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POINT CONTROL OF LINEAR HYPERBOLIC INTEGRO-DIFFERENTIAL SYSTEMS

The work focuses on the optimal control of distributed systems described by linear hyperbolic integro-differential equations with partial derivatives and Volterra-type integral components. Such integro-differential equations are a standard subject in applied mathematics and frequently arise in studies of processes in viscoelastic media (such as amorphous polymers, semi-crystalline polymers, biopolymers, metals at very high temperatures, bituminous materials, and more). The primary goal is to prove the existence of optimal control for distributed systems modeled by these equations.

We apply methods of functional analysis and the theory of distributions. The study is conducted in specially defined Hilbert spaces, where the control operator in the system's right-hand side includes generalized Dirac delta functions. We establish the main result on the existence of optimal control based on the theory of a priori estimates in negative norms, building on the foundational work of Yu. M. Berezansky and further developed by V. P. Didenko, S. I. Lyashko, and their colleagues.

We formulate an optimal control problem where the control of the system is governed by a control operator appearing in the right-hand side of the initial-boundary value problem. The control operator acts into spaces of generalized functions, modeling pointwise control of the system. Further, we propose appropriate Hilbert spaces for the problem's operator and the space of admissible controls. Moreover, we provide a priori estimates in negative norms, define generalized solutions, and prove the well-posedness of the initial-boundary value problem. Finally, a general theorem on the existence of optimal control is established and the well-definedness and weak continuity of the control operator are proved. Based on these statements, we formulate a theorem on the existence of optimal control for the problem, imposing restrictions on the admissible control set and the quality criterion to be minimized.

We prove a theorem that provides sufficient conditions for the existence of optimal control for the system considered. In particular, we demonstrate that the control operator corresponding to pointwise control is well defined and weakly continuous from the control space to the space of the right-hand sides.

K e y w o r d s: ***integro-differential equations, linear hyperbolic equation, Volterra operator, viscoelastic media, optimal control, point control.***

AMS 2020 classification: 34K06, 34K10, 34K35, 35A01, 35A02.

Introduction

Partial differential and integro-differential models of hyperbolic type for a long time have been standard objects of study in applied mathematics. This is due to the wide class of physical processes described by these equations. For certain systems, a characteristic feature is that the forces arising within the body depend not only on the current values of the deformation and/or velocity gradient, but also on the entire history of motion. For example, it is known that the behavior of certain viscoelastic materials (such as polymers, suspensions, emulsions) exhibits this "memory" property. Consequently, some models for viscoelastic materials lead to motion equations that take the form of a linear hyperbolic differential equation perturbed by a dissipative integral term of the Volterra type (Cannarsa & Sforza, 2011; Conti, Gatti, & Pata, 2008).

The list of viscoelastic materials with such properties includes (but is not limited to) amorphous polymers, semicrystalline polymers, biopolymers, metals at very high temperatures, bituminous materials, and others (Lakes, 1998; Kelly, 2013).

A lot of research has been performed to study various problems for linear and nonlinear integro-differential equations of the hyperbolic type. Starting from classical monographs (Renardy, Hrusa, & Nohel, 1987; Fabrizio & Morro, 1992; Duvaut & Lions, 1976), which provided the models and physical justification, to more recent works (Loreti & Sforza, 2010; Diagana, 2014; Engler, 1991; Gan & Yin, 2015; Karaa, Pani, & Yadav, 2015; Saedpanah, 2014; Cannarsa & Sforza, 2011; Shen, Yang, & Liu, 2014; Conti et al., 2008; Dafermos, 1970; Hulianytskyi, 2020; Anikushyn & Hranishak, 2018). A more detailed review of the cited works is provided in (Anikushyn, Lyashko, & Samosonok, 2024). We will not rewrite it here once again.

In our work, we consider the initial-boundary value problem

$$Lu := u_{tt} - \sum_{i,j=1}^{n} \left(a_{ij}(x)u_{x_j}\right)_{x_i} + \sum_{i=1}^{n} a_i(x)u_{x_i} + a(x)u + \int_0^t \sum_{i=1}^{n} \left(K_i\left(x,t,\tau\right)u_{x_i}\left(x,\tau\right)\right)_{x_i} d\tau = f, \tag{1}$$

with homogeneous Dirichlet initial and boundary conditions

$$u|_{\partial\Omega} = 0, \tag{2}$$

$$u|_{t=0} = u_t|_{t=0} = 0. \tag{3}$$

Here, $u(x,t)$ is a function that describes the state of the system in the cylindrical region $Q = \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded simply connected domain with a sufficiently regular boundary $\partial\Omega$.

A similar initial-boundary value problem for a differential operator was considered in the monograph (Lyashko, 2002), where a priori inequalities in negative norms were obtained for the operator L. Afterwards, the problem (1)–(3) was studied in (Anikushyn et al., 2024). In the cited paper a priori inequalities in negative norms and generalized solvability theorems were obtained for the problem with the integro-differential operator (1).

We outline that the indicated approach is generally an effective method for studying optimization problems for differential and integro-differential equations. Historically, the ideas of estimates in negative norms appeared in works from the 1950s (Berezanskyi, 1968) and were further developed, particularly in the works of S. I. Lyashko and his colleagues (Lyashko, 2002; Klyushin et al., 2011; Anikushyn & Nomirovskyi, 2011).

