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PREDICTION FORMULAS FOR INTEGRATED FRACTIONAL BROWNIAN MOTION

This paper investigates the projection (prediction) of increments of the integrated fractional Brownian motion (ifBm). We introduce ifBm, calculate its covariance function, and establish the stationarity of its increments. Our primary goal is to determine the coefficients for the linear prediction of a future ifBm increment based on a series of past increments. We show that the prediction coefficients exhibit complex behavior, with their signs changing depending on the Hurst index (H). For overlapping increments, this sign change occurs at a non-trivial values of H. In contrast, for non-overlapping increments, the sign change happens at H=0.5, consistent with the properties of fractional Brownian motion itself. This work provides explicit formulas where possible and extensive numerical tables, offering insights into the properties and predictability of ifBm, laying the groundwork for further research.

K e y w o r d s: ***fractional Brownian motion, integrated fractional Brownian motion, covariance function, projection problem, projection coefficients.***

AMS 2020 classification: 60G15, 60G22.

Introduction

Fractional Brownian motion (fBm) is one of the most popular processes in recent decades. The reason is that it is a stochastic process that has memory, and this property is inherent in real processes in physics, biology, electronics, economics, and finance. Investigation of stochastic representations and the main properties of fBm started with the works (Mandelbrot, & Van Ness, 1968; Barton, & Vincent Poor, 1988; Cioczek-Georges, & Mandelbrot, 1995; Dai, & Heyde, 1996; Decreusefond, & Üstünel, 1998; Carmona, & Coutin, 1998; Coutin, & Decreusefond, 1999; Decreusefond, & Üstünel, 1999; Norros, Valkeila, & Virtamo, 1999; Novikov, & Valkeila, 1999). Then, stochastic analysis of fBm and related processes, together with the main applications, was developed in the papers (Alòs, Mazet, & Nualart, 2001; Coutin, Nualart, & Tudor, 2001; Duncan, 2001; Duncan, Pasik-Duncan, & Maslowski, 2002; Biagini et al., 2002; Alòs, & Nualart, 2003; Bender, 2003b, 2003a; Benth, 2003; Boufoussi, & Ouknine, 2003; Nualart, 2003; Nualart, & Ouknine, 2003; Anh, & Inoue, 2004; Carmona, Coutin, & Montseny, 2003; Cheridito, 2001; Cheridito, Kawaguchi, & Maejima, 2003; Bender, & Elliott, 2004; Cheridito, & Nualart, 2005; Corcuera, Nualart, & Woerner, 2006; Bender, Sottinen, & Valkeila, 2008). Let us mention also expansions and transformations of fBm contained in (Dzhaparidze, & van Zanten, 2004, 2005; Jost, 2006). Of course, this list is not exhaustive. The basic properties of fractional Brownian motion have been summarized in many books and papers, and without claiming to be complete, we will mention just the books (Banna et al., 2019; Biagini et al., 2008; Beran, 1994; Nourdin, 2012; Pipiras, & Taqqu, 2017; Samorodnitsky, 2016). Although a fairly convenient and flexible model, fractional Brownian motion still has the disadvantage that fractional exponents in the covariance function make the covariance matrix very inconvenient for analytical research, calculations of the determinant, inversion and other operations. Often, problems related to calculating the determinant do not have an analytical solution at all, but only a numerical one. This point, the accompanying conditions and possible actions were discussed in detail in the papers (Malyarenko et al., 2023; Mishura, Ralchenko, & Shklyar, 2020). In particular, in paper (Mishura, Ralchenko, & Shklyar, 2020), considering applications of fractional Brownian motion to financial markets, we hypothesized that for values of the Hurst index $H \in (\frac{1}{2}, 1)$, all coefficients of projection of the increment of fBm on the subsequent increments are strictly positive. Because of the overly complex form of the determinants associated with the covariance matrix of the increments of fBm that arose when solving this problem, we were able to analytically verify our hypothesis only for a limited number of increments, and we also considered numerical examples. Taking this into account, in paper (Bodnarchuk, & Mishura, 2024) we considered, instead of fBm, an integrated Wiener process, which preserves memory but has not a complicated covariance matrix. In this case we could completely resolve the problem of projection, but the result was unexpected: in this case the signs of the coefficients alternate. Integrated fractional Brownian motion is a natural generalization of an integrated Wiener process. It is natural to average very big datasets in the periods of time. It has also proven its value, in particular, in modeling first-passage time problems in various fields such as computational neuroscience, finance, and queueing theory. Specifically, in case of computational neuroscience, it is used to model the time variability and firing rate adaptation of neurons. (see, e.g., the discussion in (Abundo, & Pirozzi, 2018, 2019)). Now, given the value of studying such integrated processes, we decided in the present paper to consider the problem of projection (prediction) of the increments of the integrated fractional Brownian motion. Why are the problems of projection (roughly speaking, to find $\mathbb{E}[\Delta I_0 \mid \Delta I_1, \dots, \Delta I_n]$, see Section 3) and prediction (roughly speaking, to find $\mathbb{E}[\Delta I_n \mid \Delta I_0, \dots, \Delta I_{n-1}]$) equivalent? Just because we consider stationary sequences, for which the corresponding coefficients are simply arranged in mutually inverse order. The paper is organized as follows: in Section 1 we give preliminaries on fBm; in Section 2 we introduce integrated fBm, calculate its covariance function and establish its properties; in Section 3 we calculate the coefficients of projection $\mathbb{E}[\Delta I_0 \mid \Delta I_1, \dots, \Delta I_n],\ n \geqslant 1$ with overlapping intervals analytically for $n = 1$, partially analytically for $n = 2$ and provide numerical results for $2 \leqslant n \leqslant 10$; in Section 4 we do the same for the non-overlapping intervals.

1. Preliminaries

A *fractional Brownian motion* (fBm) with Hurst index $H \in (0,1)$ is a centered Gaussian process $B^H = \{B_t^H\}_{t\geqslant 0}$ satisfying the following properties:

- $B_0^H = 0$ almost surely;
- For every $t, s \geqslant 0$, the covariance function is given by

$$\mathbb{E}\left[B_t^H B_s^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right);$$

- B^H has stationary increments;
- the trajectories of B^H are almost surely locally Hölder continuous of order α for every $\alpha \in (0,H)$, i.e., for every compact interval $[0,T]$, there exists a random variable $C = C(T,\omega) > 0$ such that

$$|B_t^H - B_s^H| \leqslant C|t-s|^\alpha, \quad \text{for all } s,t \in [0,T] \quad \text{a.s.}$$

The last property, in particular, implies that for any $0 \leqslant a < b < \infty$ the integral $\int_a^b B_s^H\,ds$ is well-defined in the classical Riemann sense and is a zero-mean Gaussian random variable.

2. Integrated Fractional Brownian Motion

In this section, we consider the increments of the integral of a fractional Brownian motion, more precisely, we consider the integrals $I_k = \int_k^{k+1} B_s^H\,ds$, where B^H denotes fBm with Hurst index $H \in (0,1)$, and $k \geqslant 0$ is an integer.

