case of the diagonal matrix $\hat{a}$, the solution can be constructed in the form of a convergent series by the method of separation of variables. Moreover, each term of the series is a particular solution of the homogenized problem with suitable initial conditions

Thus, two terms in the asymptotic approximation (23) for the solution of problem (1) are defined. In order to define the third term $v_2 = v_2(r,\varphi,z,r/\varepsilon,z/\varepsilon)$, we subtract equation (29) from equality (28). Then, we obtain

$$-\,\partial_\tau(a_1\,\partial_\tau\, v_2) - \partial_\xi\big(a_3\,\partial_\xi\, v_2\big) = \tilde{g}\, f - \tilde{m}\, v''_{tt} - \tilde{s}\, v'_t + r^{-2}\,\tilde{a}_2\,\partial_\varphi^2\, v + \\ + \tilde{a}_1^1\,\partial_r^2\, v + r^{-1}\,\tilde{a}_1^r\,\partial_r v + \tilde{a}_1^3\,\partial_r\partial_z\, v + r^{-1}\,\tilde{a}_1^z\,\partial_z\, v + \tilde{a}_3^3\,\partial_z^2 v + \tilde{a}_3^1\,\partial_z\partial_r v, \tag{31}$$

where the coefficients $\tilde{g}$, $\tilde{m}$, $\tilde{s}$, $\tilde{a}_2$, $\tilde{a}_1^1$, $\tilde{a}_1^r$, $\tilde{a}_1^z$, $\tilde{a}_1^3$, $\tilde{a}_1^z$, $\tilde{a}_3^3$, $\tilde{a}_3^1$ are 1-periodic functions and are orthogonal to 1 in $L^2(Y)$.

Therefore, equality (31) and therefore equality (28) are fulfilled if we choose

$$v_2 = \Phi_g\, f - \Phi_m\, v''_{tt} - \Phi_s\, v'_t + r^{-2}\,\Phi_2\,\partial_\varphi^2\, v + \Phi_1^1\,\partial_r^2\, v + \\ + r^{-1}\Phi_1^r\,\partial_r v + \Phi_1^3\,\partial_r\partial_z v + r^{-1}\Phi_1^z\,\partial_z v + \Phi_3^3\,\partial_z^2 v + \Phi_3^1\,\partial_z\partial_r v, \tag{32}$$

where, for example, the functions $\Phi_3^1 = \Phi_3^1(\tau,\xi)$ is 1-periodic solution of the following problem

$$-\,\partial_\tau(\,a_1\,\partial_\tau\,\Phi_3^1\,) - \partial_\xi(\,a_3\,\partial_\xi\,\Phi_3^1\,) = \tilde{a}_3^1 \quad in \quad Y. \tag{33}$$

The problem is quite similar to problems (27) and is solvable up to a constant fixed by the condition $\langle\langle\,\Phi_3^1\,\rangle\rangle = 0$. Hence, 1-periodic problems for Φ_g, Φ_m, Φ_s, Φ_2, Φ_1^1, Φ_1^r, Φ_1^3, Φ_1^z, Φ_3^3 can be defined in the same way as before.

This approach allows us to completely determine $v_2 = v_2(t,r,\varphi,z,r/\varepsilon,z/\varepsilon)$ and achieve the fulfillment of equation (2) up to $O(\varepsilon)$. Therefore, following (Sandrakov, 2004b), this can be verified by energetic methods that the last statement implies, for example, the following accuracy estimate

$$\left\| r^{1/2}\big(\,\partial_r u - \theta_1^\varepsilon\,\partial_r v + r^{-1}\,\partial_\varphi u - r^{-1}\,\partial_\varphi v + \partial_z u - \theta_3^\varepsilon\,\partial_z v\,\big)\right\|^2_{C^0([0,T];L^2(\Omega))} + \\ + \ \|\, u'_t - v'_t\,\|^2_{C^0([0,T];L^2(\Omega))} \ + \ \|\, u - v\,\|^2_{C^0([0,T];L^2(\Omega))} \le C\,\varepsilon, \tag{34}$$