Since then, this method has been extensively employed to investigate various issues related to well-posedness, optimal control, controllability, and numerical methods for systems described by partial differential equations. For instance, the studies (Lyashko & Nomirovskii, 2003b; Nomirovskii, 2004, 2007; Semenov, 2005; Semenov & Denisov, 2005; Tymchyshyn & Nomirovskii, 2021) address the problems of generalized solvability and optimization of parabolic systems with diverse boundary conditions in different domains, such as those with thin inclusions. The unique solvability and trajectory/final controllability of hyperbolic and pseudohyperbolic systems in various classes of distributions have been analyzed in (Lyashko, Nomirovskii, & Sergienko, 2001; Nomirovskii, 1999, 2006d, 2006c). Galerkin-type methods for the approximate solution of boundary value problems for parabolic and hyperbolic equations have been examined in (Nomirovskii, 2006b, 2006a). The papers (Lyashko & Nomirovskii, 2003a; Lyashko & Semenov, 2001; Nomirovskii & Vostrikov, 2016; Sergienko et al., 2023) discuss various optimization issues, such as those arising in domains with cuts, etc. Lastly, the paper (Semenov & Denisov, 2023) explores the controllability issues of parabolic-hyperbolic systems.

On the other hand, the works (Anikushyn, 2014; Anikushyn & Hulianytskyi, 2014; Anikushyn et al., 2024) as well as (Анікушин & Андарал, 2024; Anikushyn, 2024; Гуляницький & Семенов, 2012) use the method of a priori estimates to prove the well-posedness of various types of integro-differential equations.

In particular, the papers (Анікушин & Андарал, 2024) and (Гуляницький & Семенов, 2012) deal with Dirichlet problem for pseudo-parabolic integro-differential equation

$$a(x)u_t - \sum_{i,j=1}^{n} (a_{ij}(x)u_{tx_j})_{x_i} + b(x)u - \sum_{i,j=1}^{n} (b_{ij}(x)u_{x_j})_{x_i} + \int_0^t \sum_{i=1}^{n} (K_i(x,t,\tau)u_{x_i})_{x_i} \, d\tau = f(x,t),$$

and provide results a priory inequalities as well as results on well-posendess and existence of an optimal control of the corresponding systems for control operators operating in the spaces of generalized functions.

In (Anikushyn & Hulianytskyi, 2014) the authors investigate BVP for a parabolic integro-differential equation of the form

$$u_t + Au + Iu = f,$$

with Dirichlet boundary conditions. Here

$$Au = -\sum_{i,j=1}^{n} (a_{ij}(x)u_{x_j})_{x_i} + \sum_{i=1}^{n} a_i(x)u_{x_i} + a(x)u,$$

and I is the Volterra integral operator

$$Iu = \int_0^t - \sum_{i,j=1}^{n} (k_{ij}(x,t,\tau)u_{x_i}(x,\tau))_{x_j} + \sum_{i=1}^{n} k_i(x,t,\tau)u_{x_i}(x,\tau) + k(x,t,\tau)u(x,\tau) \, d\tau.$$

The a priory inequalities are proved and theorems on generalized solvability are provided.

The model

$$u(x,t) - \int_0^t K(t,\tau) \sum_{i=1}^{n} \frac{\partial^2 u(x,\tau)}{\partial x_i^2} d\tau = f(x,t)$$

is considered in the paper (Anikushyn, 2014). In order to get a priory inequalities, it is assumed that the kernel K is non-negative defined, i.e. for all functions u the inequality

$$\int_0^T \int_0^t K(t,\tau)u(\tau)u(t)d\tau dt \geqslant 0$$

holds.

In contrast to the previously cited papers, the work (Anikushyn, 2024) investigates the BVP for the equation (1) with mixed homogeneous Dirichlet and nonhomogeneous Neumann conditions

$$u(x,t)|_{\Gamma_1} = 0, \sigma(u,x,t) \cdot n|_{\Gamma_2} = g(x,t), u(x,0) = u_0, \; u_t(x,0) = v_0, \; x \in \Omega.$$

In the work (Anikushyn et al., 2024), the authors obtain a priori estimates for the hyperbolic integro-differential operator (1)–(3) in the same triplet of spaces, as it was done previously in (Lyashko, 2002) for the differential part only. Moreover, based on these inequalities, the existence of an optimal control was demonstrated in (Lyashko, 2002). By following the same reasoning step by step, one can obtain a theorem on the existence of optimal control for the problem with the integro-differential operator from (Anikushyn et al., 2024) in this same triplet of spaces.

In the work (Anikushyn & Hranishak, 2018), the authors consider the same problem (1)–(3), but a priori inequalities were obtained in other spaces. Due to this, the result from (Lyashko, 2002) on existence of optimal control for the hyperbolic equation cannot be directly applied. The main result of our work is to state a theorem on the existence of optimal control for the system described by the problem (1)–(3) in the spaces considered in (Anikushyn & Hranishak, 2018).

1. Main notations, spaces, and necessary results

Consider the linear operator L defined by equation (1). The sought function $u(x,t)$ satisfies the initial conditions represented by (3) and the boundary conditions given by the equation (2).

In our work, as in (Lyashko, 2002; Anikushyn et al., 2024), we assume that:

1. The kernels $K_i(x,t,\tau)$ are continuously differentiable on $[0,T]\times[0,T]$;
2. a is a continuous function on $\overline{\Omega}$;
3. The coefficients a_{ij}, a_i are continuously differentiable functions in the domain $\overline{\Omega}$, and $a_{ij}=a_{ji}$;
4. For any real numbers λ_i, the following inequality holds:

$$\sum_{i,j=1}^{n} a_{ij}\left(x\right)\lambda_i\lambda_j \geqslant \alpha\sum_{i=1}^{n}\lambda_i^2,$$

where α is a positive constant that does not depend on $x\in\overline{\Omega}$ and λ_i.

