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Геометрична рекурентність процесу неоднорідної гауссівської авторегресії

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Geometric recurrence of inhomogeneous Gaussian autoregression process

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У цій роботі досліджено гауссівську авторегресію вигляду $X_{n+1} = \alpha_{n+1}X_n + W_{n+1}$ з неоднорідними в часі центрованими нормально розподіленими приростами W_n і коефіцієнтами стиску α_n. Отримано верхню межу для математичного сподівання експоненційного моменту першого повернення ланцюгу в компакт $[-c; c]$ та верхню межу сподівання функції коефіцієнтів стиску від вказаного моменту.

Ключові слова: гауссівська авторегресія, неоднорідний ланцюг Маркова

In this paper we study Gaussian autoregression model of the form $X_{n+1} = \alpha_{n+1}X_n + W_{n+1}$. It has time-inhomogeneous centered normal increments W_n and control ratios α_n. We obtained upper bounds for expectation of exponential return time to the compact $[-c; c]$ and for expectation of the function of compressing ratios and the mentioned moment.

Key Words: Gaussian autoregression, Inhomogeneous Markov chain

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1 Introduction

Gaussian autoregression process is a base for models with a wide range of practical use. Sometimes dynamic systems we study using this model are not homogeneous in time. Instead of being constant, some controls can fluctuate around the known value. This makes it harder to establish the bounds for the variables of interest, such as return times.

In this paper we study Gaussian autoregression of the form $X_{n+1} = \alpha_{n+1}X_n + W_{n+1}$, where (W_n) are centered and normally distributed. In our case the controls fluctuate in the interval $(0; 1)$ with rare exceptions. We introduce computable upper bounds for exponential return time to the compact $[-c; c]$ and provide three simple examples of its use by applying numeric and random sequences as (α_n) controls or modifying (W_n) with random variance.

2 Main result

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and define on it a time-inhomogeneous Gaussian autoregression process $(X_n)_{n\geqslant 0}$, controlled by a sequence of compression ratios $(\alpha_n \geqslant 0)_{n\in\mathbb{N}}$:

$$X_{n+1} = \alpha_{n+1}X_n + W_{n+1}, \tag{1}$$

where $W_n \sim N(0, \sigma_n^2)$ are the independent normal increments. We denote their distribution and density functions as Φ_n and ϕ_n respectively. We also use the following notation: $P_x^n(A)$ is a probability measure generated by $(X_{n+k})_{k\geqslant 0}$ and $\mathbb{E}_x^n$ is the corresponding expectation.

We aim to develop stability conditions for the defined process. Our goal is to show that for certain family of constants $c > 0$ and for all $x \in \mathbb{R}$ we can bind the expectation $\mathbb{E}_x^n\left[\psi_{\sigma_C}^{-1}\right]$, where $\sigma_C = \inf\{n \geqslant 1 | X_n \in C\}$ is the first return time to the set $C = [-c; c]$ and ψ_n is dependent on α_n. Consider the following theorem.

Theorem. *Assume that the following conditions hold:*

1. Exists $\sigma > 0$ such that

$$\sigma = \sup_{n\in\mathbb{N}} \sigma_n \tag{2}$$

2. Exists $\epsilon \in (0; 1)$ such that only a finite number of $(\alpha_n)_{n\in\mathbb{N}}$ lies outside the half-open interval $(0; \epsilon]$.

Define

$$\psi_n := \prod_{k=1}^{n}\left(\alpha_k + \frac{1-\epsilon}{2}\right) \quad (3)$$

Then for any constant $c \geqslant \max\{1, \frac{2K}{1-\epsilon}\}$ *and for all* $x \in \mathbb{R}$ *and for all* $n \geqslant 0$:

$$\mathbb{E}_x^n\left[\psi_{\sigma_C}^{-1}\right] \leqslant 1 + |x| + \left(K - \frac{1-\epsilon}{2}\right)\mathbb{I}_C(x), \quad (4)$$

where

$$K = \frac{2\sigma}{\sqrt{2\pi}} + 1. \quad (5)$$

Proof. *We denote* $\delta = (1-\varepsilon)/2$. *Then for all* $\alpha_n \in (0;\epsilon]$ *holds* $\alpha_n + \delta < 1$. *Thus* $\forall n$

$$\prod_{k=1}^{n}\max\{\alpha_k + \delta, 1\} \geqslant 1 \quad (6)$$

From (6) immediately follows that

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}\max\{\alpha_k+\delta,1\}\right)(1-\alpha_n-\delta)^+ = \infty \quad (7)$$