Lemma 1. *The covariance function of I_k is given by the formula*

$$\mathbb{E}[I_k I_l] = \frac{1}{2}\left(I_{1,k,l} - I_{2,k,l}\right),$$

where

$$I_{1,k,l} = \frac{1}{2H+1}\Big[(l+1)^{2H+1} - l^{2H+1} + (k+1)^{2H+1} - k^{2H+1}\Big].$$

$$I_{2,k,l} = \frac{||k-l|+1|^{2H+2} - 2|k-l|^{2H+2} + ||k-l|-1|^{2H+2}}{(2H+1)(2H+2)}.$$

Proof. Indeed,

$$\mathbb{E}[I_k I_l] = \mathbb{E}\left[\int_k^{k+1} B_s^H\,ds \int_l^{l+1} B_t^H\,dt\right] = \int_k^{k+1}\int_l^{l+1} \mathbb{E}[B_s^H B_t^H]\,dt\,ds = \frac{1}{2}\int_k^{k+1}\int_l^{l+1}\left(s^{2H} + t^{2H} - |s-t|^{2H}\right)dtds =$$

$$= \frac{1}{2}\left[\int_k^{k+1}\int_l^{l+1}(s^{2H}+t^{2H})dtds - \int_k^{k+1}\int_l^{l+1}|s-t|^{2H}dtds\right] =: \frac{1}{2}\left(I_{1,k,l} - I_{2,k,l}\right).$$

Then, obviously,

$$I_{1,k,l} = \frac{1}{2H+1}\Big[(l+1)^{2H+1} - l^{2H+1} + (k+1)^{2H+1} - k^{2H+1}\Big].$$

Consider two cases. If $k = l$, then the integral $I_{2,k,l}$ simplifies to

$$I_{2,k,k} = \int_k^{k+1}\int_k^{k+1}|s-t|^{2H}\,dtds = \int_0^1\int_0^1 |s-t|^{2H}\,dtds =$$

$$= 2\int_0^1\int_0^s (s-t)^{2H}\,dtds = 2\int_0^1 \frac{s^{2H+1}}{2H+1}\,ds = \frac{2}{(2H+1)(2H+2)}.$$

Further, for $k > l$, the integral equals

$$I_{2,k,l} = \int_k^{k+1}\int_l^{l+1}(s-t)^{2H}\,dtds = \int_k^{k+1}\frac{(s-l)^{2H+1} - (s-l-1)^{2H+1}}{2H+1}ds =$$

$$= \frac{1}{2H+1}\int_{k-l}^{k-l+1}\Big[v^{2H+1} - (v-1)^{2H+1}\Big]dv = \frac{(k-l+1)^{2H+2} - 2(k-l)^{2H+2} + (k-l-1)^{2H+2}}{(2H+1)(2H+2)}.$$

Since the initial integral is symmetric in s and t, the previous case can be naturally generalized for $k < l$. Finally, we get

$$I_{2,k,l} = \frac{||k-l|+1|^{2H+2} - 2|k-l|^{2H+2} + ||k-l|-1|^{2H+2}}{(2H+1)(2H+2)},$$

and the proof is completed. □

Corollary 1. *The variance of I_k is given by*

$$\mathbb{E}[I_k^2] = \frac{(H+1)\left[(k+1)^{2H+1} - k^{2H+1}\right] - 1}{(2H+1)(H+1)}.$$

The following statements present additional properties of the discrete-time process $\{I_k, k \geqslant 0\}$.

Corollary 2. *The discrete-time process $\{I_k, k \geqslant 0\}$ is not stationary, since the covariance does not entirely depend on the difference $|k-l|$.*

Lemma 2. *The discrete-time process $\{I_k,\ k \geqslant 0\}$ has stationary increments.*

Proof. We can prove even a bit more general fact. Indeed, introduce $\Delta I_k = I_{k+1} - I_k,\ k \geqslant 0$ and consider a shift of the integration interval, not necessarily by an integer, but by any $h \geqslant 0$. Then we need to check whether $\mathbb{E}[\Delta I_k \Delta I_{k+h}]$ depends solely on h.

$$\mathbb{E}[\Delta I_k \Delta I_{k+h}] = \int_k^{k+1} \int_{k+h}^{k+h+1} \mathbb{E}[(B_{s+1}^H - B_s^H)(B_{t+1}^H - B_t^H)]\, dt\, ds = \tag{1}$$
$$= \int_k^{k+1} \int_{k+h}^{k+h+1} \frac{1}{2}\left(|s-t+1|^{2H} + |s-t-1|^{2H} - 2|s-t|^{2H}\right) dt\, ds =$$
$$= \frac{1}{2}\int_0^1 \int_0^1 \left(|u-v-h+1|^{2H} + |u-v-h-1|^{2H} - 2|u-v-h|^{2H}\right) dv\, du,$$

which is a function just of h. □

In what follows we use the notation for the covariance

$$\rho(d,H) := \mathbb{E}[\Delta I_0 \Delta I_d], d \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

Note that $\rho(d,H) = \mathbb{E}[\Delta I_k \Delta I_{k+d}],\ k, k+d \geqslant 0.$

Lemma 3. *The covariance function $\rho(d,H)$ is given by the formula*

$$\rho(d,H) = \frac{|d+2|^{2H+2} - 4|d+1|^{2H+2} + 6|d|^{2H+2} - 4|d-1|^{2H+2} + |d-2|^{2H+2}}{2(2H+2)(2H+1)}.$$

Proof. Consider the covariance $\mathbb{E}[\Delta I_k \Delta I_l]$ for $k, l \geqslant 0$ and rewrite it in the following way:

$$\mathbb{E}[\Delta I_k \Delta I_l] = \mathbb{E}[(I_{k+1} - I_k)(I_{l+1} - I_l)] = \mathbb{E}[I_{k+1} I_{l+1}] - \mathbb{E}[I_{k+1} I_l] - \mathbb{E}[I_k I_{l+1}] + \mathbb{E}[I_k I_l] =$$
$$= -I_{2,k+1,l+1} + I_{2,k+1,l} + I_{2,k,l+1} - I_{2,k,l} =$$
$$= \frac{|d+2|^{2H+2} - 4|d+1|^{2H+2} + 6|d|^{2H+2} - 4|d-1|^{2H+2} + |d-2|^{2H+2}}{2(2H+2)(2H+1)}, \quad d = |k-l|,$$

from which the result of lemma immediately follows. □

Remark 1. Note that in the case $l = k \pm 1$ $(d=1)$ the increments $\Delta I_k = I_{k+1} - I_k$ and $\Delta I_l = I_{l+1} - I_l$ are considered under overlapping intervals, while in the case $d \geqslant 2$ the underlying intervals do not overlap. This directly affects the properties of the respective covariance.

It is well-known that the increments of fractional Brownian motion are positively (negatively) correlated when $H > 1/2\,(H < 1/2)$ if the respective intervals are non-overlapping. However, as it was mentioned in Remark 1, the intervals overlap if $d = 1$ and do not overlap if $d \geqslant 2$. This affects the sign of the covariance. In the following lemma, we investigate the sign of $\rho(d,H)$.