Let C_s^∞, $s\in\{0,T\}$ denote the set of all functions $u\in C^\infty\left(\overline{Q}\right)$ that satisfy the boundary conditions (2) and the initial conditions, which are given by the following equation

$$u|_{t=s}=u_t|_{t=s}=\ldots=0. \tag{4}$$

We will consider these spaces C_0^∞ and C_T^∞ as the domains of the operator L and its adjoint L^*, respectively.

Following (Anikushyn & Hranishak, 2018), let H_0^2, S_0^1, and V_0^0 denote the completions of the set C_0^∞ by the norms

$$\|u\|_{H_0^2}^2=\int_Q u_{tt}^2+\sum_{i=1}^n u_{tx_i}^2 dQ,\quad \|u\|_{S_0^1}^2=\int_Q u_t^2+\sum_{i=1}^n u_{x_i}^2 dQ+\int_\Omega u_t^2|_{t=T}+\sum_{i=1}^n u_{x_i}^2|_{t=T}\,d\Omega,$$

$$\|u\|_{V_0^0}^2=\int_Q u^2+\sum_{i=1}^n\left(\int_0^t u_{x_i}(\xi)d\xi\right)^2 dQ+\int_\Omega\sum_{i=1}^n\left(\int_0^T u_{x_i}(\xi)d\xi\right)^2 d\Omega,$$

respectively, and let H_T^2, S_T^1, V_T^0 denote the completions of the set C_T^∞ by the norms

$$\|v\|_{H_T^2}^2=\int_Q v_{tt}^2+\sum_{i=1}^n v_{tx_i}^2 dQ,\quad \|v\|_{S_T^1}^2=\int_Q v_t^2+\sum_{i=1}^n v_{x_i}^2 dQ+\int_\Omega v_t^2|_{t=0}+\sum_{i=1}^n v_{x_i}^2|_{t=0}\,d\Omega,$$

$$\|v\|_{V_T^0}^2=\int_Q v^2+\sum_{i=1}^n\left(\int_t^T v_{x_i}(\xi)d\xi\right)^2 dQ+\int_\Omega\sum_{i=1}^n\left(\int_0^T v_{x_i}(\xi)d\xi\right)^2 d\Omega,$$

respectively.

Negative spaces with respect to $L_2(Q)$ are denoted by $\left(H_0^2\right)^-$, $\left(V_0^0\right)^-$, $\left(S_0^1\right)^-$, $\left(H_T^2\right)^-$, $\left(V_T^0\right)^-$, $\left(S_T^1\right)^-$, respectively. Also, by $\langle\cdot,\cdot\rangle_{(V_T^0)^-\times V_T^0}$ we denote the natural bilinear form on $(V_T^0)^-\times V_T^0$. Corresponding bilinear forms are also introduced for other pairs of spaces. The scalar product in the space $L_2(Q)$ will be denoted by $(\cdot,\cdot)_{L_2(Q)}$.

In the work (Anikushyn & Hranishak, 2018), the following inequalities are stated:

Lemma 1. *For some positive constants c_1,c_2, the following inequalities hold for all $u\in H_0^2$ and $v\in H_T^2$:*

$$\|u\|_{S_0^1}\leqslant c_1\|Lu\|_{(V_T^0)^-}\leqslant c_2\|u\|_{H_0^2}, \tag{5}$$

$$\|v\|_{S_T^1}\leqslant c_1\|L^*v\|_{(V_0^0)^-}\leqslant c_2\|v\|_{H_T^2}. \tag{6}$$

A solution of problem (1)–(3) will be understood in the following sense (Lyashko, 2002; Anikushyn & Hranishak, 2018):

Definition 1. The solution of the problem (1)–(3) with the right-hand side $f\in(V_T^0)^-$ is called a function $u(x,t)\in S_0^1$ for which there exists a sequence of functions $u_i(x,t)\in C_0^\infty, i=1,2,...$, such that

$$\|u-u_i\|_{S_0^1}\to 0,\|Lu_i-f\|_{(V_T^0)^-}\to 0, i\to\infty.$$

Using the inequalities (5), (6), and applying the general approach from (Lyashko, 2002), the following theorem is asserted in (Anikushyn & Hranishak, 2018):

Theorem 1. *For any $f\in(V_T^0)^-$ there exists a unique generalized solution to the problem* (1)–(3) *in the sense of Definition 1.*

2. Optimal control problem

For simplicity, in the sequel we assume (Lyashko, 2002) that the domain Ω is cylindrical with respect to the variable x_1, i.e., $\Omega = [a, b] \times \Omega'$. Denote $Q' = \Omega' \times (0, T)$ and further we introduce the auxiliary space $\mathcal{W}(Q')$, which is the completion of the set $C^\infty(Q')$ with respect to the norm

$$\|\varphi\|_{\mathcal{W}} = \left(\int_{Q'} \varphi^2 + (\varphi')^2 \, dQ' \right)^{\frac{1}{2}}. \tag{7}$$

Here and in the sequel, φ' denotes the derivative of the function φ with respect to the variable t.