Next, we show that exists

$$G = \sup_{n\geqslant 0}\mathbb{E}_x^n\left[W_n^+\right] < \infty \quad (8)$$

Let us calculate the expectation in (8):

$$\begin{aligned} &\int_0^{+\infty} y\Phi_n(dy) = \\ &= \frac{1}{\sqrt{2\pi}\sigma_n}\int_0^{+\infty} y\exp\left(-\frac{y^2}{2\sigma_n^2}\right)dy = \\ &= \frac{1}{\sqrt{2\pi}\sigma_n}\int_0^{+\infty}\exp\left(-\frac{y^2}{2\sigma_n^2}\right)d\left(\frac{y^2}{2}\right) = \\ &= \frac{1}{\sqrt{2\pi}\sigma_n}\int_0^{+\infty}\exp\left(-t/\sigma_n^2\right)dt = \\ &= \frac{\sigma_n}{\sqrt{2\pi}} \end{aligned} \quad (9)$$

From the first condition of the theorem we obtain:

$$\begin{aligned} G &= \sup_{n\geqslant 0}\mathbb{E}_x^n\left[W_n^+\right] = \\ &= \sup_{n\geqslant 0}\frac{\sigma_n}{\sqrt{2\pi}} = \frac{\sigma}{\sqrt{2\pi}} \end{aligned} \quad (10)$$

Since (7) and (8) hold true, we can apply Theorem 1 from [1] and get the desired boundary for $\mathbb{E}_x^n\left[\psi_{\sigma_C}^{-1}\right]$: *for any constant* $c \geqslant \max\{1, K/\delta\}$, *for all* $x \in \mathbb{R}$, $t \geqslant 0$ *holds:*

$$\mathbb{E}_x^n\left[\psi_{\sigma_C}^{-1}\right] \leqslant 1 + |x| + \left(K - \frac{1-\epsilon}{2}\right)\mathbb{I}_C(x), \quad (11)$$

where $C = [-c; c]$ *and* $K = \frac{2\sigma}{\sqrt{2\pi}} + 1$. $\square$

We can go a bit further by setting a stricter condition for (α_n). *Observe:*

Corollary. *Suppose* $0 < \varepsilon = \sup\{\alpha_n\} < 1$. *Set* $\psi = (\frac{\varepsilon+1}{2})^{-1}$, *then for all* $x \in \mathbb{R}$ *holds:*

$$\mathbb{E}_x^n\left[\psi^{\sigma_C}\right] \leqslant 1 + |x| + \left(K - \frac{1-\epsilon}{2}\right)\mathbb{I}_C(x) \quad (12)$$

Proof. *The setup of* ψ *obviously meets the condition of the Corollary 1 from [1]:*

$$\psi_n = \prod_{k=1}^{n}(\alpha_k + \delta) \leqslant \psi^{-n}. \quad \square \quad (13)$$

Let us now take a look on examples.

3 Examples

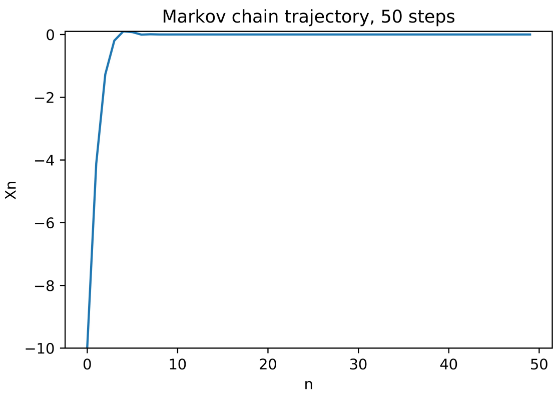
In this section we provide results of simulations to approve the Theorem statement. We consider 3 examples. The code is available via the link: https://github.com/gamble27/geometric-recurrence-inhomogenous-gaussian-autoregression