with $\theta_1^\varepsilon = \theta_1(r/\varepsilon, z/\varepsilon)$ and $\theta_3^\varepsilon = \theta_3(r/\varepsilon, z/\varepsilon)$, where $\theta_1(\tau,\xi) = 1 + \partial_\tau\Phi^1 + \partial_\xi\Phi^1$, $\theta_3(\tau,\xi) = 1 + \partial_\tau\Phi^3 + \partial_\xi\Phi^3$, $\Phi^1 = \Phi^1(\tau,\xi)$ and $\Phi^3 = \Phi^3(\tau,\xi)$ are the solutions of problem (27), u is the solution of problem (1) with coefficients from (22), v is the solution of homogenized problem with coefficients from (30) and the constant C does not depend on ε when $0 < \varepsilon \leq \varepsilon_0$ for the corresponding ε_0.

Thus, just like before, we have a fairly good accuracy estimate for solutions and flows for sufficiently small ε. In this case, for modeling wave processes (correspondingly, heat processes when $m^\varepsilon = 0$) in such composites, instead of solving problem (1) (for example, numerically through computer simulations), it is possible to solve the homogenized problem for equation (29) with guaranteed accuracy.

It is useful to note that representation (32) as well as (14) contain quite a lot of terms. Therefore, when proving the accuracy estimate (34), it will be necessary to estimate the contribution of each such term. In fact, in order to prove the estimate, it suffices to use only two terms

$$u_a^1 = v(t,r,\varphi,z) + \varepsilon\, v_1(t,r,\varphi,z,r/\varepsilon,z/\varepsilon)$$

in the asymptotic approximation (23) for the solution of problem (1). Further details can be found in (Sandrakov, 2004b), where the main idea is that the right side of the equation for $w = u - u_a^1$ can contain a term $div\,(\varepsilon F)$, which is estimable using integration by parts. But, without understanding the structure of the function $v_2 = v_2(t,r,\varphi,z,r/\varepsilon,z/\varepsilon)$ it would be impossible to find out the exact form of homogenized equation (29), which is the basis of the method here.

5. Models for processes in cylindrical layered composites, the layers of which are formed by helixes

We have considered composites which are the union of thin cylindrical tubes, which are formed by a large number of rings separated by a material. Some transformers have a cylindrical layer structure around the core, the layers of which consist of conductor turns in the form of helixes. Thus, one can naturally consider a model for a cylindrical layered composite, the layers of which are formed by a helix with many turns separated by some material (or several materials, for example, an insulator and air). This case can be considered by the same homogenization method.

Indeed, let $z \in (0,l)$ and $\varphi \in [0,2\pi)$ just like before, consider a new coordinate

$$\zeta = z + \varepsilon\,(\varphi/2\pi).$$

It would be possible to use this coordinate as new instead of z in equation (2). But, we will use the coordinate only in the coefficients defined by (21). Let, as before, we have

$$\Omega = \cup_{k=1}^{N} B_k \times (0,l) = \cup_{k=1}^{N} \{\, (x,y) \in \mathbb{R}^2 :\ \kappa + \varepsilon\,(k-1) \leq \sqrt{x^2+y^2} < \kappa + \varepsilon\, k \,\} \times (0,l),$$

where B_k is a ring of thickness ε and suitable radii for $k = 1,\dots,N$. Remark that by fixing a positive $\kappa < r_0 < R$, we can consider the following curve in Cartesian coordinates

$$x = r_0\cos(t),\quad y = r_0\sin(t),\quad z = \varepsilon\,(t/2\pi)\quad for\quad 0 \leq t < \infty, \tag{35}$$

which is called a cylindrical helix and passes through one of the tubes $B_k \times (0,l)$, since $0 < r_0 < R$. Consequently, the curve $r = \sqrt{x^2+y^2} = r_0,\ z = \varepsilon\,(\varphi/2\pi)$ is the cylindrical helix also for the parameter $\varphi = t$.

