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METHOD OF INTERPOLATION USING ROOT-FRACTIONAL-RATIONAL FUNCTIONS OF DIFFERENT ORDERS

The possibility of interpolating different kinds of mathematical functions using different order root-fractional-rational functions, namely, second, third, and fourth, is considered in the article. Generally, it is shown that root-fractional-rational functions are given a precision interpolation with a small value of error even for stiff analytical dependences. Root-fractional-rational functions from second to fourth order are considered, and corresponding analytical relations for defining polynomial coefficients in the nominator and the denominator are given. It is also proven that the number of necessary points for interpolation corresponds to the value 2n + 1, where n is the order of the root-fractional-rational function. Examples of interpolation of different functions for electrodynamics problems, simulation of magnetic lenses, probability tasks, and fuzzy-logic tasks are given; the error of interpolation for all considered examples is also defined. All presumptions of theoretical analysis are tested and verified using the original elaborated computer software, created using the Python programming language means. In the most considered examples, the resulting error of interpolation is smaller than a few percent. The graphic results of testing the proposed method of interpolation are also given.

Keywords: ***interpolation; root-fractional-rational function; electrodynamic tasks; functions of probability theory; fuzzy logic; membership functions.***

AMS 2020 classification: 26A06, 26A09.

Introduction

Today, interpolation methods are widely used to solve complex problems in computational mathematics, applied physics, and mathematical statistics, to simplify computational algorithms, and to effectively process large amounts of numerical information. But the general problem with using this approach is that the accuracy of describing the behavior of rigid functions using polynomial dependencies is usually not adequate, since polynomial functions often produce large outliers with extrema in smooth sections of the interpolated functional dependency. In this case, the interpolation accuracy can be very low, and the difference between the original data and the interpolation result exceeds hundreds of percent (Chapra, & Canale, 2014; Jain, M., Iengar, & Jain, R., 2010; Mathews, & Fink, 1998; Bronshtein et al., 2007; Phillips, 2023; Epperson, 2013). Moreover, under this condition, even a qualitative description of a set of numerical data using polynomial interpolation methods is incorrect. In the classical theory of interpolation, this problem is solved using splines through interpolation on small segments of the function argument using known derived values, but the general disadvantage of this approach is that different functions are used in different sections of the argument change, and the number of such sections can be several thousand and even be more (Chapra, & Canale, 2014; Jain, M., Iengar, & Jain, R., 2010; Mathews, & Fink, 1998; Bronshtein et al., 2007; Phillips, 2023; Epperson, 2013).

In the years 2019–2020 in paper (Melnyk, & Pochynok, 2020), a slightly different approach to the interpolation of data sets of stiff functions that are described as ravine dependencies associated with the use of root-polynomial functions has been proposed. Initially, the proposed approach has been applied to interpolating the boundary trajectories of electron beams during their transportation in an ionized gas. In paper (Melnyk, & Pochynok, 2020) has been proposed the mathematical relation for root-polynomial functions, which is generally written as:

$$r(z)=\sqrt[n]{C_n z^n + C_{n-1} z^{n-1} + \ldots + C_1 z + C_0}, \tag{1}$$

where z is the longitudinal coordinate, r is the radius of the boundary trajectory of the beam, n is the degree of the polynomial and the order of the root function, $C_0 - C_n$ is the value of the polynomial coefficients.

Research carried out and its results, reflected in paper (Melnyk, & Pochynok, 2020), showed that usually root polynomial functions (1) give an extremely small error in the interpolation of the boundary trajectories of the electron beams, no more than a fraction of a percent. Also, in paper (Melnyk, & Pochynok, 2020), simple analytical relations have been given for calculating the coefficients of root polynomial functions C_{nj} for values of the order of the function n from 2 to 6. Subsequently, in paper (Melnyk, & Pochynok, 2023), algorithms for approximating ravine dependencies using root-polynomial functions also have been considered. The paper (Melnyk, & Pochynok, 2023) considered methods of approximation by the reference points, and by the tangents to ravine functions corresponded to a given set of reference points.

However, studies carried out in (Melnyk, & Pochynok, 2023) showed that not even for all ravine and vertex functions, the numerical interpolation algorithms converge, considered in (Melnyk, & Pochynok, 2020). It was justified that for ravine functions, such algorithms converge in the case of dependencies with a narrow minimum region, and for functions with a plateau, especially in the case when the values of the numerical data are close to zero, the interpolation algorithms diverge. On the contrary, for vertex functions, the convergence of the developed algorithms is guaranteed in the case of a wide maximum area, that is, for functions with a plateau. It was also shown in (Melnyk, & Pochynok, 2023) that this problem of algorithms divergence can be avoided by using the method of mirroring the function values relative to the line corresponding to their average value along the ordinate axis. Then the corresponding values of $r(z)_{mir}$ are calculated through the following relation (Melnyk, & Pochynok, 2023):

$$r(z)_{\text{mir}} = \frac{r(z)_{\max} + r(z)_{\min}}{2} - \sqrt[n]{C_n z^n + C_{n-1} z^{n-1} + \ldots + C_1 z + C_0}, \tag{2}$$

where $r(z)_{max}$ – maximal, and $r(z)_{min}$ – minimal values of function $r(z)$ for considered digital data set.

However, in papers (Melnyk, & Pochynok, 2023), it was shown that using the root polynomial function (1) from the second to the sixth order, it is possible to solve the problem of interpolation of the boundary trajectory of a short-focus electron beam propagated in an ionized gas with a very small error, less than a fraction of a percent. Solving this problem is extremely important from a practical point of view for analyzing the parameters of electron beams that are formed by high-voltage glow discharge electron guns.