Example 1. Control by numeric sequences. *Consider first the simplest model, where the control is maintained by clearly decreasing sequences of variances and compression ratios:*

$$X_{n+1} = \alpha_{n+1}X_n + W_{n+1},$$

where $X_0 = -10$, $\alpha_n = \frac{1}{n+1}$, $\sigma_n = 2^{-n}$. *From our initial conditions we obtain* $\epsilon = \sigma = \frac{1}{2}$, *hence* $c = 1.67 \geqslant 4\left(\frac{1}{\sqrt{2\pi}} + 1\right)$

We generate 10^4 *trajectories, which look like the one on image 1.*



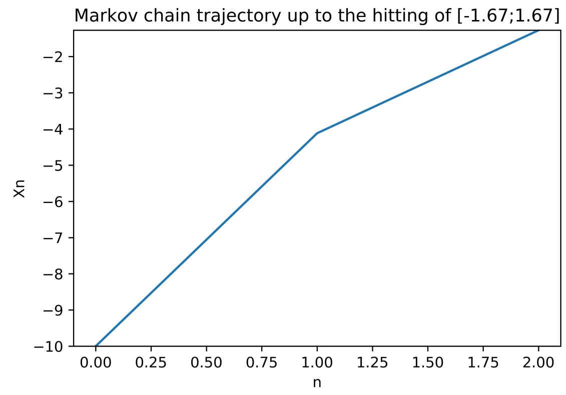


Fig. 1: Single trajectory of the Markov chain

We would also like to take a look on distribution of the first return time. See image 2:

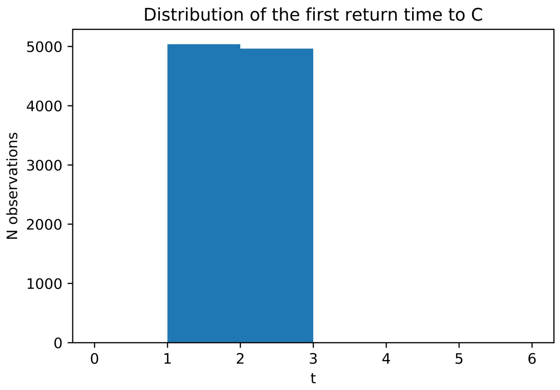


Fig. 2: Distribution of the first return time of the chain to the compact $[-1; 1]$

Lastly, we would like to vary X_0 values and see how the exponential moment of first return would act and compare it to the Corollary. In this case we get: $\psi = \frac{4}{3}$, $\quad K = \frac{1}{\sqrt{2\pi}} + 1$ and

$$\mathbb{E}_x^n \left(\frac{4}{3}\right)^{\sigma_C} \leqslant 1 + |x| + \left(\frac{1}{\sqrt{2\pi}} + \frac{1}{4}\right) \mathbb{I}_C(x)$$

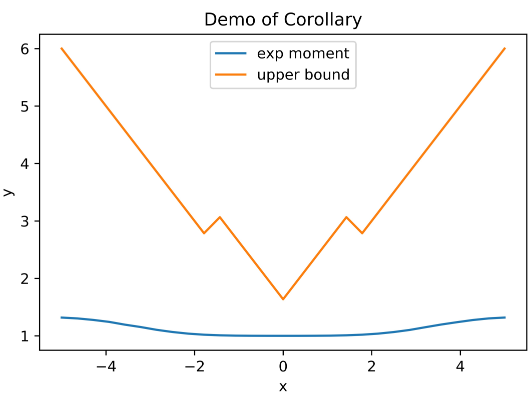


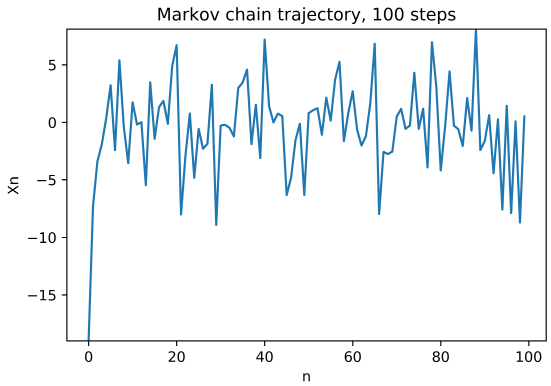
Fig. 3: Demonstration of the Corollary

As we can see on the picture 3, the upper bound works correctly: the horizontal axis x is a chosen starting point X_0, while the vertical axis y are functions of the argument X_0 - exponential moment of the first return time to C and its upper bound.

