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Козображення матричних алгебр Манна

Corepresentations of Munn matrix algebras

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Нехай \mathcal{A} — алгебра над полем K , m і n — натуральні числа і $P = (p_{ji})$ — фіксована $n \times m$ матриця над \mathcal{A} . K -векторний простір усіх $m \times n$ матриць над алгеброю \mathcal{A} можна перетворити в алгебру відносно наступної операції (\circ): $B \circ C = BPC$. Ця алгебра називається матричною алгеброю Манна над \mathcal{A} із сендвіч-матрицею P . Алгебри такого типу виникли як узагальнення напівгрупових алгебр матричних напівгруп Ріса, які, в свою чергу, тісно пов'язані з простими напівгрупами. У цій статті описано твірні та визначальні співвідношення матричних алгебр Манна із регулярною сендвіч-матрицею.

Ключові слова: козображення, твірні елементи та визначальні співвідношення, некомутативний поліном, алгебра Манна, напівгрупа Ріса, регулярна сендвіч-матриця

Let \mathcal{A} be an algebra over a field K , m and n natural numbers and $P = (p_{ji})$ a fixed $n \times m$ matrix over \mathcal{A} . The K -vector space of all $m \times n$ matrices over the algebra \mathcal{A} can be made into an algebra with respect to the following operation (\circ): $B \circ C = BPC$. This algebra is called the Munn matrix algebra over \mathcal{A} with sandwich matrix P . The algebras of such type arose as generalizations of semigroup algebras of Rees matrix semigroups which in turn are closely related to simple semigroups.

This article describes the generators and defining relations of Mann matrix algebras with a regular sandwich matrix.

Key Words: corepresentation, generators and defining relations, noncommutative polynomial, Mann algebra, Rees semigroup, regular sandwich matrix

1 Introduction

Munn matrix algebras arose as generalizations of semigroup algebras of Rees matrix semigroups which in turn are closely related to simple semigroups (see [1]).

Let G be a finite group and $G^0 = G \cup 0$ the group with zero (the adjunction of a zero element 0 to G means, by definition, that $00 = 0g = g0 = 0$ for all $g \in G$, what make $G \cup 0$ a semigroup with zero). For I and J be nonempty finite sets, denote by $M^0(G; I, J)$ the set of all $I \times J$ matrices over G^0 and by $M_1^0(G; I, J)$ its subset consisting of the matrices having at most one non-zero entry.

Now let $P = (p_{ji})$ be arbitrary fix $J \times I$ matrix over G^0 such that each row and each column consists at least one non-zero entry (such a matrix is called regular). The set $M_1^0(G, I, J)$ can be made into a semigroup with respect to the following operation (\circ): $B \circ C = BPC$. This semigroup

is called the Rees $I \times J$ matrix semigroup over G^0 with sandwich matrix P , and is denoted by $\mathcal{M}^0(G; I, J; P)$.

By the Rees theorem a finite semigroup is completely 0-simple if and only if it is isomorphic with Rees matrix semigroup over a group with zero (a semigroup is called simple if it does not properly contain any two-sided ideal, and 0-simple if it additionally contains a primitive idempotent).

The Munn algebras are defined in the same way as the Rees semigroups, but instead of a group with zero one need to take an algebra. A particular case of the Mann algebras are the contracted semigroup algebras of Rees semigroups.

In this paper we study corepresentations of Mann algebras.

2 Generators and defining relations of Munn algebras

Through the paper K be a field and \mathcal{A} a finite-dimensional unital algebra over K . In the definition of the Mann algebra (see Introduction) we put $I = \{1, 2, \dots, n\}$, $J = \{1, 2, \dots, m\}$ ($n, m \in \mathbb{N}$).

Definition. Let \mathcal{A} be an algebra over K . Let m and n be natural numbers and let $P = (p_{ji})$ be a fixed $n \times m$ matrix over \mathcal{A} . The K -vector space of all $m \times n$ matrices over the algebra \mathcal{A} can be made into an algebra with respect to the following operation (\circ): $B \circ C = BPC$. This algebra is called the *Munn $m \times n$ matrix algebra over \mathcal{A} with sandwich matrix P* and is denoted by $\mathcal{M}(\mathcal{A}; m, n; P)$.

By analogy with the Rees semigroups, the matrix P could be called regular if each row and each column consists at least one invertible entry. But since after replacing the matrix P by an equivalent matrix $P' = XPY$ (with invertible X and Y) we obtain an isomorphic Mann algebra, it suffices to require that there be at least one invertible entry. Namely such matrices P we call regular and will assume without loss of generality that p_{11} is invertible in \mathcal{A} . For the same reason, we can somewhat simplify matrix P , assuming that p_{11} is equal to the identity element of \mathcal{A} and the remaining entries of the first row and first column are equal to zero. Just such a variant is considered in Theorem 1.

Recall that a corepresentation (or presentation) of an algebra \mathcal{A} is, by definition, a pair $\langle A|R \rangle$, where A is a set of generators and R a set of defining relations for the given set of generators. Then one writes $\mathcal{A} = \langle A|R \rangle$. For convenience, the relations (non-commutative polynomials in elements of A) are written not only in the form $f = 0$, but also in the form $f_1 = f_2$.

The following theorem defines corepresentation of any Munn algebras $\mathcal{M}(\mathcal{A}; m, n; P)$ with regular sandwich matrix.