Consider the optimal control problem for the system (1)–(3). Control operator appears the right-hand side of eq. (1):

$$Lu = f + \mathcal{A}h, \ f \in (V_0^T)^-. \tag{8}$$

Here, the control operator $\mathcal{A}$ has the form

$$\mathcal{A}h = \sum_{k=1}^{d} \delta(x_1 - x_{1,k}) \otimes \varphi_k(x', t), \tag{9}$$

where

$$x_1, x_{1,k} \in [a, b], x' = (x_2, x_3, ..., x_n) \in \Omega', \varphi_k \in \mathcal{W}(Q').$$

Here, $h = \{(x_{1,k}, \varphi_k)\}_{k=1}^d$ represents a fixed control belonging to a certain set of admissible controls $\mathcal{U}$ from the control space $\mathcal{H} = ([a, b] \times \mathcal{W}(Q'))^d$. The solutions of the problem (8) are subject to the functional $J(h) = \Phi(u(h))$, which needs to be minimized under the condition $h \in \mathcal{U}$.

The mapping $(\mathcal{A}h)(\cdot)$ is defined as a functional that acts on smooth functions v in $\overline{Q}$ as follows:

$$(\mathcal{A}h)(v) = \sum_{k=1}^{d} \int_{Q'} v(x_{1,k}, x', t)\varphi_k(x', t) \, dQ'.$$

Based on the estimates (5), (6), a general theorem on optimal control existence (Lyashko, 2002) has the following form:

Theorem 2. *Let the state of the system be determined as the solution of the problem* (8)*, and the following conditions hold:*

1. *The quality criterion $\Phi(\cdot) : S_0^1 \to \mathbb{R}$ is weakly lower semicontinuous with respect to the state of the system $u(x, t, h)$ (if $u_i \rightharpoonup u^*$ in S_0^1, then $\liminf_{i\to\infty} \Phi(u_i) \geqslant \Phi(u^*)$), and is bounded from below;*
2. *$\mathcal{H}$ is a Hilbert space and the set of admissible controls $\mathcal{U} \subset \mathcal{H}$ is weakly compact in $\mathcal{H}$;*
3. *The operator $\mathcal{A}$ acts weakly continuously from $\mathcal{H}$ to $(V_T^0)^-$;*
4. *The operators L, L^* satisfy the estimates* (5)*,* (6)*.*

Then, there exists an optimal control for the system (8)*.*

In the next section, we will show that the mapping $\mathcal{A}$, defined by the equation (9), is weakly continuous. Based on this, using the mentioned theorem 2, and taking into account the inequalities (5), (6), we arrive at the main result of our work:

Theorem 3. *Let the state of the system be described by the solution of the problem* (1)–(3) *and the specified control space $\mathcal{H}$, and let the right-hand side of equation* (1) *has the form $f + \mathcal{A}h$ under the condition* (9)*. Suppose also that the set of admissible controls $\mathcal{U}$ is closed, convex, and bounded in $\mathcal{H}$, and the quality criterion $\Phi(\cdot) : S_0^1 \to \mathbb{R}$ is weakly lower semicontinuous with respect to the state of the system $u(t, x, h)$ and bounded from below. Then, there exists an optimal control for the given system.*

3. Proof of necessary statements

In this section, we use the following notation

$$\overline{v}(x, t) = \int_T^t v(x, \xi) \, d\xi, \quad \overline{v}(s, x', t) = \int_T^t v(s, x', \xi) \, d\xi.$$

To properly define the functional $\mathcal{A}h$, we first prove the following statement.

Lemma 2. *The linear functional $l_A(v) = (\mathcal{A}h)(v)$ can be extended to be continuous on the space V_T^0.*

Proof. Integrating by parts with respect to t and applying the Cauchy-Schwarz inequality, we have (recall that φ_k' denotes the derivative of the function φ_k with respect to t):

$$\int_{Q'} v(x_{1,k}, x', t) \, \varphi_k(x', t) \, dQ' = \int_{\Omega'} \int_0^T v(x_{1,k}, x', t) \, \varphi_k(x', t) \, dt \, d\Omega' = \tag{10}$$

$$= \int_{\Omega'} \overline{v}(x_{1,k}, x', t) \, \varphi_k(x', t) \Big|_0^T d\Omega' - \int_{\Omega'} \int_0^T \overline{v}(x_{1,k}, x', t) \, \varphi_k'(x', t) \, dt d\Omega' =$$

$$= \int_{\Omega'} -\overline{v}(x_{1,k}, x', 0)\, \varphi_k(x', 0) d\Omega' - \int_{\Omega'} \int_0^T \overline{v}(x_{1,k}, x', t)\, \varphi_k'(x', t)\, dt d\Omega' \leqslant$$
$$\leqslant \|\overline{v}(x_{1,k}, x', 0)\|_{L_2(\Omega')} \|\varphi_k(x', 0)\|_{L_2(\Omega')} + \left| \int_{\Omega'} \int_0^T \overline{v}(x_{1,k}, x', t)\, \varphi_k'(x', t)\, dt\, d\Omega' \right|.$$

Note that for the functions $v \in C_T^\infty$, the Cauchy-Schwarz inequality implies

$$\left|\overline{v}(x_{1,k}, x', t)\right| = \left| \int_t^T v(x_{1,k}, x', \xi)\, d\xi \right| = \left| \int_t^T \int_a^{x_{1,k}} v_{x_1}(s, x', \xi)\, ds\, d\xi \right| = \tag{11}$$
$$= \left| \int_a^{x_{1,k}} \int_t^T v_{x_1}(s, x', \xi)\, d\xi\, ds \right| \leqslant \left(\int_a^{x_{1,k}} ds \int_a^{x_{1,k}} \left(\int_t^T v_{x_1}(s, x', \xi)\, d\xi \right)^2 ds \right)^{\frac{1}{2}} \leqslant$$
$$\leqslant (b-a)^{\frac{1}{2}} \left(\int_a^b \left(\int_t^T v_{x_1}(s, x', \xi)\, d\xi \right)^2 ds \right)^{\frac{1}{2}}.$$