Example 2. Control with random variance.

We take the next step and introduce some randomness in the model controls. In equation 1 we set compress ratios again as a sequence: $\alpha_n = \frac{1}{n+1}$ and $\sigma_n \sim U[\{5, 3, 1\}]$. We take $\epsilon = \frac{1}{2}$ and $\sigma = 5$. Let us choose c:

$$c = 16 \geqslant 4\left(\frac{10}{\sqrt{2\pi}} + 1\right)$$



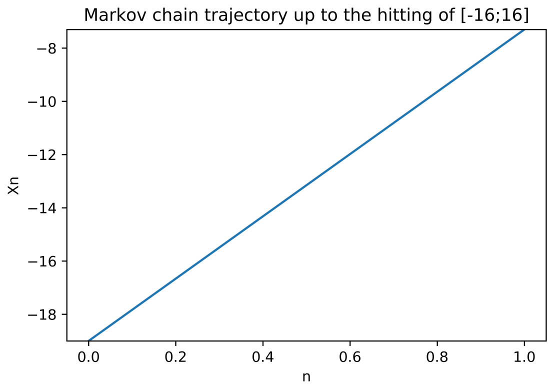


Fig. 4: Single trajectory of the Markov chain

In the simulation we vary the staring point X_0 from 0 to 25; sample trajectories can be found on picture (6). Then the Corollary holds again for $\phi = \frac{2}{3}$ and $K = \frac{10}{\sqrt{2\pi}} + 1$:

$$\mathbb{E}_x^n \left(\frac{2}{3}\right)^{\sigma_C} \leqslant 1 + |x| + \left(\frac{10}{\sqrt{2\pi}} - 3\right) \mathbb{I}_C(x) \quad (14)$$

We can clearly see it on the picture (5):

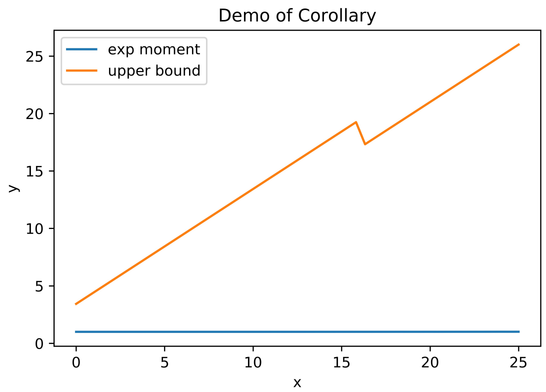


Fig. 5: Demonstration of inequality (14)

Example 3. Control with random compression. *In this example we consider the random coefficient autoregressive process (RCA), defined in example 2.1.7 [2]:*

$$X_{n+1} = \alpha_{n+1} X_n + W_{n+1},$$

where $\alpha_n \sim U[0.1; 0.9]$, $\quad \sigma_n = 1 + 0.03\frac{n}{n+1}$. *Thus we have constants* $\sigma = 1.03$ *and* $\epsilon = 0.9$. *It gives us*

$$c = 17 \geqslant 20\left(\frac{2.06}{\sqrt{2\pi}} + 1\right)$$

Again we generate 10^4 *trajectories to explore the behavior of given Markov chain.*