Electorn guns of this type have a number of advantages over traditional guns with hot cathodes (Melnyk, 2005; Melnyk, 2006) and today they are widely used in modern welding technologies (Melnyk, 2006; Melnyk et al., 2022; Druzhinin et al., 2015), the deposition of ceramic coatings for various purposes (Prikhna et al., 2022a; Grechanyuk et al., 2022; Zakharov et al., 2020; Zakharov, Rozenko, & Ilchenko, 2019), and the melting of refractory metals and non-metallic materials in order to purify them and improve their quality (Akhonin et al., 2018; Paton, Akhonin, & Berezos, 2018; Kemmotsu, Nagai, & Maeda, 2011). The main industries where high-voltage glow discharge electron guns are effectively used today are the electronics industry (Druzhinin et al., 2014; Druzhinin et al., 2015; Zakharov et al., 2020; Zakharov, Rozenko, & Ilchenko, 2019; Kemmotsu, Nagai, & Maeda, 2011), instrument making (Melnyk et al., 2022; Zakharov et al., 2020; Zakharov, Rozenko, & Ilchenko, 2019), mechanical engineering, production of high-energy equipment and electric vehicles, automotive, aviation, and space industries (Prikhna et al., 2022b). In general, methods for simulating electron beams formed in gas-discharge devices with gas ionization and beam interaction with ionized plasma are based on the basic principles of the theory of gas discharge (Melnyk, 2005; Smirnov, 2015; Lieberman, & Lichtenberg, 1994; Raizer, 1991).

There are also other approaches to the simulation of gas-discharge devices that form intensive electron beams for technological purposes. In particular, the works (Tseluyko et al., 2008; Rudychev, V., Lazurik, & Rudychev, Y., 2021) describe algorithms for the analysis of high-energy relativistic electron beams that are formed in ionized plasma. These methods are based on direct measurement of beam parameters, analysis of experimental data, and their subsequent statistical processing using modern computer technologies and suitable algorithms.

Thus, previous studies have shown that, in any case, the class of functions that are subject to interpolation and approximation using relations (1) and (2) is extremely limited; usually, these can only be ravine and vertex functions with a small, stiff degree. And even for the class of such functions, the simple numerical algorithms proposed and described in (Melnyk, & Pochynok, 2020; Melnyk, & Pochynok, 2023) often diverge. Therefore, the aim of this work is the development and software implementation of an interpolation algorithm for another, more wide class of interpolation functions, in particular root-fractional-rational ones.

1. Analysis of the results of previous research and general formulation of the problem

As substantiated in papers (Melnyk, & Pochynok, 2023), the convergence of interpolation methods based on the use of relations (1, 2) significantly depends on the maximum value of the derivative for the set of interpolated data and is not always guaranteed. A significant limitation is also that such interpolation methods are only suitable for ravine and vertex numerical dependencies with one global extremum. For ravine functions with a narrow minimum region, if their values in this region tend to zero, the proposed algorithms for interpolation by root polynomial functions also diverge.

Therefore, the aim of this paper is to analyze the possibilities of using other root-fractional-rational functions to solve interpolation problems.

In the general case, one can write the root-fraction-rational function as follows:

$$r(z) = \sqrt[n]{\frac{z^n + C_1 z^{n-1} + \ldots + C_{n-1} z + C_n}{C_{n+1} z^n + \ldots + C_{2n} z + C_{2n+1}}}, \tag{3}$$

where $C_1 \ldots C_{2n+1}$ – the corresponded values of coefficients of root-fraction-rational function.

The provided research shown, that the root-fractional-rational functions written in the form (3) can be used to solve a wide range of practical engineering problems, in particular.

1. To interpolate the magnetic field of focusing lenses in tasks of electron optics (Schiller, Heisig, & Panzer, 1982; Szilagyi, 2012; Humphries, 2013; Kasper, & Hawkes, 2018; Grivet, 1972).

2. To interpolate the boundary trajectory of electron beams in the case of their transportation in equipotential channels with a change in pressure along the transportation channel. From the point of view of the physical formulation and the numerical algorithms used, such a problem is considered as a problem of nonlinear electron optics (Schiller, Heisig, & Panzer, 1982; Szilagyi, 2012; Humphries, 2013; Kasper, & Hawkes, 2018; Grivet, 1972; El-Kareh, 2012).

3. Interpolation of the values of probability distribution and probability density functions in problems of probability theory and mathematical statistics. The main feature here is the high degree of stiffness of such functions and the impossibility of obtaining an analytical expression for the inverse function in a simple algebraic form (Wentzel, & Ovcharov, 2002; Gubner, 2006).

4. Interpolation of membership function values in fuzzy logic problems. Typically, such functions are specified as piecewise continuous on separate intervals of argument values since they do not have an analytical description over the entire interval of numerical values (Zadeh, & Aliev, 2018). Under this condition, to describe the membership functions over the entire numerical interval of values of the argument, one usually uses either a piecewise continuous description of functions on finite numerical intervals of the argument according to given conditions or corresponding arithmetic-logical expressions (Melnyk, & Luntovskyy, 2022).