In order to estimate the first term on the right-hand side of (10), we use inequality (11). Integrating squared inequality (11) over the region Ω we obtain

$$\|\overline{v}(x_{1,k}, x', 0)\|^2_{L_2(\Omega')} = \int_{\Omega'} \left(\int_0^T v(x_{1,k}, x', \xi)\, d\xi \right)^2 d\Omega' \leqslant \tag{12}$$
$$\leqslant (b-a) \int_{\Omega'} \int_a^b \left(\int_0^T v_{x_1}(s, x', \xi)\, d\xi \right)^2 ds\, d\Omega' \leqslant (b-a)\|v\|^2_{V_T^0}.$$

Additionally, it is well-known (see, e.g., (Berezansky, 1968)), there exists a constant c_Ω, depending only on the domain Ω, such that for any $x' \in \Omega'$, the following inequality holds:

$$\varphi^2(x', 0) \leqslant c_\Omega \int_0^T \varphi^2(x', t) + (\varphi'(x', t))^2 dt.$$

Thus, integrating the last inequality by x' over the region Ω', one can easily obtain the following inequality, which establishes the relationship between the norms $\|\cdot\|_{L_2(\Omega')}$ and $\|\cdot\|_{\mathcal{W}}$.

$$\|\varphi|_{t=0}\|_{L_2(\Omega')} = \int_{\Omega'} \varphi^2(x', 0)\, d\Omega' \leqslant c_\Omega \int_{Q'} \varphi^2(x, t) + (\varphi'(x, t))^2\, dQ' = c_\Omega \|\varphi\|_{\mathcal{W}}. \tag{13}$$

Now we use the Cauchy-Schwarz inequality and eqution (11) once again for estimating of the second term on the right-hand side of enequality (10). We have

$$\left| \int_{\Omega'} \int_0^T \overline{v}(x_{1,k}, x', t)\, \varphi_k'(x', t)\, dt\, d\Omega' \right|^2 \leqslant \int_{\Omega'} \int_0^T \overline{v}^2(x_{1,k}, x', t)\, dt\, d\Omega' \int_{\Omega'} \int_0^T (\varphi_k'(x', t))^2\, dt\, d\Omega' \leqslant \tag{14}$$
$$\leqslant (b-a) \int_Q \left(\int_t^T v_{x_1}(s, x', \xi) d\xi \right)^2 dQ \int_{Q'} (\varphi_k'(x', t))^2 dQ' \leqslant (b-a)\|v\|^2_{V_T^0} \|\varphi\|^2_{\mathcal{W}}.$$

Applying (12), (13), and (14) to the right-hand side of inequality (10), we derive

$$\left| \int_{Q'} v(x_{1,k}, x', t)\, \varphi_k(x', t)\, dQ' \right| \leqslant c\|v\|_{V_T^0} \|\varphi_k\|_{\mathcal{W}},$$

and hence,

$$|l_A(v)| \leqslant \sum_{k=1}^d \left| \int_{Q'} v(x_{1,k}, x', t)\, \varphi_k(x', t)\, dQ' \right| \leqslant c \sum_{k=1}^d \|\varphi_k\|_{\mathcal{W}} \|v\|_{V_T^0}. \tag{15}$$

Now, the theorem on the extension of a linear functional by continuity states that the bounded functional l_A can be extended from the linear manifold C_T^∞ to the entire space V_T^0 while preserving the norm. This concludes the proof. □

Note that since $f \in (V_T^0)^-$, utilizing Lemma 2, we have $F = f + \mathcal{A}h \in (V_T^0)^-$. Then, by Theorem 1, there exists a unique solution $u(x, t, h)$ of the problem (1)–(3) with the right-hand side $F = f + \mathcal{A}h$ in the sense of Definition 1.

Now we will prove the statement about the weak continuity of the operator $\mathcal{A}$.

Lemma 3. *The mapping $\mathcal{A}$ defined by equation* (9) *is weakly continuous.*

Proof. Let $h^{(m)} = \{(x_{1,k}^{(m)}, \varphi_k^{(m)})\}_{k=1}^d$ be a weakly convergent sequence in the space $\mathcal{H}$, i.e., $h^{(m)} \rightharpoonup h^*$, and denote $h^* = \{(x_{1,k}^*, \varphi_k^*)\}_{k=1}^d$. From weak convergence we derive that $x_{1,i}^{(m)} \to x_{1,i}^*$ in the space $\mathbb{R}$ and $\varphi_k^{(m)} \rightharpoonup \varphi_k^*$ in the space $\mathcal{W}(Q')$. We need to show that $\mathcal{A}h^{(m)}$ weakly convergent to $\mathcal{A}h^*$ in the space $(V_T^0)^-$.