Lemma 4. *The function $\rho : \mathbb{N}_0 \times (0,1) \to \mathbb{R}$ has the following properties:*

(i) for $d \geqslant 2$, $\rho(d,H)$ is negative (positive) for $H < \frac{1}{2}$ $(H > \frac{1}{2})$, and zero otherwise.

(ii) $\rho(1,H)$ is negative (positive) for $H < \alpha_H$ $(H > \alpha_H)$, and zero otherwise, where $\alpha_H \approx 0.2626$.

Proof. Similarly to (1),

$$\mathbb{E}[\Delta I_k \Delta I_l] = \mathbb{E}\left[(I_{k+1}-I_k)(I_{l+1}-I_l)\right] =$$
$$= \mathbb{E}\left[\left(\int_{k+1}^{k+2} B_s^H\,ds - \int_k^{k+1} B_s^H\,ds\right)\left(\int_{l+1}^{l+2} B_t^H\,dt - \int_l^{l+1} B_t^H\,dt\right)\right] =$$
$$= \mathbb{E}\left[\int_0^1 (B_{u+k+1}^H - B_{u+k}^H)\,du \int_0^1 (B_{v+l+1}^H - B_{v+l}^H)\,dv\right] = \int_0^1\int_0^1 \mathbb{E}\left[(B_{u+k+1}^H - B_{u+k}^H)(B_{v+l+1}^H - B_{v+l}^H)\right] du\,dv.$$

Inside of the expectation we have a product of two increments. If $d = |k-l| \geqslant 2$, the intervals are non-overlapping, and we get the first statement.

We are left to consider the case $d = 1$, for which the intervals overlap. Using the expression for ρ, obtained in Lemma 4, we get:

$$\rho(1,H) = \frac{3^{2H+2} - 4\cdot 2^{2H+2} + 7}{2(2H+2)(2H+1)}.$$

Since the denominator is strictly positive, the sign of $\rho(1,H)$ is determined by the sign of the numerator. Let us define the numerator as a function of H

$$f(H) = 3^{2H+2} - 4\cdot 2^{2H+2} + 7.$$

We need to analyze the sign of $f(H)$ for $H \in (0,1)$. First, observe the limits at boundaries

$$\lim_{H\to 0} f(H) = 0, \qquad \lim_{H\to 1} f(H) = 24.$$

We compute the derivative to study monotonicity

$$f'(H) = \frac{d}{dH}\left(3^{2H+2} - 4\cdot 2^{2H+2} + 7\right) = 18\log 3\cdot 9^H - 32\log 2\cdot 4^H.$$

Setting $f'(H) = 0$ gives the critical point β_H

$$18\log 3\cdot 9^{\beta_H} = 32\log 2\cdot 4^{\beta_H} \Rightarrow \beta_H = \frac{\log\left(\frac{16\log 2}{9\log 3}\right)}{\log\frac{9}{4}} \approx 0.14157.$$

Since $g(H) = \left(\frac{9}{4}\right)^H$ is strictly increasing, if $H < \beta_H$, then $f'(H) < 0$ and if $H > \beta_H$, then $f'(H) > 0$. Therefore, $f(H)$ is strictly decreasing on $(0,\beta_H)$ and strictly increasing on $(\beta_H, 1)$, attaining a local minimum at $H = \beta_H$. Since $f(H)$ starts at 0, becomes negative, and then increases up to a positive value, it changes the sign exactly once at some point α_H. Solving the equation $f(\alpha_H) = 3^{2\alpha_H+2} - 4\cdot 2^{2\alpha_H+2} + 7 = 0$ numerically yields $\alpha_H \approx 0.2626$. Therefore, for $0 < H < \alpha_H$, the function $f(H)$ starts at 0, decreases to a minimum at β_H, and increases back to 0 at α_H. Thus, $f(H) < 0$ on $(0,\alpha_H)$. For $H > \alpha_H$, the function $f(H)$ continues to increase from $f(\alpha_H) = 0$. Thus, $f(H) > 0$ on $(\alpha_H, 1)$. For $H = \alpha_H$, $f(H) = 0$.

Since the sign of $\rho(1,H)$ is the same as the sign of $f(H)$, we conclude that $\rho(1,H)$ is negative for $H \in (0,\alpha_H)$, positive for $H \in (\alpha_H, 1)$, and zero at $H = \alpha_H$, where $\alpha_H \approx 0.2626$. □

3. Coefficients of projection

In this section we consider the projection of ΔI_0 onto $(\Delta I_1,\ldots,\Delta I_n)$, that is, the conditional expectation $\mathbb{E}[\Delta I_0 \mid \Delta I_1,\ldots,\Delta I_n]$. Note that the joint distribution of $(\Delta I_0,\ldots,\Delta I_n)$ is centered and Gaussian. Applying the theorem of normal correlation, we obtain the following representation

$$\mathbb{E}[\Delta I_0 \mid \Delta I_1,\ldots,\Delta I_n] = \sum_{k=1}^n b_k^{(n)}\Delta I_k, \quad n \geqslant 1, \tag{2}$$

where the coefficients $b_k^{(n)} \in \mathbb{R}$. Since the increments are stationary (by Lemma 2), clearly $\mathbb{E}[\Delta I_k \mid \Delta I_{k+1},\ldots,\Delta I_{k+n}]$ contains the same coefficients. Now, let us consider the coefficients $b_k^{(n)}$ in the projection (2). The approach to obtain the coefficients $b_k^{(n)}$ is to solve the system of linear equations, which we get by multiplying both sides of (2) by ΔI_l, $l \geqslant 1$ and taking the expectation. We get the system

$$\rho_l = \sum_{k=1}^n b_k^{(n)}\rho_{|k-l|}, \quad l = 1,\ldots,n,$$

where $\rho_d := \rho(d,H) = \mathbb{E}[\Delta I_0 \Delta I_d]$ and is given explicitly by Lemma 3.

Let us define the matrix $A^{(n)} \in \mathbb{R}^{n\times n}$ with entries $A_{ij} = \rho_{|i-j|}$ and the vector $\mathbf{r}^{(n)} = (\rho_1,\ldots,\rho_n)^T$. Then

$$A^{(n)}\mathbf{b}^{(n)} = \mathbf{r}^{(n)}, \quad \text{and} \quad b_k^{(n)} = \frac{\det A_k^{(n)}}{\det A^{(n)}}, \tag{3}$$

where $A_k^{(n)}$ is the matrix obtained from $A^{(n)}$ by replacing the kth column with the vector $\mathbf{r}^{(n)}$. Note that det $A^{(n)}$ is strictly positive, it cannot be proved similarly to Theorem 1.1 from (Banna et al., 2019). Therefore, the sign of each coefficient is determined by the det $A_k^{(n)}$. Now we will proceed as follows: we will calculate analytically those coefficients for which this is possible, and we will carry out the rest of the analysis by numeric. Let us write down the expressions for ρ_0, ρ_1, ρ_2, where $\rho_d := \rho(d, H)$, using the formula obtained in Lemma 3:

$$\rho_0 = \frac{8 \cdot 4^H - 8}{4(H+1)(2H+1)}, \quad \rho_1 = \frac{9 \cdot 9^H - 16 \cdot 4^H + 7}{4(H+1)(2H+1)}, \quad \rho_2 = \frac{16 \cdot 16^H - 36 \cdot 9^H + 24 \cdot 4^H - 4}{4(H+1)(2H+1)}.$$