2. Main obtained results and analytical relations for determining the coefficients of root-fractional-rational functions

Let us assume that a vector of reference points along the coordinate z $\mathbf{V_z}$ = [$z_1, z_2, \ldots, z_{2n}, z_{2n-1}$] and a vector of function values $r(z)$ $\mathbf{V_r}$ = [$r_1, r_2, \ldots, r_{2n}, r_{2n-1}$], which are correspond to these points, are given. Under this condition, the determination of the coefficients $C_1 \ldots C_{2n+1}$ of the root-fractional-rational function (3) is carried out by analytically solving the corresponding set of linear equations:

$$C_1 z_j^{n-1} + \ldots + C_{n-1} z_j + C_n - \left(r_j\left(z_j\right)\right)^n \times \left(C_{n+1} z_j^n + \ldots + C_{2n} z_j^n + C_{2n+1}\right) = \\ = \sum_{i==1}^{n} C_i z_j^i - \left(r_j\left(z_j\right)\right)^n \sum_{i=n+1}^{2n+1} C_i z_j^i = -z_j^n; j \in N, j \le 2n+1. \tag{4}$$

A generalized technique for the analytical solution of sets of linear equations of this type was described in papers (Melnyk, & Pochynok, 2020). The set of equations is solved using the classical Gauss–Seidel method with the sequential elimination of the C_l variables, but to order at each iteration the coefficients k in front of the C_l variables and the coefficients b on the right side of the equations, such a simple ordered indexing system is used. Let us write $k_{m,j,l}$, and $b_{k,j}$, where m is the number of the current iteration, j is the number of the equation according to formula (4), and l is the number of coefficient C according to formula (3). Thus, the first values of the coefficients $k_{1,j,l}$ and $b_{1,j}$ are specified by the coordinates of points from the vectors $\mathbf{V_z}$ and $\mathbf{V_r}$, and the values of these coefficients at the next iterations are calculated iteratively as follows:

$$k_{m,j,l} = \frac{k_{m-1,j,l} - k_{m-1,j_e,l_e}}{k_{m-1,j_e,l} - k_{m-1,j_e,l_e}}, b_{m,j} = \frac{b_{m-1,j} - b_{m-1,j_e}}{k_{m-1,j_e,l} - k_{m-1,j_e,l_e}}, \tag{5}$$

where j_e – number of the equation that is subtracted from other equations of the set and, thus, excluded from consideration in subsequent iterations, and l_e – the number of the coefficient C, which is excluded from the set of equations and which, in connection with this, is not considered at subsequent iterations.

After calculating all the values of the intermediate coefficients $k_{1,j,l}$ and $b_{1,j}$, the coefficients C that are being sought are found through iterative relations (5), using them sequentially in reverse order.

From a computational point of view, the main feature of iterative relations (5) is that they contain only simple arithmetic operations of subtraction and division, which the computer processor can easily perform at the hardware level (Melnyk, & Luntovskyy, 2022). An important advantage of the proposed algorithm is also that calculations using formula (5) for different values of j and l are carried out using the same ratios, so they can be performed by parallel computing. These features of the proposed algorithm greatly simplify the writing of program code, especially with the use of modern declarative, functional, structural, and matrix programming tools (Melnyk, & Luntovskyy, 2022; McKinney, 2023; Chollet, 2022; Lutz, 2013). When using parallel computing, the speed of solving the interpolation problem can be increased in proportion to the number of generated parallel threads (Melnyk, & Luntovskyy, 2022).

During the provided research and studies, the following iterative analytical relations were obtained to search for the coefficients of root-fractional rational functions of the second, third, and fourth orders. For example, for second order function: $r(z) = \sqrt{\dfrac{z^2 + C_1 z + C_2}{C_3 z^2 + C_4 z + C_5}}$, the following iterative relations have been obtained:

$$\begin{gathered}
k_{1,1,1} = z_1 - z_5; k_{1,1,3} = \left(r_5 z_5\right)^2 - \left(r_1 z_1\right)^2; k_{1,1,4} = r_5^2 z_5 - r_1^2 z_1; k_{1,1,5} = r_5^2 - r_1^2; b_{11} = z_5^2 - z_1^2; \\
k_{1,2,1} = z_2 - z_5; k_{1,2,3} = \left(r_5 z_5\right)^2 - \left(r_2 z_2\right)^2; k_{1,2,4} = r_5^2 z_5 - r_2^2 z_2; k_{1,2,5} = r_5^2 - r_2^2; b_{12} = z_5^2 - z_2^2; \\
k_{1,3,1} = z_3 - z_5; k_{1,3,3} = \left(r_5 z_5\right)^2 - \left(r_3 z_3\right)^2; k_{1,3,4} = r_5^2 z_5 - r_3^2 z_3; k_{1,3,5} = r_5^2 - r_3^2; b_{13} = z_5^2 - z_3^2; \\
k_{1,4,1} = z_4 - z_5; k_{1,4,3} = \left(r_5 z_5\right)^2 - \left(r_4 z_4\right)^2; k_{1,4,4} = r_5^2 z_5 - r_4^2 z_4; k_{1,4,5} = r_5^2 - r_4^2; b_{14} = z_5^2 - z_4^2; \\
k_{2,1,1} = \frac{k_{1,1,1}}{k_{1,1,5}}; k_{2,1,3} = \frac{k_{1,1,3}}{k_{1,1,5}}; k_{2,1,4} = \frac{k_{1,1,4}}{k_{1,1,5}}; b_{2,1} = \frac{b_{1,1}}{k_{1,1,5}};\ k_{2,2,1} = \frac{k_{1,2,1}}{k_{1,2,5}}; k_{2,2,3} = \frac{k_{1,2,3}}{k_{1,2,5}}; k_{2,2,4} = \frac{k_{1,2,4}}{k_{1,2,5}}; b_{2,2} = \frac{b_{1,2}}{k_{1,2,5}}; \\
k_{2,3,1} = \frac{k_{1,3,1}}{k_{1,3,5}}; k_{2,3,3} = \frac{k_{1,3,3}}{k_{1,3,5}}; k_{2,3,4} = \frac{k_{1,3,4}}{k_{1,3,5}}; b_{2,3} = \frac{b_{1,3}}{k_{1,3,5}};\ k_{2,4,1} = \frac{k_{1,4,1}}{k_{1,4,5}}; k_{2,4,3} = \frac{k_{1,4,3}}{k_{1,4,5}}; k_{2,4,4} = \frac{k_{1,4,4}}{k_{1,4,5}}; b_{2,4} = \frac{b_{1,4}}{k_{1,4,5}}; \\
k_{3,1,1} = \frac{k_{2,1,1} - k_{2,4,1}}{k_{2,1,4} - k_{2,4,4}}; k_{3,1,3} = \frac{k_{2,1,3} - k_{2,4,3}}{k_{2,1,4} - k_{2,4,4}}; b_{3,1} = \frac{b_{2,1} - b_{2,4}}{k_{2,1,4} - k_{2,4,4}};\ k_{3,2,1} = \frac{k_{2,2,1} - k_{2,4,1}}{k_{2,2,4} - k_{2,4,4}}; k_{3,2,3} = \frac{k_{2,2,3} - k_{2,4,3}}{k_{2,2,4} - k_{2,4,4}}; \\
b_{3,2} = \frac{b_{2,2} - b_{2,4}}{k_{2,2,4} - k_{2,4,4}};\ k_{3,3,1} = \frac{k_{2,3,1} - k_{2,4,1}}{k_{2,3,4} - k_{2,4,4}}; k_{3,3,3} = \frac{k_{2,3,3} - k_{2,4,3}}{k_{2,3,4} - k_{2,4,4}}; b_{3,3} = \frac{b_{2,3} - b_{2,4}}{k_{2,3,4} - k_{2,4,4}};\ k_{4,1,1} = \frac{k_{3,1,1} - k_{3,3,1}}{k_{3,1,3} - k_{3,3,3}}; \\
b_{4,1} = \frac{b_{3,1} - b_{3,3}}{k_{3,1,3} - k_{3,3,3}};\ k_{4,2,1} = \frac{k_{3,2,1} - k_{3,3,1}}{k_{3,2,3} - k_{3,3,3}}; b_{4,2} = \frac{b_{3,2} - b_{3,3}}{k_{3,2,3} - k_{3,3,3}};\ C_1 = \frac{b_{4,1} - b_{4,2}}{k_{4,1,1} - k_{4,2,1}};\ C_3 = b_{4,1} - k_{4,1,1}\frac{b_{4,1} - b_{4,2}}{k_{4,1,1} - k_{4,2,1}}; \\
C_4 = b_{3,1} - k_{3,1,1}\frac{b_{4,1} - b_{4,2}}{k_{4,1,1} - k_{4,2,1}} - k_{3,1,3}\left(b_{4,1} - k_{4,1,1}\frac{b_{4,1} - b_{4,2}}{k_{4,1,1} - k_{4,2,1}}\right);\ C_5 = b_{2,1} - k_{2,1,1}C_1 - k_{2,1,3}C_3 - k_{2,1,4}C_4; \\
C_2 = r_1^2\left(z_1^2 C_3 + z_1 C_4 + C_5\right) - z_1 C_2 - z_1^2.
\end{gathered} \tag{6}$$

The relations, obtained for third and fourth-order root-fractional rational functions (3), are generally similar, but sophisticated. Let's consider now typical examples of using such relations to solve practical engineering problems related to the interpolation of numerical data.

It should be also pointed out, that convergence of proposed algorithm and error of calculation is generally depend on position of base points. The divergence is really possible, when the values of interpolated dataset in the region of minimum is closed to 0 because of possibility of negative values under the root. The condition $r_l(z) > 0$ is necessary to convergence the proposed algorithm, but in the case $r_l(z) \approx 0$ the divergence is possible, because the value $\frac{z^n + C_1 z^{n-1} + \ldots + C_{n-1} z + C_n}{C_{n+1} z^n + \ldots + C_{2n} z + C_{2n+1}}$ under the root can be negative. Corresponded example for homomorphic function will be considered in the next part of the paper. In this case the best solution is using of positive deviation for interpolated dataset. As for coefficients $C_1 \ldots C_{2n+1}$, with the positive values of r they are always calculated correct and additional limitation doesn't need. The approach, described here is similar, as for root-polynomial functions, which has been considered carefully on the mathematical point of view in papers (Melnyk, & Pochynok, 2020; Melnyk, & Pochynok, 2023).

3. Test examples for solving interpolation tasks using root-fractional-rational functions

3.1. First example

It is well known that the axial distribution of the magnetic field of a short focusing lens can be calculated with fairly high accuracy using the Galejs model (Schiller, Heisig, & Panzer, 1982; Szilagyi, 2012; Humphries, 2013; Kasper, & Hawkes, 2018; Grivet, 1972):

$$B_{z0} = \frac{1{,}26 \cdot 10^{-6} I_l N_l}{2 S_l \pi} \left(\frac{(z_l - P_l) + \frac{S_l}{2}}{\sqrt{\left(\frac{D_l}{3}\right)^2 + \left((z_l - P_l) + \frac{S_l}{2}\right)^2}} - \left(\frac{(z_l - P_l) - \frac{S_l}{2}}{\sqrt{\left(\frac{D_l}{3}\right)^2 + \left((z_l - P_l) - \frac{S_l}{2}\right)^2}} \right) \right), \tag{7}$$

where B_{z0} – magnetic field induction, I_l – lens current, N_l – number of turns, z_l – lens thickness, R_l – winding radius relative to the axis of symmetry of the lens, D_l – winding diameter, $D_l = 2R_l$, S_l – non-magnetic gap size, P_l – position of the lens symmetry plane along the longitudinal coordinate z relative to the point $z = 0$. The corresponding geometric parameters of the short focusing magnetic lens are shown in Fig. 1.