First, note that by inequality (16) from lemma 2, for the functional $\mathcal{A}h^{(m)}$, we have the following estimate

$$\|\mathcal{A}h^{(m)}\|_{(V_T^0)^-} \leqslant c\sum_{k=1}^{d}\|\varphi_k^{(m)}\|_{\mathcal{W}}. \tag{16}$$

Since, for each k, the sequence $\varphi_k^{(m)}$ is weakly convergent in the space $\mathcal{W}$, it is bounded in norm of this space. Therefore, the sequence $\mathcal{A}h^{(m)}$ is bounded in the norm of the space $(V_T^0)^-$. Using this, according to the criterion of weak convergence, it is sufficient to prove that $\langle\mathcal{A}h^{(m)},v\rangle_{(V_T^0)^-\times V_T^0}\to\langle\mathcal{A}h^*,v\rangle_{(V_T^0)^-\times V_T^0}$ for all v from some dense subset in the space V_T^0. We will show the validity of this statement for all $v\in C_T^\infty$. To this end, consider the auxiliary functional

$$\mathcal{T}h^{(m)}=\sum_{k=1}^{d}\delta(x_1-x_{1,k}^*)\otimes\varphi_k^{(m)}(x',t).$$

For any function $v\in C_T^\infty$, we write the equality

$$\left\langle\mathcal{A}h^{(m)}-\mathcal{A}h^*,v\right\rangle_{(V_T^0)^-\times V_T^0}=\left\langle\mathcal{A}h^{(m)}-\mathcal{T}h^{(m)},v\right\rangle_{(V_T^0)^-\times V_T^0}+\left\langle\mathcal{T}h^{(m)}-\mathcal{A}h^*,v\right\rangle_{(V_T^0)^-\times V_T^0}, \tag{17}$$

and show that both terms on the right-hand side of the last equality tend to zero.

Let us start with the first term. Note that for each fixed k, it is easy to see that

$$\left\|\left(\delta(x_1-x_{1,k}^{(m)})-\delta(x_1-x_{1,k}^*)\right)\otimes\varphi_k^{(m)}(x',t)\right\|_{(V_T^0)^-}=\sup_{v\in C_T^\infty}\frac{1}{\|v\|_{V_T^0}}\int_{Q'}\left(v|_{x_1=x_{1,k}^{(m)}}-v|_{x_1=x_{1,k}^*}\right)\varphi_k^{(m)}dQ'= \tag{18}$$

$$=\sup_{v\in C_T^\infty}\frac{1}{\|v\|_{V_T^0}}\int_{Q'}\int_{x_{1,k}^*}^{x_{1,k}^{(m)}}v_{x_1}(s,x',t)ds\,\varphi_k^{(m)}(x',t)dQ'.$$

Transform the last integral similarly to lemma 2. First, integrating by parts with respect to the variable t, we get

$$\int_0^T\int_{x_{1,k}^*}^{x_{1,k}^{(m)}}v_{x_1}(s,x',t)\,ds\,\varphi_k^{(m)}(x',t)\,dt= \tag{19}$$

$$=\int_T^t\int_{x_{1,k}^*}^{x_{1,k}^{(m)}}v_{x_1}(s,x',\xi)dsd\xi\varphi_k^{(m)}(x',t)\Big|_{t=0}^{t=T}-\int_0^T\int_T^t\int_{x_{1,k}^*}^{x_{1,k}^{(m)}}v_{x_1}(s,x',\xi)dsd\xi(\varphi_k^{(m)}(x',t))'dt=$$

$$=\int_0^T\int_{x_{1,k}^*}^{x_{1,k}^{(m)}}v_{x_1}(s,x',\xi)dsd\xi\varphi_k^{(m)}(x',0)-\int_0^T\int_T^t\int_{x_{1,k}^*}^{x_{1,k}^{(m)}}v_{x_1}(s,x',\xi)dsd\xi(\varphi_k^{(m)}(x',t))'dt=$$

$$=\int_{x_{1,k}^*}^{x_{1,k}^{(m)}}\int_0^T v_{x_1}(s,x',\xi)d\xi ds\varphi_k^{(m)}(x',0)-\int_0^T\int_{x_{1,k}^*}^{x_{1,k}^{(m)}}\int_T^t v_{x_1}(s,x',\xi)d\xi ds(\varphi_k^{(m)}(x',t))'dt=$$

$$=-\int_{x_{1,k}^*}^{x_{1,k}^{(m)}}\overline{v}_{x_1}(s,x',0)ds\varphi_k^{(m)}(x',0)-\int_0^T\int_{x_{1,k}^*}^{x_{1,k}^{(m)}}\overline{v}_{x_1}(s,x',t)ds(\varphi_k^{(m)}(x',t))'dt.$$

By virtue of Cauchy-Schwarz inequality, we have

$$\left|\int_{x_{1,k}^*}^{x_{1,k}^{(m)}}\overline{v}_{x_1}(s,x',0)ds\right|^2\leqslant\int_{x_{1,k}^*}^{x_{1,k}^{(m)}}ds\int_{x_{1,k}^*}^{x_{1,k}^{(m)}}\overline{v}_{x_1}^2(s,x',0)ds\leqslant|x_{1,k}^*-x_{1,k}^{(m)}|\int_a^b\overline{v}_{x_1}^2(s,x',0)ds, \tag{20}$$

$$\left|\int_{x_{1,k}^*}^{x_{1,k}^{(m)}}\overline{v}_{x_1}(s,x',t)ds\right|^2\leqslant\int_{x_{1,k}^*}^{x_{1,k}^{(m)}}ds\int_{x_{1,k}^*}^{x_{1,k}^{(m)}}\overline{v}_{x_1}^2(s,x',t)ds\leqslant|x_{1,k}^*-x_{1,k}^{(m)}|\int_a^b\overline{v}_{x_1}^2(s,x',t)ds. \tag{21}$$