Theorem 1. *Let \mathcal{A} be a finite-dimensional algebra with identity e over a field K and $\langle A|R \rangle$ be its corepresentation with*

$$A = \{a_0 = e, a_1, \dots, a_s\},$$

$$R = \{f_1 = g_1, \dots, f_r = g_r\},$$

where $a_1, \dots, a_s \neq 0$ and $f_i = f_i(a_1, \dots, a_s)$, $g_i = g_i(a_1, \dots, a_s)$ are non-commutative polynomials over K .

Let $P = (p_{ji})$ be an $n \times m$ matrix over \mathcal{A} such that $p_{11} = e, p_{1i} = 0, p_{j1} = 0$ ($i = 2, \dots, m; j = 2, \dots, n$). For $n, m \in \mathbb{N}$, introduce the sets

$$\Gamma = \{\gamma_2, \dots, \gamma_m\}, \quad \Lambda = \{\lambda_2, \dots, \lambda_n\}.$$

Then we have the corepresentation $\langle M|\bar{R} \rangle$ for the Munn algebra $\mathcal{M} = \mathcal{M}(\mathcal{A}$, where $M = A \cup \Gamma \cup \Lambda$ and \bar{R} consists of the following relations:

- (1) the relation from R ;
- (2) $a_0 a_i = a_i a_0 = a_i$ ($i = 0, \dots, s$);
- (3) $a_0 \gamma_i = 0, \gamma_i a_0 = \gamma_i$ ($i = 2, \dots, m$);
- (4) $a_0 \lambda_j = \lambda_j, \lambda_j a_0 = 0$ ($j = 2, \dots, n$);
- (5) $\gamma_i \gamma_{i'} = 0$ ($i, i' = 2, \dots, m$);
- (6) $\lambda_j \lambda_{j'} = 0$ ($j, j' = 2, \dots, n$);
- (7) $\lambda_j \gamma_i = p_{ji}$ ($i = 2, \dots, m; j = 2, \dots, n$).

Proof. First, indicate which element of \mathcal{M} corresponds to each generator.

For $x \in \mathcal{A}$ and $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$, denote by $(x)_{ij}$ the matrix from \mathcal{M} with its (i, j) th entry being x , and its remaining entries being 0; the expression $(0)_{ij}$ mean the $I \times J$ zero matrix 0. Then, by the above definition,

$$(b)_{ij} \circ (c)_{i'j'} = (bp_{j'i'}c)_{ij'} \quad (1)$$

and in particular (taking into account the equality $p_{11} = 1$)

$$(bc)_{ij} = (b)_{i1} \circ (c)_{1j}. \quad (2)$$

As the generators indicated in the condition of the theorem, we take the following matrices (as elements of \mathcal{M}):

$$a_k = (a_k)_{11}, \quad \gamma_i = (e)_{i1}, \quad \lambda_j = (e)_{1j} \quad (3)$$

for $k \in \{0, 1, \dots, s\}, i \in \{2, \dots, m\}, j \in \{2, \dots, n\}$.

Prove that these elements form a system of generators of \mathcal{M} .

Denote by \bar{A}^e all words in the set $A \setminus e$ (as an alphabet) and put $\bar{A} = \bar{A}^e \cup e$, $(\bar{A}) = \{(\bar{a})_{ij} \mid \bar{a} \in \bar{A}, i = 1, \dots, m, j = 1, \dots, n\}$. Since the elements of \bar{A} form a system of generators of \mathcal{A} as a vector space, all elements from (\bar{A}) form a system of generators of \mathcal{M} as a vector space. Hence to prove that the elements from M form a system of generators of the algebra \mathcal{M} , it suffices to show that each element from (\bar{A}) is a product of elements from M , and according to (2), it suffices to check this for elements of the form $(a_r)_{i1}$ with $i \neq 1$ and $(a_r)_{1j}$ with $j \neq 1$. Using equality (2) again, we have $(a_r)_{i1} = \gamma_i(a_r)_{11}$ and $(a_r)_{1j} = (a_r)_{11}\lambda_j$. So the elements from M form system of generators of the algebra \mathcal{M} .

Now we show that the set of relations \bar{R} defines the algebra \mathcal{M} . In other words, there is no relation between generators from the set M which would not follow from the relations of \bar{R} . Assume the contrary and let F be a non-commutative polynomial in variables from M such that $F = 0$ and it is not a consequence of the relations from the set M .

It follows from the equations (2) and (3) (see the condition of the theorem) that $a_r \gamma_i = 0, \gamma_i a_r = \gamma_i$ for each $a_r \in A$. Therefore every word in M containing γ_i is equal to γ_i .

The same is true for λ_j (because of equalities (2) and (4)). Taking into account also equalities (5), (6) and (7), we have that

$$F = f(a_0, a_1, \dots, a_s) +$$

Список використаних джерел

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$$+ \sum_{2 \leq i \leq m} x_i \gamma_i + \sum_{2 \leq j \leq n} y_j \lambda_j + \sum_{\substack{2 \leq i \leq m \\ 2 \leq j \leq n}} z_{ij} \gamma_i \lambda_j,$$

where $x_i, y_j, z_{ij} \in K$.

Since the non-zero parts of the matrices that correspond (according to the definitions) generators from M do not intersect, then, substituting these matrices into the equality $F = 0$, we obtain that all x_i, y_j, z_{ij} are equal to zero. Hence the equality $F = 0$ turned into the equality $f(a_0, a_1, \dots, a_s) = 0$ and we came to a contradiction, since this relation is a consequence of relations from R .

Theorem 1 is proved.

References

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