Interpolate the Galejs function (9) with a fourth-order root-fractional rational function and determine the coefficients of this function C_0 – C_9 for the following parameters of the lens model: $I_l = 3$ A, $N_l = 1500$, $z_l = 40$ mm, $R_l = 60$ mm, $D_l = 40$ mm, S_l – 20 mm, P_l – 0,1 m. Calculate the interpolation error, determine the maximum and average value of the interpolation error in the interval [0.1 m; 0.13 m].

The result of the interpolation of dependence (7) for the given parameters of the focusing lens by the fourth-order root-fraction-rational function of is shown in Fig. 2. The position of the basic calculation points z_1 – z_9 has been chosen as follows: $z_1 = 0.1$ m, $z_2 = 0.1015$ m, $z_3 = 0.103$ m, $z_4 = 0.1075$ m, $z_5 = 0.112$ m, $z_6 = 0.115$ m, $z_7 = 0.121$ m, $z_8 = 0.127$ m, $z_9 = 0.129$ m. The corresponding interpolation results are shown in Fig. 2. From the presented results it is clear that the interpolation error relative to the basic calculated values of the Galejs function (9) does not exceed 0.125%. Under this condition, the relative average error in the interval [0.1 m; 0.13 m] is 0.0143%.

The coefficients of the fourth-order root-fractional-rational function, calculated from relations (8), for this example have the following values:

$C_1 = –0.47038$ m; $C_2 = 0.08212$ m^2; $C_3 = –6.2884 \cdot 10^{-3}$ m^3; $C_4 = – 1.7746 \cdot 10^{-4}$ m^4; $C_5 = –4.6373 \cdot 10^4$ m^{-4};
$C_6 = 1.9268 \cdot 10^4$ m^{-3}; $C_7 = –3.04108 \cdot 10^3$ m^{-2}; $C_8 = 216.733$ m^{-1}; $C_9 = – 5.91715$.

3.2. Second example

Describe the error integral erf(z) (Wentzel, *& Ovcharov, 2002;* Gubner, 2006) by a second-order root-fractional-rational function. Interpolate using the following reference points: $z_1 = 0.01$; $z_2 = 0.5$; $z_3 = 1.1$; $z_4 = 1.9$; $z_5 = 3$.

The interpolation result, has been obtained for the selected parameters of error function is shown in Fig. 3.

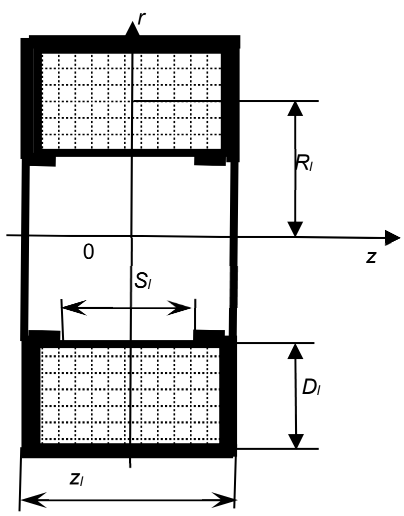


Fig. 1. Geometric parameters of the short focusing magnetic lens

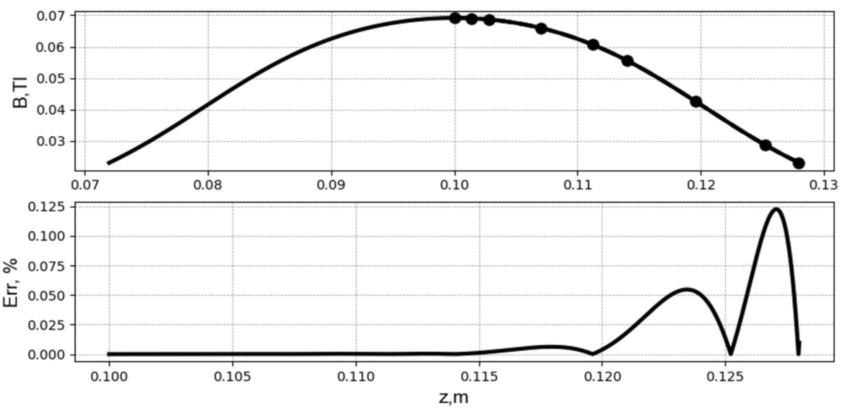


Fig. 2. The result of interpolation of the axial magnetic field distribution of a short focusing lens with a fourth-order root-fractional-rational function for first example

From the given graphical dependencies, it is clear that the maximum value of the interpolation error is observed near the value of the function argument $z = 0$ and is about 0.4 %. The average interpolation error is 0.1 %. The coefficients of the second-order root-fractional-rational function, calculated from relations (6), have the following values: $C_1 = 0.07482$; $C_2 = -6.9629 \cdot 10^{-4}$; $C_3 = 1.1851$; $C_4 = -0.88147$; $C_5 = 1.20276$.

From the point of view of further research of the described interpolation methods, it is interesting that for a root-fractional-rational function of the second order an inverse function $z(r) = r^{-1}(z)$ is existed. The corresponding analytical relationship for the functional dependence $z(r)$ can be written in the following form:

$$z(r) = \frac{C_1 - r^2 C_4}{2 \cdot \left(1 - r^2 C_3\right)} \pm \sqrt{\frac{\left(C_1 - r^2 C_4\right)^2}{4 \cdot \left(1 - r^2 C_3\right)^2} - \frac{C_2 - r^2 C_5}{1 - r^2 C_3}}. \quad (8)$$