Case $n = 1$: In this case, the matrix $A^{(n)}$ and the vector $\mathbf{r}^{(n)}$ contain only one element each, and they are equal

$$A^{(1)} = \rho_0 = \frac{8 \cdot 4^H - 8}{4(H+1)(2H+1)}, \quad r^{(1)} = \rho_1 = \frac{9 \cdot 9^H - 16 \cdot 4^H + 7}{4(H+1)(2H+1)}.$$

Solving the equation $A^{(1)} b_1^{(1)} = r^{(1)}$ for $b_1^{(1)}$, we get that

$$b_1^{(1)} = \frac{\rho_1}{\rho_0} = \frac{9 \cdot 9^H - 16 \cdot 4^H + 7}{8 \cdot 4^H - 8}.$$

Since ρ_0 is a variance, it is strictly positive for all $H \in (0,1)$. Therefore, the sign of $b_1^{(1)}$ is determined by the ρ_1 in the numerator. The next result is a direct consequence of Lemma 4.

Proposition 1. *The coefficient $b_1^{(1)}$ is negative (positive) for $H < H_{1,1}$ $(H > H_{1,1})$, where $H_{1,1} := \alpha_H \approx 0.2626$.*

Case $n = 2$: In this case

$$A^{(2)} = \begin{pmatrix} \rho_0 & \rho_1 \\ \rho_1 & \rho_0 \end{pmatrix}, \quad \mathbf{r}^{(2)} = \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}.$$

Therefore

$$b_1^{(2)} = \frac{\rho_1(\rho_0 - \rho_2)}{\rho_0^2 - \rho_1^2}, \quad b_2^{(2)} = \frac{\rho_0\rho_2 - \rho_1^2}{\rho_0^2 - \rho_1^2}.$$

As mentioned previously, the sign of each projection coefficient is determined by its numerator, since the denominators are the determinants of a covariance matrix and therefore always positive.

Proposition 2. *The coefficient $b_1^{(2)}$ is negative (positive) for $H < H_{2,1}$ $(H > H_{2,1})$, where $H_{2,1} := \alpha_H \approx 0.2626$, and the coefficient $b_2^{(2)}$ is negative (positive) for $H < H_{2,2}$ $(H > H_{2,2})$, where $H_{2,2} \approx 0.8171$.*

Proof. Note that $\rho_0 > \rho_2$ for all $H \in (0,1)$, because of the general variance-covariance inequality. So, the sign of the coefficient $b_1^{(2)}$ is determined by ρ_1, which we addressed earlier. Therefore, according to Lemma 4, $b_1^{(2)}$ is negative for $H < \alpha_H$ and positive afterward, where $\alpha_H \approx 0.2626$.

For $b_2^{(2)}$, its sign is determined by the function $\rho_0\rho_2 - \rho_1^2 := g_2(H)$, which can be written in closed form as

$$g_2(H) = \frac{-81 \cdot 81^H + 128 \cdot 64^H - 192 \cdot 16^H + 162 \cdot 9^H - 17}{16(H+1)^2(2H+1)^2} = \frac{g_2^*(H)}{16(H+1)^2(2H+1)^2}.$$

Due to the complex nature of $g_2^*(H)$, a complete analytical study of its sign on the interval $(0,1)$ or even establishing its behavior on certain subintervals using elementary analytical tools seems to be an unsolvable problem.

Consequently, we rely on numerical methods to analyze the behavior of function $g_2^*(H)$. These numerical investigations, which are further supported by a plot of $b_2^{(2)}$ presented at (Fig. 1), indicate that $g_2^*(H)$ has a unique root within the interval $(0,1)$ at point $H_{2,2} \approx 0.8171$. The numerical analysis shows that $g_2^*(H) < 0$ for $H < H_{2,2}$ and $g_2^*(H) > 0$ for $H > H_{2,2}$. □

Remark 2. While a comprehensive analytical proof of the behavior of $g_2^*(H)$ on the interval $(0,1)$ is not provided due to its complexity, we can investigate its characteristics around specific points of interest using Taylor expansions. This approach does not serve as a general proof of the function's behavior across the entire interval but offers insights into its local properties. In our primary consideration, $H \in (0,1)$, but since $g_2^*(H)$ is continuous, we can extend the domain to $H \in [0,1]$ to include boundary points. To explore the behavior near $H = 0$, we can employ Taylor expansion. It can be verified that $g_2^*(0) = 0$. The first and second derivatives of $g_2^*(H)$ are

$$g_2^{*\prime}(H) = -81 \cdot 81^H \log 81 + 128 \cdot 64^H \log 64 - 192 \cdot 16^H \log 16 + 162 \cdot 9^H \log 9.$$

$$g_2^{*\prime\prime}(H) = -81 \cdot 81^H (\log 81)^2 + 128 \cdot 64^H (\log 64)^2 - 192 \cdot 16^H (\log 16)^2 + 162 \cdot 9^H (\log 9)^2.$$

Evaluating at $H = 0$, we find that $g_2^{*\prime}(0) = 0$ and $g_2^{*\prime\prime}(0) < 0$. The Taylor expansion for $g_2^*(H)$ around $H_0 = 0$ is thus

$$g_2^*(H) = g_2^*(0) + g_2^{*\prime}(0)H + \frac{g_2^{*\prime\prime}(0)}{2}H^2 + o(H^2).$$

Since $g_2^{*\prime\prime}(0) < 0$, this suggests that $g_2^*(H)$ is negative for small positive values of H. To understand the function's behavior around the numerically identified root $H_{2,2} \approx 0.8171$, where $g_2^*(H_{2,2}) \approx 0$, we can examine its first derivative. It can be checked that $g_2^{*\prime}(H_{2,2}) > 0$. The Taylor expansion around $H_0 = H_{2,2}$ is

$$g_2^*(H) = g_2^*(H_{2,2}) + g_2^{*\prime}(H_{2,2})(H - H_{2,2}) + o(H - H_{2,2}).$$

Since $g_2^*(H_{2,2}) = 0$ and $g_2^{*\prime}(H_{2,2}) > 0$, this expansion indicates that the function is increasing as it passes through this root, transitioning from negative to positive values. Furthermore, direct evaluation shows that $g_2^*(1) = 0$. Of course, while the fact that we dropped the denominator from the analysis affected the position of the critical points, it did not affect the roots and thus the behavior of the sign of $b_2^{(2)}$ we were initially interested in.

Numerical results

In this section, we present the numerically computed coefficients $b_k^{(n)}$ for $n = 1, \dots, 10$. The results below are listed for the Hurst indices $H = 0.1,\ 0.4,\ 0.7,\ 0.9$ (Tab. 1-4).