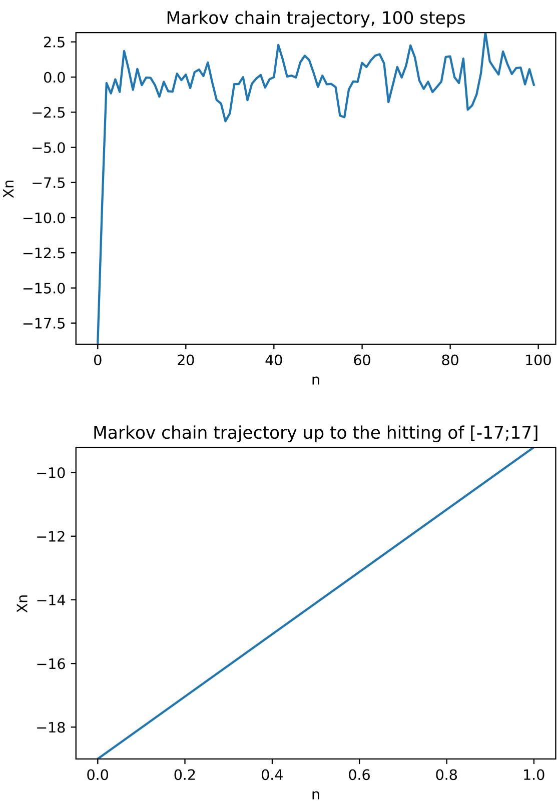


Fig. 6: Single trajectory of the Markov chain

We can clearly see that the return time to the set $C = [-17; 17]$ *is exponentially bounded. To show that more explicitly, we generate a pool of trajectories, which start at* $X_0 = -19$ *and consider the return time distribution. Let us take a look on the time of the first return to set* $C = [-17; 17]$. *For this, we run* 10^4 *trajectory simulations and figure out the first return time to* C.

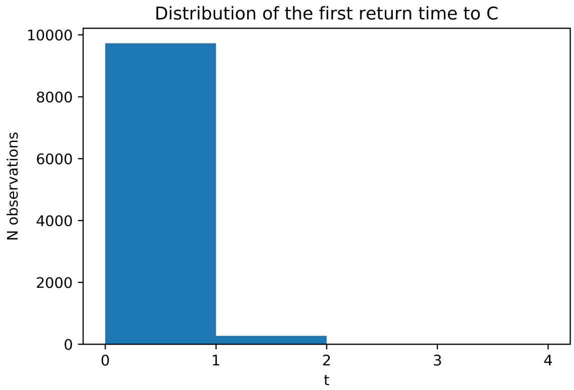


Fig. 7: Distribution of the first return time of the chain to the compact $[-1; 1]$

On picture (7) we can see the distribution of the return time to C. Now we vary the starting point X_0 *in simulation and show that Corollary holds on the picture (8): on the horizontal axis we can see the starting point and on vertical - exponential moment and its upper bound, just as it holds theoretically. In this example to obtain* $\phi = \frac{20}{19}$ *and* $K = \frac{2.06}{\sqrt{2\pi}} + 1$. *Then the following holds:*

$$\mathbb{E}_x^n \left(\frac{20}{19}\right)^{\sigma_C} \leqslant 1 + |x| + \left(\frac{2.06}{\sqrt{2\pi}} + \frac{19}{20}\right) \mathbb{I}_C(x) \quad (15)$$

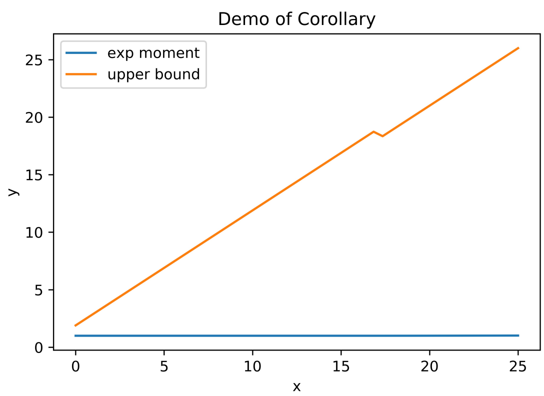


Fig. 8: Demonstration of inequality (15)

4 Conclusion

In this paper we developed the upper bound for time-inhomogeneous Gaussian autoregression in *general and in a simplified form under stricter condition. We have shown that our results can be applied to simple models with known controls as well as to their random variance and RCA extensions. However, developed bound is not very sharp, except its value at 0.*

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