Applying (20),(21) and Cauchy-Schwarz inequality to the right-hand side of (19) we derive

$$\left|\int_0^T\int_{x_{1,k}^*}^{x_{1,k}^{(m)}}v_{x_1}(s,x',t)ds\,\varphi_k^{(m)}(x',t)dt\right|\leqslant$$

$$\leqslant|x_{1,k}^*-x_{1,k}^{(m)}|^{\frac12}\left(\left(\int_a^b\overline{v}_{x_1}^2(s,x',0)ds\right)^{\frac12}|\varphi_k^{(m)}(x',0)|+\int_0^T\left(\int_a^b\overline{v}_{x_1}^2(s,x',t)ds\right)^{\frac12}|(\varphi_k^{(m)}(x',t))'|dt\right)\leqslant$$

$$\leqslant|x_{1,k}^*-x_{1,k}^{(m)}|^{\frac12}\left(\left(\int_a^b\overline{v}_{x_1}^2(s,x',0)ds\right)^{\frac12}|\varphi_k^{(m)}(x',0)|+\left(\int_0^T\int_a^b\overline{v}_{x_1}^2(s,x',t)dsdt\int_0^T((\varphi_k^{(m)}(x',t))')^2dt\right)^{\frac12}\right).$$

Integrating the last inequality by $x' \in \Omega'$ and applying Cauchy-Schwarz once again we arrive at

$$\left| \int_{Q'} \int_{x^*_{1,k}}^{x^{(m)}_{1,k}} v_{x_1}(s,x',t)\, ds\, \varphi^{(m)}_k(x',t)\, dQ' \right| \leqslant \tag{22}$$

$$\leqslant |x^*_{1,k} - x^{(m)}_{1,k}|^{\frac{1}{2}} \int_{\Omega'} \left(\left(\int_a^b \overline{v}^2_{x_1}(s,x',0) ds \right)^{\frac{1}{2}} |\varphi^{(m)}_k(x',0)| + \left(\int_0^T \int_a^b \overline{v}^2_{x_1}(s,x',t) ds dt \int_0^T ((\varphi^{(m)}_k(x',t))')^2\, dt \right)^{\frac{1}{2}} d\Omega' \leqslant \right.$$

$$\leqslant |x^*_{1,k} - x^{(m)}_{1,k}|^{\frac{1}{2}} \left(\left(\int_\Omega \overline{v}^2_{x_1}(x,0) d\Omega \int_{\Omega'} (\varphi^{(m)}_k(x',0))^2 d\Omega' \right)^{\frac{1}{2}} + \left(\int_Q \overline{v}^2_{x_1}(x,t) dQ \int_{Q'} ((\varphi^{(m)}_k(x',t))')^2\, dQ' \right)^{\frac{1}{2}} \right) =$$

$$= |x^*_{1,k} - x^{(m)}_{1,k}|^{\frac{1}{2}} \left(\|\overline{v}_{x_1}|_{t=0}\|_{L_2(\Omega)} \|\varphi^{(m)}_k|_{t=0}\|_{L_2(\Omega')} + \|\overline{v}_{x_1}\|_{L_2(Q)} \|(\varphi^{(m)}_k)'\|_{L_2(Q')} \right) \leqslant$$

$$\leqslant |x^*_{1,k} - x^{(m)}_{1,k}|^{\frac{1}{2}} \left(\|v\|_{V^0_T} \|\varphi^{(m)}_k|_{t=0}\|_{L_2(\Omega')} + \|v\|_{V^0_T} \|(\varphi^{(m)}_k)'\|_{L_2(Q')} \right).$$

The last inequality holds due to the fact that

$$\|v\|^2_{V^0_T} = \|v\|^2_{L_2(Q)} + \sum_{i=1}^n \|\overline{v}_{x_i}\|^2_{L_2(Q)} + \sum_{i=1}^n \|\overline{v}_{x_i}|_{t=0}\|^2_{L_2(\Omega)}.$$

Finally, exploiting inequality (13) for the right-hand side of (22), we obtain

$$\left| \int_{Q'} \int_{x^*_{1,k}}^{x^{(m)}_{1,k}} v(s,x',t)\, ds\, \varphi(x',t)\, dQ' \right| \leqslant (c_\Omega + 1)|x^*_{1,k} - x^{(m)}_{1,k}|^{\frac{1}{2}} \|v\|_{V^0_T} \|\varphi^{(m)}_k\|_{\mathcal{W}}.$$

Using the last inequality in (18) we conclude that

$$\left\| \left(\delta(x_1 - x^{(m)}_{1,k}) - \delta(x_1 - x^*_{1,k}) \right) \otimes \varphi^{(m)}_k(x',t) \right\|_{(V^0_T)^-} =$$

$$= \sup_{v \in C^\infty_T} \frac{1}{\|v\|_{V^0_T}} \int_{Q'} \int_{x^*_{1,k}}^{x^{(m)}_{1,k}} v_{x_1}(s,x',t) ds\, \varphi^{(m)}_k(x',t) dQ' \leqslant$$

$$\leqslant \sup_{v \in C^\infty_T} \frac{1}{\|v\|_{V^0_T}} (c_\Omega + 1)|x^*_{1,k} - x^{(m)}_{1,k}|^{\frac{1}{2}} \|v\|_{V^0_T} \|\varphi^{(m)}_k\|_{\mathcal{W}} = (c_\Omega + 1)|x^*_{1,k} - x^{(m)}_{1,k}|^{\frac{1}{2}} \|\varphi^{(m)}_k\|_{\mathcal{W}}.$$