Now, we define the coefficients of equation (2) by the equalities

$$m^\varepsilon = m(r/\varepsilon,\zeta/\varepsilon),\quad s^\varepsilon = s(r/\varepsilon,\zeta/\varepsilon),\quad a_i^\varepsilon = a_i(r/\varepsilon,\zeta/\varepsilon),\quad g^\varepsilon = g(r/\varepsilon,\zeta/\varepsilon) \tag{36}$$

for $i = 1,2,3$. Here the functions $m = m(\tau,\xi)$, $s = s(\tau,\xi)$, $a_i = a_i(\tau,\xi)$ and $g = g(\tau,\xi)$ for $i = 1,2,3$ are 1-periodic with respect to $(\tau,\xi) \in \mathbb{R}^2$ and are defined by equalities (21) on the periodicity cell $Y = (0,1)^2$.

Note, for example, that if $m^\varepsilon = m(r/\varepsilon,\zeta/\varepsilon) = m_0$ for $r = r_0$, $\varphi = 0$ and $z = z_0$, then $m^\varepsilon = m(r/\varepsilon, 1 + z_0/\varepsilon) = m_0$ for $r = r_0$, $\varphi = 2\pi$ and $z = z_0$ by 1-periodicity, and we can assume that $\varphi \in [\,0,2\pi)$ in (36). In fact, the latter possibility follows from the periodicity of the trigonometric functions in (35) with the period 2π.

Further, to construct an asymptotic expansion, one may choose an asymptotic approximation for the solution of problem (1) with coefficients (36) in the form

$$u_a = v(t,r,\varphi,z) + \varepsilon\, v_1(t,r,\varphi,z,r/\varepsilon,\zeta/\varepsilon) + \varepsilon^2\, v_2(t,r,\varphi,z,r/\varepsilon,\zeta/\varepsilon),$$

where the functions $v_1(t,r,\varphi,z,\tau,\xi)$ and $v_2(t,r,\varphi,z,\tau,\varphi)$ are 1-periodic relative to $(\tau,\xi) \in \mathbb{R}^2$ and the first term does not depend on $\tau = r/\varepsilon$ and $\xi = z/\varepsilon$.

Hence, the differentiation formula (24) should now be replaced by the formulas

$$\begin{aligned}
&\partial_r\, w(t,r,\varphi,z,r/\varepsilon,\zeta/\varepsilon) = \left(\varepsilon^{-1}\partial_\tau\, w(t,r,\varphi,z,\tau,\xi) + \partial_r\, w(t,r,\varphi,z,\tau,\xi)\right)\big|_{\tau=r/\varepsilon,\xi=\zeta/\varepsilon},\\
&\partial_z\, w(t,r,\varphi,z,r/\varepsilon,\zeta/\varepsilon) = \left(\varepsilon^{-1}\partial_\xi\, w(t,r,\varphi,z,\tau,\xi) + \partial_z\, w(t,r,\varphi,z,\tau,\xi)\right)\Big|_{\tau=r/\varepsilon,\xi=\zeta/\varepsilon},\\
&\partial_\varphi w\left(t,r,\varphi,z,\frac{r}{\varepsilon},\frac{\zeta}{\varepsilon}\right) = \left(\left(\partial_\varphi + (2\,\pi)^{-1}\partial_\xi\right) w(t,r,\varphi,z,\tau,\xi)\right)\Big|_{\tau=\frac{r}{\varepsilon},\xi=\frac{\zeta}{\varepsilon}} =\\
&= \left(\partial_\varphi\, w(t,r,\varphi,z,\tau,\xi) + (2\,\pi)^{-1}\partial_\xi\, w(t,r,\varphi,z,\tau,\xi)\right)\Big|_{\tau=r/\varepsilon,\xi=\zeta/\varepsilon},
\end{aligned} \tag{37}$$