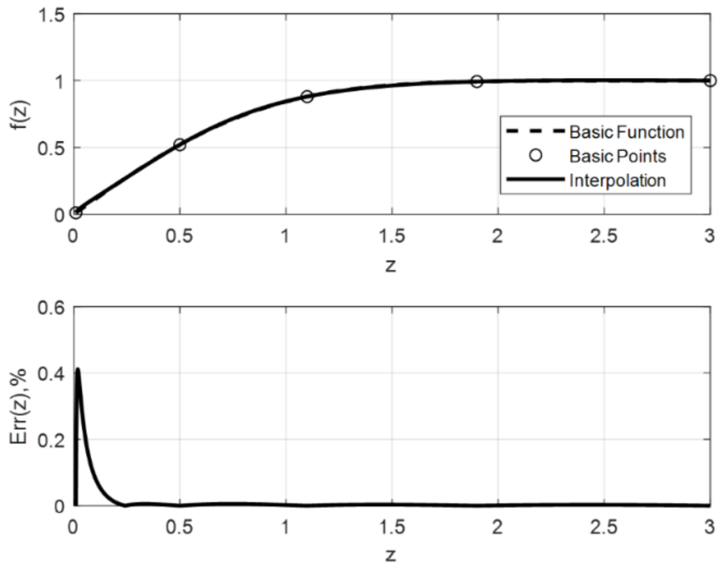


Fig. 3. Interpolation of the function erf(z) by a second-order root-fractional-rational function for second example

For example, calculating the values of the inverse function in the error function $z(r) = \mathrm{erf}^{-1}(z)$ using formula (8) makes it possible to construct an effective generator of random numbers distributed according to a normal law. It should also be noted that relation (8) is universal and can be used for any set of numerical data. Only the coefficients of the root-fractional-rational function $C_1 - C_5$ change.

It should be noted here that for a homomorphic single-valued functions $r(z)$, the sign – in front of the square root in relation (8) is an incorrect solution. It yields negative values of the inverse function $z(r)$, and by this reason is discarded. The corresponding dependencies for this example are shown in Fig. 4. On the other hand, for multi-valued polymorphic functions, both solutions should be considered, each of which is valid for the corresponding values of the argument of the original function $r(z)$.

3.3. Third example

Describe a П- shaped fuzzy logic membership function defined by the following arithmetic-logical expressions (Zadeh, & Aliev, 2018):

$$F_S(z;a,b) = 1 \cdot (z \le a) + \frac{1 + \cos\left(\dfrac{\pi \cdot (z - b)}{b - a}\right)}{2} \cdot (a < z \le b) + 0 \cdot (z > b);$$

$$F_Z(z;c,d) = 0\cdot(z \le c) + \frac{1+\cos\left(\frac{\pi\cdot(z-c)}{d-c}\right)}{2}\cdot(c < z \le d) + 1\cdot(z > d); \tag{9}$$
$$F_\Pi(z;a,b,c,d) = F_Z(z;c,d)\,F_S(z;a,b), a < b < c < d.$$

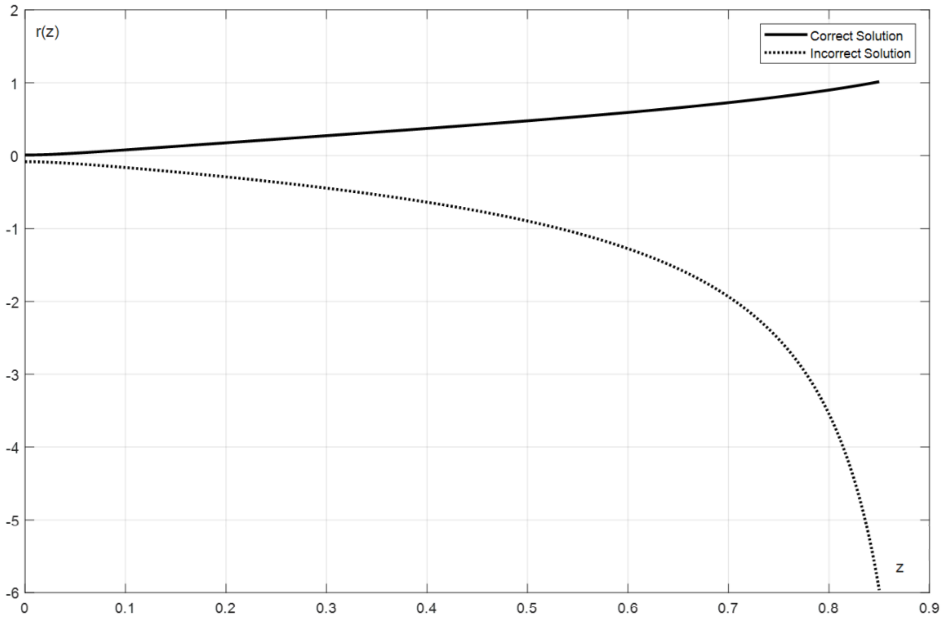


Fig. 4. Correct and incorrect solutions for inverse function *z*(*r*) corresponding to relation (8) for second example

Calculations have to be carried out for the following values of the parameters of function: $a = 1$, $b = 2$, $c = 3$, and $d = 4$. Interpolation has to be provided using a fourth-order root-fractional-rational function on the interval of values of the argument z [1,5; 3,5].

To solve this task, the following reference interpolation points have been selected: z_1 = 1.5; z_2 =1.6; z_3 = 1.9; z_4 = 2.0; z_5 = 2.5; z_6 = 2.9; z_7 = 3.15; z_8 = 3.4; z_9 = 3.5.

The result of interpolation for a given set of numerical data is presented in Fig. 5. The interpolation error in this case does not exceed 1.8 %, and the average error over the entire interpolation interval is 0.5326%. The coefficients of the fourth-order root-fractional rational function for this example have the following values: C_1 = – 9.987; C_2 = 35.1493; C_3 = – 51.006; C_4 = 26.148; C_5 = 2.5061; C_6 = –25.134466; C_7 = 91.95123; C_8 = –145.085; C_9 = 84.18057.