Since the sequences $\varphi^{(m)}_k$ are weakly convergent in $\mathcal{W}(Q')$, then those sequences are bounded in the space $\mathcal{W}(Q')$. Based on the fact $|x^*_{1,k} - x^{(m)}_{1,k}| \to 0$, as $m \to \infty$, we further obtain

$$\|\mathcal{A}h^{(m)} - \mathcal{T}h^{(m)}\|_{(V^0_T)^-} \leqslant \sum_{k=1}^d \left\| \left(\delta(x_1 - x^{(m)}_{1,k}) \otimes - \delta(x_1 - x^*_{1,k}) \right) \otimes \varphi^{(m)}_k(x',t) \right\| \to 0, \tag{23}$$

as $m \to \infty$. Finally, since norm converges implies weak convergence we claim that

$$\left\langle \mathcal{A}h^{(m)} - \mathcal{T}h^{(m)}, v \right\rangle_{(V^0_T)^- \times V^0_T} \to 0,\ m \to \infty. \tag{24}$$

Let us now consider the second term in (17). It can be shown, that weak convergence in $\mathcal{W}(Q')$ implies weak convergence in $L_2(Q')$. Therefore, $\varphi^{(m)}_k \rightharpoonup \varphi^*_k$ in $L_2(Q')$, and thus for any fixed $v \in C^\infty_T$ we have

$$\langle \mathcal{T}h^{(m)} - \mathcal{A}h^*, v \rangle_{(V^0_T)^- \times V^0_T} = \left\langle \sum_{k=1}^d \delta(x_1 - x^*_{1,k}) \otimes \left(\varphi^{(m)}_k(x',t) - \varphi^*_k(x',t) \right), v \right\rangle_{(V^0_T)^- \times V^0_T} = \tag{25}$$

$$= \sum_{k=1}^d \left(\int_{Q'} v(x^*_{1,k}, x', t) \left(\varphi^{(m)}_k(x',t) dQ' - \varphi^*_k(x',t) \right) dQ' \right) = \sum_{k=1}^d \left(\left(v|_{x_1 = x^*_{1,k}}, \varphi^{(m)}_k - \varphi^*_k \right)_{L_2(Q')} \right) \to 0.$$

Putting together (17), (24) and (25) we finally conclude that $\langle \mathcal{A}h^{(m)}, v \rangle_{(V^0_T)^- \times V^0_T} \to \langle \mathcal{A}h^*, v \rangle_{(V^0_T)^- \times V^0_T}$. $\square$

Discussion and conclusions

In this work, we examined a special case of control in the right-hand side of an integral-differential model corresponding to point control. The novelty of the obtained result lies in the fact that the weak continuity of the control operator $\mathcal{A}$ and, consequently, the theorem on the existence of an optimal control, has been proven in new spaces $\mathcal{H}$ and $(V^0_T)^-$. The next logical, yet still unimplemented step would be to develop and justify the convergence of analogs of the Galerkin method for the numerical determination of weak solutions and optimal control in the spaces $(V^0_T)^-$ and S^1_0, as well as investigation of controllability/asymptotic controllability, etc.

Authors' contribution: Andrii Anikushyn — ideas, methodology, optimal control problem statement, proofs; Khrystyna Hranishak — proofs.

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ТОЧКОВЕ КЕРУВАННЯ ЛІНІЙНИМИ ГІПЕРБОЛІЧНИМИ ІНТЕГРО-ДИФЕРЕНЦІАЛЬНИМИ СИСТЕМАМИ

Розглянуто задачі оптимального керування розподіленими системами, що описуються лінійними інтегро-диференціальними рівняннями гіперболічного типу із частинними похідними й інтегральними складовими типу Вольтерра. Інтегро-диференціальні рівняння такого типу давно стали стандартним об'єктом вивчення прикладної математики і часто виникають під час дослідження процесів у в'язкопружних середовищах (аморфні полімери, напівкристалічні полімери, біополімери, метали за дуже високих температур, бітумні матеріали тощо). Мета пропонованої роботи полягає у доведенні існування оптимального керування розподіленими системами, що описуються відповідними моделями.

Основні результати отримано за допомогою методів функціонального аналізу та теорії узагальнених функцій. Дослідження проведено у спеціально визначених гільбертових просторах. Оператор керування системою у правій частині містить узагальнені δ-функції Дірака. Головний результат щодо існування оптимального керування встановлено, ґрунтуючись на теорії апріорних оцінок у негативних нормах, що було започатковано у роботах Ю. М. Березанського, розвинуто В. П. Діденком, С. І. Ляшком та його колегами.

Наведено постановку задачі оптимального керування. Керування системою відбувається за допомогою оператора керування, що входить у праву частину початково-крайової задачі. Оператор керування діє у просторах узагальнених функцій, що моделює точкове керування системою. Для дослідження поставленої задачі запропоновано гільбертові простори, у яких діє оператор задачі, та простір допустимих керувань. Наведено апріорні оцінки в негативних нормах, означення узагальнених розв'язків, властивості щодо коректності постановки початково-крайової задачі, загальну теорему про існування оптимального керування. Доведено властивості про коректність визначення оператора керування, його слабку неперервність. На основі отриманих тверджень сформульовано теорему про існування оптимального керування для поставленої задачі. Зокрема, накладаються обмеження на допустиму множину керувань та критерій якості, що мінімізується.

Для розглянутої задачі оптимального керування встановлено теорему про достатні умови існування оптимального керування відповідною системою. Зокрема, доведено коректність визначення наведеного оператора керування, що відповідає точковому керуванню, та його слабку неперевність із простору керувань у простір правих частин.

Ключові слова: ***інтегро-диференціальне рівняння, лінійне гіперболічне рівняння, оператор Вольтерра, в'язкопружні середовища, оптимальне керування, точкове керування.***

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