Table 1

Coefficients $b_k^{(n)}$ for H=0.1

n/k	1	2	3	4	5	6	7	8	9	10
1	-0.1409	-	-	-	-	-	-	-	-	-
2	-0.1637	-0.1621	-	-	-	-	-	-	-	-
3	-0.1808	-0.1793	-0.1050	-	-	-	-	-	-	-
4	-0.1897	-0.1946	-0.1204	-0.0854	-	-	-	-	-	-
5	-0.1957	-0.2031	-0.1341	-0.0986	-0.0700	-	-	-	-	-
6	-0.1999	-0.2089	-0.1421	-0.1108	-0.0817	-0.0596	-	-	-	-
7	-0.2030	-0.2132	-0.1478	-0.1181	-0.0925	-0.0700	-0.0519	-	-	-
8	-0.2053	-0.2164	-0.1521	-0.1236	-0.0993	-0.0798	-0.0612	-0.0459	-	-
9	-0.2072	-0.2189	-0.1553	-0.1276	-0.1044	-0.0861	-0.0701	-0.0544	-0.0412	-
10	-0.2088	-0.2210	-0.1580	-0.1309	-0.1083	-0.0909	-0.0759	-0.0626	-0.0490	-0.0374

Table 2

Coefficients $b_k^{(n)}$ for H=0.4

n/k	1	2	3	4	5	6	7	8	9	10
1	0.1377	-	-	-	-	-	-	-	-	-
2	0.1500	-0.0893	-	-	-	-	-	-	-	-
3	0.1485	-0.0868	-0.0163	-	-	-	-	-	-	-
4	0.1481	-0.0890	-0.0125	-0.0252	-	-	-	-	-	-
5	0.1477	-0.0892	-0.0141	-0.0226	-0.0174	-	-	-	-	-
6	0.1474	-0.0896	-0.0143	-0.0240	-0.0151	-0.0156	-	-	-	-
7	0.1472	-0.0898	-0.0146	-0.0242	-0.0163	-0.0136	-0.0133	-	-	-
8	0.1471	-0.0899	-0.0148	-0.0245	-0.0165	-0.0147	-0.0116	-0.0118	-	-
9	0.1469	-0.0901	-0.0150	-0.0247	-0.0168	-0.0148	-0.0125	-0.0102	-0.0105	-
10	0.1468	-0.0902	-0.0151	-0.0248	-0.0169	-0.0151	-0.0126	-0.0111	-0.0091	-0.0095

Table 3

Coefficients $b_k^{(n)}$ for H=0.7

n/k	1	2	3	4	5	6	7	8	9	10
1	0.5091	-	-	-	-	-	-	-	-	-
2	0.5212	-0.0238	-	-	-	-	-	-	-	-
3	0.5235	-0.0730	0.0945	-	-	-	-	-	-	-
4	0.5200	-0.0704	0.0755	0.0363	-	-	-	-	-	-
5	0.5186	-0.0735	0.0784	0.0149	0.0410	-	-	-	-	-
6	0.5173	-0.0740	0.0760	0.0172	0.0250	0.0309	-	-	-	-
7	0.5164	-0.0747	0.0755	0.0151	0.0271	0.0165	0.0278	-	-	-
8	0.5158	-0.0750	0.0748	0.0147	0.0253	0.0183	0.0153	0.0241	-	-
9	0.5152	-0.0754	0.0745	0.0142	0.0250	0.0167	0.0170	0.0129	0.0216	-
10	0.5148	-0.0756	0.0741	0.0139	0.0245	0.0164	0.0155	0.0144	0.0116	0.0194

Table 4

Coefficients $b_k^{(n)}$ for H=0.9

n/k	1	2	3	4	5	6	7	8	9	10
1	0.8212	-	-	-	-	-	-	-	-	-
2	0.8078	0.0163	-	-	-	-	-	-	-	-
3	0.8048	-0.1323	0.1840	-	-	-	-	-	-	-
4	0.7906	-0.1221	0.1219	0.0772	-	-	-	-	-	-
5	0.7841	-0.1324	0.1322	0.0103	0.0845	-	-	-	-	-
6	0.7788	-0.1331	0.1239	0.0187	0.0352	0.0629	-	-	-	-
7	0.7752	-0.1351	0.1228	0.0116	0.0427	0.0187	0.0568	-	-	-
8	0.7724	-0.1360	0.1207	0.0111	0.0367	0.0253	0.0189	0.0489	-	-
9	0.7703	-0.1368	0.1196	0.0094	0.0363	0.0200	0.0249	0.0150	0.0439	-
10	0.7685	-0.1374	0.1187	0.0087	0.0348	0.0196	0.0202	0.0204	0.0135	0.0394

The situation with the signs of the coefficients, to some extent, reflects the situation with overlapping intervals. However, it is also a mixture with the situation of projection of the increments of fBm: for small values of H all coefficients are negative, as in the case of increments of fBm, but for $H > \frac{1}{2}$ the coefficients become strictly positive, except for the second one, and for the increments of fBm they are all positive.

To better illustrate the behavior of the coefficients, we present the plots of the coefficients as functions of H for $n = 2,\ 3$ (Fig. 1).

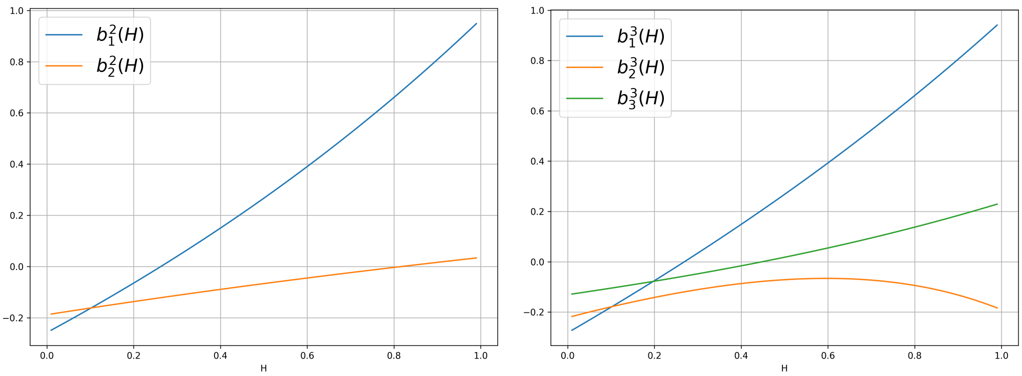

Fig. 1. LEFT – Case $n = 2$: $b_2^{(1)}$, $b_2^{(2)}$ as functions of H. RIGHT – Case $n = 3$: $b_3^{(1)}$, $b_3^{(2)}$, $b_3^{(3)}$ as functions of H

4. Projection of "non-overlapping" increments

Our previous analysis focused on the general form of the projection, which involved increments on the overlapping intervals, leading to complexities in coefficient behavior. Let us now consider the projection $\mathbb{E}[\Delta I_0 \mid \Delta I_2, \Delta I_4, \ldots, \Delta I_{2p}]$, where $p \geqslant 1$. The conditional expectation then can be written as follows

$$\mathbb{E}[\Delta I_0 \mid \Delta I_2, \Delta I_4, \ldots, \Delta I_{2p}] = \sum_{k=1}^{p} \tilde{b}_k^{(p)} \Delta I_{2k}, \quad p \geqslant 1. \tag{4}$$