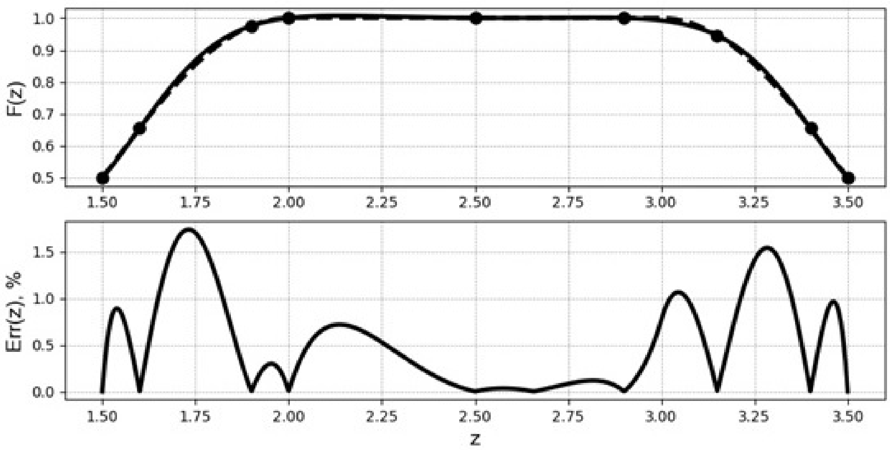


Fig. 5. Interpolation of the П-shaped fuzzy logic membership function specified by arithmetic-logical relations (9) for third example

4. Features of the software implementation of the developed interpolation algorithm

The program code intended for calculating the coefficients of root-fractional-rational functions using obtained corresponded relations, similar to (6), has been written in the Python programming language using advanced developed libraries for performing mathematical calculations and displaying graphical data from the pip module. Therefore, additional system tools have been used to connect the corresponding advanced programming possibilities. In particular, a virtual environment has been created using the system commands of the Windows 10 operating system to define corresponding virtual variables. After that, in this virtual environment, virtual memory was allocated to create a virtual disk. To perform these system functions, the venv software module has been used (Robbins*,* 2022; McKinney, 2023; Chollet, 2022; Lutz, 2013).

The main features of writing program code were as follows.

1. In general, the program is written using well-known means of structural, declarative and functional programming in one program module.

2. Mathematical operations with data arrays have been performed using advanced developed mathematical libraries of the numpy module (Robbins*,* 2022; McKinney, 2023; Chollet, 2022; Lutz, 2013). To operations with matrices, effective matrix programming tools have been used (Robbins*,* 2022; McKinney, 2023; Chollet, 2022; Lutz, 2013). Using of such approach is greatly simplifies writing, reading and improving program code.

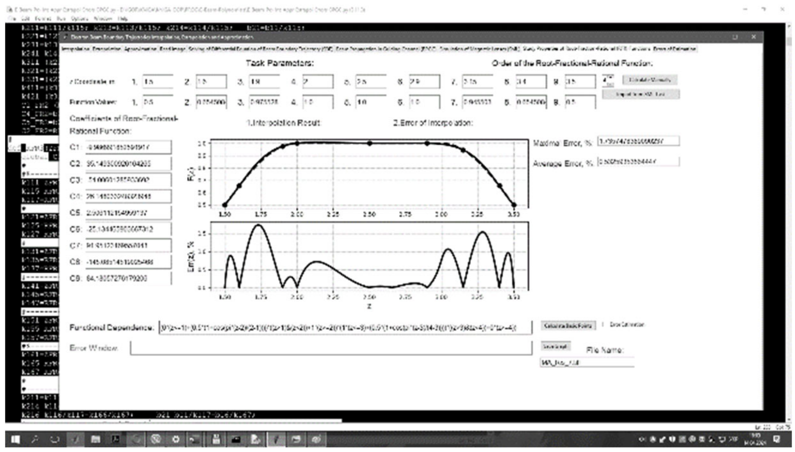
3. To display function graphs in the created graphical user interface window, developed graphic libraries of the matplotlib module have been used (Robbins*,* 2022; McKinney, 2023; Chollet, 2022; Lutz, 2013).

4. Since the created software package is primarily intended to solve a wide range of practical problems related to the simulation of electron beams, to systematize the functions of the program and to simplify the work of users with it, each of these realized functions is implemented on separate tabs.

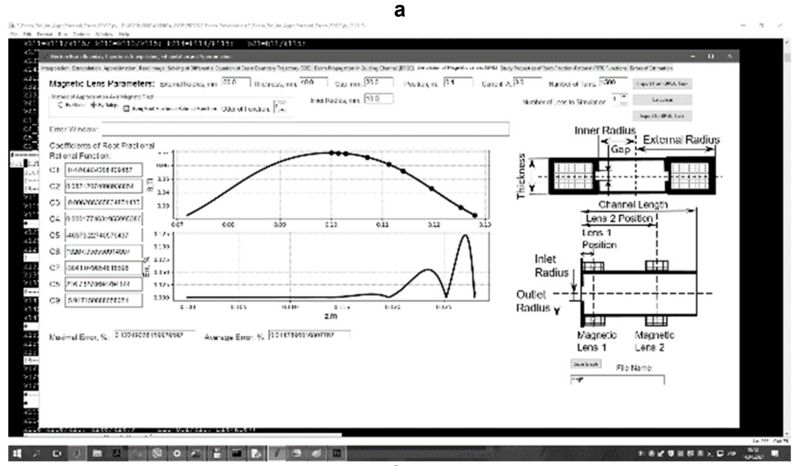
5. To solve complex problems of simulation electron beams, data exchange between separate, local tasks is provided. At the software level, data exchange between internal functions is implemented through the directive of data description global (*Robbins,* 2022; McKinney, 2023; Chollet, 2022; Lutz, 2013). For example, the results of calculating the magnetic field of a focusing lens and their interpolation can be used to calculate the boundary trajectory of an electron beam. The algorithm for calculating the trajectory of electron beams as they propagate in an ionized gas, have been realized in this software, was considered and described in detail in (Robbins*,* 2022; McKinney, 2023; Chollet, 2022; Lutz, 2013).