because $\xi = \zeta/\varepsilon = (z/\varepsilon) + (2\,\pi)^{-1}\varphi$. Taking the last formula into account, we obtain

$$\partial_\varphi\left(a_2^\varepsilon\,\partial_\varphi\, v\right) = a_2(\tau,\xi)\,\partial_\varphi^2\, v + (2\,\pi)^{-1}\left(\partial_\xi a_2(\tau,\xi)\right)\partial_\varphi\, v,$$

since v does not depend on τ and ξ. Consequently, retaining only the essential terms, we get (28), where $r^{-2}a_2\,\partial_\varphi^2 v$ is replaced by $r^{-2}a_2\,\partial_\varphi^2 v + r^{-2}(2\,\pi)^{-1}(\partial_\xi a_2)\,\partial_\varphi v$ for $\tau = r/\varepsilon$, $\xi = \zeta/\varepsilon$, in accordance with (37).

Further, we get the 1-periodic problems (27) and equality (28), in which the term $r^{-2}a_2\,\partial_\varphi^2 v$ is replaced by the term

$$r^{-2}a_2\,\partial_\varphi^2 v + r^{-2}(2\,\pi)^{-1}(\partial_\xi a_2)\,\partial_\varphi v$$

due to (37). Applying the mean to the new equality (28), we get exactly the homogenized equation (29), since $\langle\langle \partial_\xi(a_2)\rangle\rangle = -\langle\langle (a_2)\,\partial_\xi 1\rangle\rangle = 0$. Here we have used the integration by parts and the 1-periodicity of the function $a_2(\tau,\xi)$.

Thus, one more term $r^{-2}\,\Phi_2^1\,\partial_\varphi v$ should be added to definition (32), where, in addition to problem (33) and similar problems for Φ_g, Φ_m, Φ_s, Φ_2, Φ_1^1, Φ_1^r, Φ_1^3, Φ_1^z, Φ_3^3, the function Φ_2^1 is a 1-periodic solution of the following problem

$$-\partial_\tau(a_1\,\partial_\tau\,\Phi_2^1) - \partial_\xi(a_3\,\partial_\xi\,\Phi_2^1) = (2\,\pi)^{-1}(\partial_\xi\,a_2) \quad in \quad Y.$$

Again, the solution to this problem exists and is unique up to a constant, which allows us to completely determine $v_2 = v_2(t, r, \varphi, z, r/\varepsilon, \zeta/\varepsilon)$ and achieve the fulfillment of equation (2) up to $O(\varepsilon)$. Therefore, following (Sandrakov, 2004b), this can be verified by energetic methods that the accuracy estimate (34) is valid. Similar results are valid for the case of heat processes (when formally $m^\varepsilon = 0$ and the condition $u'_t|_{t=0} = u_1$ is removed). The results are natural, since at the micro level cylindrical layered composites, the layers of which are formed by a large number of rings or rings of helixes, are indistinguishable, at least for wave and heat flows.

The interested reader may consider as a useful exercise the case in which (instead of equalities (22) or (36)) the coefficients in equation (2) are given by the following relations

$$m^\varepsilon = m(\rho/\varepsilon, \zeta/\varepsilon), \quad s^\varepsilon = s(\rho/\varepsilon, \zeta/\varepsilon), \quad a_i^\varepsilon = a_i(\rho/\varepsilon, \zeta/\varepsilon), \quad g^\varepsilon = g(\rho/\varepsilon, \zeta/\varepsilon)$$

where the functions $m = m(\tau, \xi)$, $s = s(\tau, \xi)$, $a_i = a_i(\tau, \xi)$ and $g = g(\tau, \xi)$ for $i = 1,2,3$ are defined by equalities (21) and the following coordinates $\rho = r + \varepsilon\,(\varphi/2\pi)$, $\zeta = z + \varepsilon\,(\varphi/2\pi)$ are used. Here, as usual, one can take $(r, \varphi, z) \in (0, R) \times [0, 2\pi) \times (0, l)$. We only note that the corresponding curves will be conical helical lines. An inquisitive reader can also try to consider this problem using the multiscale convergence method.