To obtain the coefficients $\tilde{b}_k^{(p)}$, we multiply both sides of the projection equation (4) by ΔI_{2l} for $l = 1, \ldots, p$, and take the expectation. This yields the following system of linear equations

$$\mathbb{E}[\Delta I_0 \Delta I_{2l}] = \sum_{k=1}^{p} \tilde{b}_k^{(p)} \mathbb{E}[\Delta I_{2k} \Delta I_{2l}].$$

Using the definition $\rho_d = \mathbb{E}[\Delta I_0 \Delta I_d]$ and the stationarity property $\mathbb{E}[\Delta I_i \Delta I_j] = \rho_{|i-j|}$, we get

$$\rho_{2l} = \sum_{k=1}^{p} \tilde{b}_k^{(p)} \rho_{|2k-2l|}, \quad l = 1, \ldots, p.$$

Let us define the matrix $\tilde{A}^{(p)} \in \mathbb{R}^{p \times p}$ with entries

$$\tilde{A}_{ij} = \rho_{|2i-2j|} = \rho_{2|i-j|}, \quad i, j \in \{1, \ldots, p\}.$$

Then the matrix $\tilde{A}^{(p)}$ has the following form

$$\tilde{A}^{(p)} = \begin{pmatrix} \rho_0 & \rho_2 & \rho_4 & \cdots & \rho_{2(p-1)} \\ \rho_2 & \rho_0 & \rho_2 & \cdots & \rho_{2(p-2)} \\ \rho_4 & \rho_2 & \rho_0 & \cdots & \rho_{2(p-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{2(p-1)} & \rho_{2(p-2)} & \rho_{2(p-3)} & \cdots & \rho_0 \end{pmatrix}.$$

Let the vector of coefficients be $\tilde{\mathbf{b}}^{(p)} = (\tilde{b}_1^{(p)}, \ldots, \tilde{b}_p^{(p)})^T$, and vector $\tilde{\mathbf{r}}^{(p)}$ be defined as $\tilde{\mathbf{r}}^{(p)} = (\rho_2, \rho_4, \ldots, \rho_{2p})^T$. Then

$$\tilde{A}^{(p)} \tilde{\mathbf{b}}^{(p)} = \tilde{\mathbf{r}}^{(p)}, \text{ and } \tilde{b}_k^{(p)} = \frac{\det \tilde{A}_k^{(p)}}{\det \tilde{A}^{(p)}}, \tag{5}$$

where $\tilde{A}_k^{(p)}$ is the matrix obtained from $\tilde{A}^{(p)}$ by replacing the kth column with the vector $\tilde{\mathbf{r}}^{(p)}$. The covariance function $\rho_d = \rho(d, H) = \mathbb{E}[\Delta I_0 \Delta I_d]$ is given by Lemma 3, as before.

We will conduct our further analysis of the coefficients in a way similar to that we did in the previous section. Specifically, we will consider the cases $p = 1$ and $p = 2$, for which we write down useful expressions, namely ρ_0, ρ_2, ρ_4, in the following closed form

$$\rho_0 = \frac{8 \cdot 4^H - 8}{4(H+1)(2H+1)}, \quad \rho_2 = \frac{16 \cdot 16^H - 36 \cdot 9^H + 24 \cdot 4^H - 4}{4(H+1)(2H+1)},$$

$$\rho_4 = \frac{36 \cdot 36^H - 100 \cdot 25^H + 96 \cdot 16^H - 36 \cdot 9^H + 4 \cdot 4^H}{4(H+1)(2H+1)}.$$

Case $p = 1$: In this case, the projection coefficient $\tilde{b}_1^{(1)}$ is given by $\rho_0 \tilde{b}_1^{(1)} = \rho_2$. Thus, we trivially get

$$\tilde{b}_1^{(1)} = \frac{\rho_2}{\rho_0} = \frac{16 \cdot 16^H - 36 \cdot 9^H + 24 \cdot 4^H - 4}{8 \cdot 4^H - 8}.$$

As before, ρ_0 is a variance and thus always positive. The behavior of $\tilde{b}_1^{(1)}$ is then defined by the numerator ρ_2.

Proposition 3. *The coefficient $\tilde{b}_1^{(1)}$ is negative (positive) for $H < \frac{1}{2}$ $(H > \frac{1}{2})$.*

Proof. By Lemma 4, we know that ρ_2, and thus $\tilde{b}_1^{(1)}$, is negative (positive) for $H < \frac{1}{2}$ $(H > \frac{1}{2})$. □

Case $p = 2$: Here, the coefficients $\tilde{b}_1^{(2)}$ and $\tilde{b}_2^{(2)}$ are given by the system of equations $\tilde{A}^{(2)} \tilde{\mathbf{b}}^{(2)} = \tilde{\mathbf{r}}^{(2)}$, where

$$\tilde{A}^{(2)} = \begin{pmatrix} \rho_0 & \rho_2 \\ \rho_2 & \rho_0 \end{pmatrix}, \quad \tilde{r}^{(2)} = \begin{pmatrix} \rho_2 \\ \rho_4 \end{pmatrix}.$$

The coefficients $\tilde{\mathbf{b}}^{(2)}$ equal

$$\tilde{b}_1^{(2)} = \frac{\rho_2(\rho_0 - \rho_4)}{\rho_0^2 - \rho_2^2}, \quad \tilde{b}_2^{(2)} = \frac{\rho_0\rho_4 - \rho_2^2}{\rho_0^2 - \rho_2^2}.$$

As repeatedly emphasized, the denominators are the determinants of covariance matrices, hence strictly positive. Therefore, the behavior of the coefficients is defined by the numerators $\rho_2(\rho_0 - \rho_4) =: f_1(H),\ \rho_0\rho_4 - \rho_2^2 =: f_2(H)$ with the following closed form

$$f_1(H) = \frac{f_1^*(H)}{(H+1)^2(2H+1)^2}, \quad f_2(H) = \frac{f_2^*(H)}{(H+1)^2(2H+1)^2},$$

where f_1^*, f_2^* are given by

$$\begin{aligned} f_1^*(H) = &-36 \cdot 576^H + 100 \cdot 400^H + 81 \cdot 324^H - 96 \cdot 256^H - 225 \cdot 225^H + 198 \cdot 144^H + 150 \cdot 100^H - \\ &-81 \cdot 81^H - 140 \cdot 64^H + 54 \cdot 36^H - 25 \cdot 25^H + 22 \cdot 16^H + 9 \cdot 9^H - 13 \cdot 4^H + 2, \\ f_2^*(H) = &-16 \cdot 256^H + 90 \cdot 144^H - 50 \cdot 100^H - 81 \cdot 81^H + 72 \cdot 36^H + 50 \cdot 25^H - 74 \cdot 16^H + 10 \cdot 4^H - 1. \end{aligned}$$

Proposition 4. *The coefficients $\tilde{b}_1^{(2)}, \tilde{b}_2^{(2)}$ are negative (positive) for $H < \frac{1}{2}$ $(H > \frac{1}{2})$.*

Proof. (i) In case of $\tilde{b}_1^{(2)}$, we use the same reasoning as in the previous section, since the expression in the numerator simply factorizes to $\rho_2(\rho_0 - \rho_4)$. We know that $\rho_0 > \rho_4$ by the general variance-covariance inequality and thus the behavior of the numerator is defined by ρ_2, which was discussed previously. Hence, $\tilde{b}_1^{(2)}$ is negative for $H < \frac{1}{2}$ and positive for $H > \frac{1}{2}$.