6. The obtained graphic dependencies can be saved in a file on the local computer disk in the tiff graphic format. For this, the standard functions of the matplotlib module graphic library are used (*Robbins,* 2022; McKinney, 2023; Chollet, 2022; Lutz, 2013). To create and record a graphic file, the corresponding elements of the graphic interface are used on each tab, in particular, the event processing button and the text window for entering the file name (Robbins*,* 2022; McKinney, 2023; Chollet, 2022; Lutz, 2013).

The Study Properties of Root-Fractional-Rational Function tab, where the properties of mathematical functions describing sets of data for interpolation are studied, is shown in Fig. 6 a, and the Simulation of Magnetic Lens tab, on which the axial distribution of the magnetic field of short focusing lenses is interpolated, presented, correspondently, in Fig. 6 b.



a



b

Fig. 6. Tabs Study Properties of Root-Fractional-Rational Function (a) and Simulation of Magnetic Lens (b) of the graphical user interface of the developed software package, has been elaborated for working with root-polynomial and root-fractional-rational functions

(Melnyk et al., 2023). Screen copies

Discussion and conclusions

Theoretical studies and test computer experiments with the developed software have shown that when using root-fractional rational functions of the second, third and fourth orders, it is possible, with high accuracy, to interpolate the values of various functions, in particular, those that have a high stiff degree. Often for this class of functions there are no simple analytical expressions, and they can only be described by numerical methods associated with solving complex systems of integral or differential equations. Testing of the developed interpolation algorithm have been carried out for a wide class of mathematical functions of various types used in various fields of engineering modeling, including problems in electrical engineering and electronic optics, in probability theory and mathematical statistics, as well as in applied problems of fuzzy logic theory. These test studies have shown that even for functions with a high stiff degree, the maximum interpolation error does not exceed several percent, and in most cases it can be range of a fraction of a percent. The theoretical results obtained in the performance of this work may be of interest to a wide range of mathematicians who are involved in functional analysis and mathematical processing of large data sets, in particular, in methods of numerical and statistical analysis.

Authors' contribution: Melnyk Igor – conceptualization and main ideas; Pochynok Alina – mathematical calculations and finding analytical dependences for coefficients of root-fractional-rational functions; Skrypka Mykhailo – realization the conception of simulation in virtual environment and writing Python programming code; Olga Demyanchenko – study the mathematical properties of root-fractional-rational functions and particularities of its convergence.

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МЕТОД ІНТЕРПОЛЯЦІЇ З ВИКОРИСТАННЯМ КОРЕНЕВО-ДРОБОВО-РАЦІОНАЛЬНИХ ФУНКЦІЙ РІЗНОГО ПОРЯДКУ

Розглянуто можливість інтерполяції різних видів математичних функцій з використанням коренево-дробово-раціональних функцій різного порядку, а саме другого, третього та четвертого. Показано, що коренево-дробово-раціональні функції надають точну інтерполяцію з малою похибкою навіть жорстких аналітичних залежностей. Розглянуто коренево-дробово-раціональні функції від другого до четвертого порядку та наведено відповідні аналітичні співвідношення для визначення поліноміальних коефіцієнтів у чисельнику та знаменнику. Описано аналітичний метод розв'язування системи лінійних алгебраїчних рівнянь для пошуку коефіцієнтів коренево-дробово-раціональних функцій, оснований на відомому методі виключення змінних Гаусса – Зейделя, який відрізняється тим, що аналітичні обчислення проведені ітерайційно та, для систематизації поставленої математичної задачі, усі коефіцієнти пронумеровані. Нумерація змінних містить три коефіцієнти: номер ітерації, номер рівняння та номер змінної у цьому рівнянні. Такий систематизований підхід до розв'язування задачі у значній мірі спростив проведення аналітичних розрахунків. Також доведено, що кількість необхідних точок для інтерполяції відповідає значенню 2n+1, де n – порядок коренево-дробово-раціональної функції. Наведено приклади інтерполяції різних функцій для задач електродинаміки, моделювання магнітних лінз, імовірнісних задач і задач нечіткої логіки, а також визначено похибку інтерполяції для всіх розглянутих прикладів. Зокрема, подано результати інтерполяції для розподілу магнітного поля короткої лінзи, для функції помилок erf(x), для експоненціальної функції, для розподілу Бернуллі з різними коефіцієнтами та для П-подібної функції належності. Усі припущення теоретичного аналізу перевірені за допомогою оригінально розробленого програмного забезпечення для ЕОМ, створеного засобами мови програмування Python. Для більшості розглянутих прикладів отримана похибка інтерполяції становить менше кількох відсотків. Наведено також графічні результати тестування, отримані для запропонованого методу інтерполяції.

Ключові слова: ***інтерполяція, коренево-дробово-раціональна функція, задачі електродинаміки, функції теорії ймовірностей, нечітка логіка, функції належності.***

Автори заявляють про відсутність конфлікту інтересів. Спонсори не брали участі в розробленні дослідження; у зборі, аналізі чи інтерпретації даних; у написанні рукопису; в рішенні про публікацію результатів.

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