Discussion and conclusions

Initial boundary value problems for dynamic parabolic heat conduction equations and hyperbolic wave equations in periodic media having a cylindrical structure are considered. Analytical methods and an algorithm for determining asymptotic expansions and homogenized problems, the solutions of which provide approximate multi-scale asymptotic solutions of such problems, have been established. Asymptotic accuracy estimates are presented, which are actual in numerical calculations and computer simulation for models with guaranteed accuracy. Thus, it is shown in sufficient detail that homogenization methods can be useful for modeling wave and thermal processes in a cylinder consisting of alternating layers with different characteristics, which are similar to capacitors and transformers. This approach can also be implemented for modeling filtration processes in cylindrical filters. Undoubtedly, the extension of the methods to Maxwell's equations is very relevant and promising for modeling processes in various microdevices. The approach developed here can really be generalized to such a system of equations and is asymptotically accurate, since it takes into account the dependence on the parameters and geometry of the layers quite essentially.

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МОДЕЛЮВАННЯ І ГОМОГЕНІЗАЦІЯ ПРОЦЕСІВ У ЦИЛІНДРИЧНИХ ШАРУВАТИХ КОМПОЗИТАХ, ТАКИХ ЯК КОНДЕНСАТОРИ ТА ТРАНСФОРМАТОРИ

Конденсатори та трансформатори є допоміжними пристроями, які широко використовуються в сучасному житті. Нині спостерігається тенденція до мініатюризації таких пристроїв. Отже, виникає природна проблема моделювання теплових і електромагнітних процесів у них. Розуміння моделей зазначених процесів дає змогу, наприклад, оптимізувати або здешевити такі пристрої. Проте геометрія таких пристроїв дуже специфічна. Звичайний конденсатор, наприклад, являє собою згорнуту в циліндр довгу стрічку, укриту ізолятором. Безпосереднє моделювання процесів у такому пристрої складне, оскільки існує дуже багато шарів, що призводить до рівнянь зі швидко осцилюючими коефіцієнтами. З іншого боку, до таких рівнянь можна застосувати методи теорії гомогенізації й отримати рівняння зі сталими коефіцієнтами, розв'язки яких апроксимують розв'язки рівнянь, які розглядаємо. Зрозуміло, що значно простіше розв'язувати рівняння зі сталими коефіцієнтами як чисельними, так й аналітичними методами. Як перший крок у цьому напрямі, у пропонованому дослідженні розглянуто задачу моделювання хвильових і теплових процесів у циліндрі, що складається з великої кількості шарів з різними характеристиками, які чергуються. За допомогою асимптотичних методів теорії гомогенізації цю задачу наближено зведено до моделювання процесів в однорідному (гомогенізованому) циліндрі. Наведено асимптотичні розклади та гомогенізовані задачі, розв'язки яких визначають наближену асимптотику розв'язків задачі, яку ми розглядаємо. Наведено формули для розрахунку характеристик однорідного циліндра й оцінки точності наближень. Подано також оцінки точності цих апроксимацій, що необхідні для чисельних розрахунків і комп'ютерного моделювання таких проблем з гарантованою точністю. Крім того, наведені наближення, безсумнівно, корисні для розуміння хвильових і теплових процесів у пристроях, подібних до конденсаторів і трансформаторів. Зазначений підхід можна розглядати як основу для моделювання загальніших електромагнітних і теплових процесів, що моделюються рівняннями Максвелла в поєднанні з рівнянням теплопровідності.

Ключові слова: ***початково-крайові задачі, гомогенізовані задачі, хвильові рівняння, рівняння теплопровідності, асимптотичні розвинення, коефіцієнти теплопровідності, коефіцієнти швидкості поширення хвиль.***

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