(ii) The determination of the sign of $\tilde{b}_2^{(2)}$ for $H < \frac{1}{2}$ follows from Lemma 4, which establishes that $\rho_0\rho_4 - \rho_2^2 < 0$ within this domain, since ρ_0 is strictly positive and $\rho_2 < 0,\ \rho_4 < 0$ for $H < \frac{1}{2}$. However, the case $H > \frac{1}{2}$ requires a different approach, which is not accessible at the moment. Alternatively, we present a plot of $\tilde{b}_2^{(2)}$ at (Fig. 2) to convey its behavior. □

Remark 3. Similarly to Remark 2, we can investigate the behavior of f_2^* around interesting points using Taylor expansions. First, we check the behavior near $H = 0$. We verify that $f_2^*(0) = 0$, $f_2^{*\prime}(0) = 0$, $f_2^{*\prime\prime}(0) < 0$ and then apply the Taylor expansion around $H_0 = 0$, which immediately gives us that $f_2^*(H)$ starts negative for small positive H. Next, we analyze the function near $H = \frac{1}{2}$. We make sure that $f_2^*(\frac{1}{2}) = 0$ and $f_2^{*\prime}(\frac{1}{2}) > 0$. Since $f_2^{*\prime}(\frac{1}{2}) > 0$, the Taylor expansion around $H_0 = \frac{1}{2}$ shows that the function starts positive for $H > \frac{1}{2}$. Finally, we analyze the function near $H = 1$. We ensure that $f_2^*(1) = 0$ and $f_2^{*\prime}(1) < 0$. Since $f_2^{*\prime}(1) < 0$, the function is decreasing at $H = 1$. This shows that for $H < 1$ and $H \approx 1$, $f_2^*(H) > f_2^*(1) = 0$, so the function was positive and decreases towards zero at $H = 1$.

Numerical results

Here we present the numerically computed coefficients $\tilde{b}_k^{(p)}$ for $p = 1, \ldots, 10$. The results below are listed for the Hurst indices $H = 0.1,\ 0.4,\ 0.7,\ 0.9$ (Tab. 5-8).

Table 5

Coefficients $\tilde{b}_k^{(p)}$ for H=0.1

p/k	1	2	3	4	5	6	7	8	9	10
1	-0.1391	-	-	-	-	-	-	-	-	-
2	-0.1462	-0.0513	-	-	-	-	-	-	-	-
3	-0.1476	-0.0552	-0.0267	-	-	-	-	-	-	-
4	-0.1480	-0.0561	-0.0291	-0.0165	-	-	-	-	-	-
5	-0.1482	-0.0564	-0.0298	-0.0181	-0.0112	-	-	-	-	-
6	-0.1483	-0.0566	-0.0300	-0.0186	-0.0124	-0.0081	-	-	-	-
7	-0.1483	-0.0567	-0.0301	-0.0188	-0.0127	-0.0091	-0.0062	-	-	-
8	-0.1484	-0.0567	-0.0302	-0.0188	-0.0129	-0.0093	-0.0069	-0.0049	-	-
9	-0.1484	-0.0567	-0.0302	-0.0189	-0.0130	-0.0094	-0.0071	-0.0055	-0.0040	-
10	-0.1484	-0.0568	-0.0302	-0.0189	-0.0130	-0.0095	-0.0072	-0.0057	-0.0044	-0.0033

Table 6

Coefficients $\tilde{b}_k^{(p)}$ for H=0.4

p/k	1	2	3	4	5	6	7	8	9	10
1	-0.0686	-	-	-	-	-	-	-	-	-
2	-0.0708	-0.0314	-	-	-	-	-	-	-	-
3	-0.0714	-0.0328	-0.0202	-	-	-	-	-	-	-
4	-0.0717	-0.0333	-0.0212	-0.0148	-	-	-	-	-	-
5	-0.0719	-0.0335	-0.0216	-0.0156	-0.0116	-	-	-	-	-
6	-0.0720	-0.0337	-0.0218	-0.0159	-0.0123	-0.0095	-	-	-	-
7	-0.0720	-0.0338	-0.0220	-0.0161	-0.0126	-0.0101	-0.0081	-	-	-
8	-0.0721	-0.0339	-0.0221	-0.0162	-0.0127	-0.0104	-0.0086	-0.0070	-	-
9	-0.0721	-0.0339	-0.0221	-0.0163	-0.0128	-0.0105	-0.0088	-0.0074	-0.0061	-
10	-0.0722	-0.0340	-0.0222	-0.0164	-0.0129	-0.0106	-0.0089	-0.0076	-0.0065	-0.0055

Table 7

Coefficients $\tilde{b}_k^{(p)}$ for H=0.7

p/k	1	2	3	4	5	6	7	8	9	10
1	0.2416	-	-	-	-	-	-	-	-	-
2	0.2172	0.1008	-	-	-	-	-	-	-	-
3	0.2105	0.0864	0.0663	-	-	-	-	-	-	-
4	0.2073	0.0821	0.0559	0.0495	-	-	-	-	-	-
5	0.2053	0.0799	0.0527	0.0413	0.0395	-	-	-	-	-
6	0.2040	0.0786	0.0509	0.0387	0.0327	0.0329	-	-	-	-
7	0.2031	0.0776	0.0499	0.0372	0.0305	0.0271	0.0282	-	-	-
8	0.2024	0.0770	0.0491	0.0363	0.0293	0.0252	0.0232	0.0246	-	-
9	0.2018	0.0765	0.0486	0.0357	0.0285	0.0241	0.0215	0.0202	0.0219	-
10	0.2014	0.0761	0.0481	0.0352	0.0279	0.0234	0.0205	0.0187	0.0179	0.0197

Table 8

Coefficients $\tilde{b}_k^{(p)}$ for H=0.9

p/k	1	2	3	4	5	6	7	8	9	10
1	0.6797	-	-	-	-	-	-	-	-	-
2	0.5226	0.2311	-	-	-	-	-	-	-	-
3	0.4880	0.1527	0.1500	-	-	-	-	-	-	-
4	0.4715	0.1360	0.0965	0.1095	-	-	-	-	-	-
5	0.4621	0.1276	0.0848	0.0689	0.0862	-	-	-	-	-
6	0.4560	0.1228	0.0788	0.0598	0.0534	0.0711	-	-	-	-
7	0.4517	0.1195	0.0751	0.0550	0.0460	0.0435	0.0604	-	-	-
8	0.4485	0.1172	0.0727	0.0521	0.0420	0.0372	0.0367	0.0526	-	-
9	0.4460	0.1155	0.0710	0.0502	0.0396	0.0339	0.0313	0.0317	0.0465	-
10	0.4441	0.1142	0.0697	0.0488	0.0379	0.0318	0.0283	0.0269	0.0279	0.0417

Here we can see that the behavior of the projection coefficients in the non-overlapping scenario is consistent with the properties of fBm: all coefficients are negative for $H < \frac{1}{2}$, and all coefficients are positive for $H > \frac{1}{2}$.

As in the previous case, we present the plots of the coefficients as functions of H for $p = 2,\ 3$ (Fig. 2).

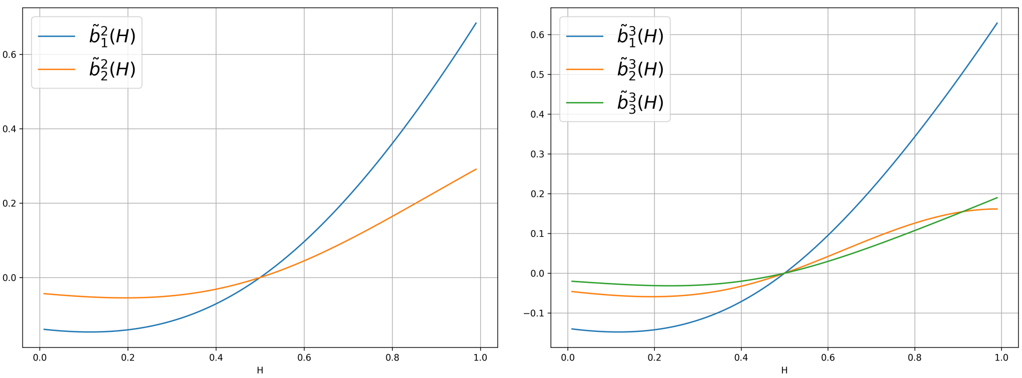

Fig. 2. LEFT – Case $p = 2$: $\tilde{b}_2^{(1)}$, $\tilde{b}_2^{(2)}$ as functions of H. RIGHT – Case $p = 3$: $\tilde{b}_3^{(1)}$, $\tilde{b}_3^{(2)}$, $\tilde{b}_3^{(3)}$ as functions of H

5. Calculation of the Prediction Error

In this section, we derive the formula for the Mean Squared Error (MSE) of the projection discussed in Section 3. The error of the prediction is defined as $e_n = \Delta I_0 - \mathbb{E}[\Delta I_0 \mid \Delta I_1, \ldots, \Delta I_n]$. Since the process is centered, the MSE is simply the variance of this error, $\text{MSE}_n = \mathbb{E}[e_n^2]$, and it equals

$$\text{MSE}_n = \rho_0 - \sum_{k=1}^{n} \rho_k b_k^{(n)}$$

for overlapping, and

$$\text{MSE}_p = \rho_0 - \sum_{k=1}^{p} \rho_{2k} \tilde{b}_k^{(p)}$$

for non-overlapping intervals, where the values of coefficients are given by (3) and (5). In Tab. 9, we can see that the MSE decreases as the number of predictors grows, both in the overlapping and non-overlapping scenarios. It is interesting to see that MSE decreases as H approaches its boundary points $H = 0$ and $H = 1$ and increases when H is close to $1/2$. The behaviour around $H = 1$, where fBm and consequently ifBm degenerate to $\xi \cdot \phi(t)$, where $\phi(t) = t$ for fBm, and $\phi(t) = t^2/2$ for ifBm, is very clear. The same is about $H = 1/2$ because the terms of prediction are independent for non-overlapping intervals. When H approaches 0, fBm tends in distribution to white noise, and therefore the norm of projection increases, and consequently MSE decreases.

Table 9

Mean Squared Error (MSE) for the Overlapping Case

n	H = 0.01	H = 0.4	H = 0.7	H = 0.99
1	0.0259	0.5770	0.5952	0.0366
2	0.0250	0.5724	0.5949	0.0366
3	0.0246	0.5723	0.5895	0.0347
4	0.0243	0.5719	0.5888	0.0344
5	0.0242	0.5717	0.5878	0.0340
6	0.0240	0.5716	0.5872	0.0338
7	0.0239	0.5715	0.5868	0.0336
8	0.0239	0.5714	0.5864	0.0335
9	0.0238	0.5714	0.5861	0.0334
10	0.0238	0.5713	0.5859	0.0333

Mean Squared Error (MSE) for the Non-Overlapping Case

p	H = 0.01	H = 0.4	H = 0.7	H = 0.99
1	0.0266	0.5854	0.7566	0.0696
2	0.0266	0.5848	0.7489	0.0637
3	0.0266	0.5846	0.7456	0.0614
4	0.0266	0.5845	0.7437	0.0602
5	0.0266	0.5844	0.7426	0.0595
6	0.0266	0.5843	0.7418	0.0591
7	0.0266	0.5843	0.7412	0.0587
8	0.0266	0.5843	0.7407	0.0585
9	0.0266	0.5842	0.7404	0.0583
10	0.0266	0.5842	0.7401	0.0581

Discussion and conclusions

In this work, we have successfully investigated the projection of increments of integrated fractional Brownian motion (ifBm). We began by formally introducing ifBm, deriving its covariance function, and establishing the key property that its increments are stationary. Our primary contribution lies in the determination of the coefficients for the linear prediction of an ifBm increment, ΔI_0, based on a series of subsequent increments, $(\Delta I_1, ..., \Delta I_n)$. We derived exact analytical formulas for the projection onto a single subsequent increment in both overlapping and non-overlapping scenarios. For projections onto two increments, we provided partially analytical results. The corresponding code for reproducing these results is provided in our repository[1].

To handle more complex cases, we computed numerical solutions for projections onto as many as ten increments for both interval configurations. A significant finding of this study is the complex behavior of these projection coefficients, particularly the sign changes observed as a function of the Hurst index H. This reveals a non-trivial dependence structure, especially when the intervals overlap. However, the covariance structure of the fractional Brownian motion poses a significant problem in terms of formal analysis, and thus needs new approaches.

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[1] https://github.com/RadicallyUprooted/prediction_formulas_for_integrated_fbm

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ФОРМУЛИ ПРОГНОЗУ ДЛЯ ІНТЕГРОВАНОГО ДРОБОВОГО БРОУНІВСЬКОГО РУХУ

Досліджено проєкцію (прогноз) приростів інтегрованого дробового броунівського руху (ідБр). Уведено ідБр, обчислено його коваріаційну функцію та встановлено стаціонарність його приростів. Основна мета роботи — визначити коефіцієнти для лінійного прогнозу майбутнього приросту ідБР на основі ряду минулих приростів. Показано, що коефіцієнти прогнозу демонструють складну поведінку, зі зміною знака залежно від показника Хюрста (H). Для приростів на інтервалах, що перетинаються, ця зміна знака відбувається за нетривіальних значень H. На відміну від цього, для приростів на інтервалах, що не перетинаються, зміна знака відбувається за H=0.5, що узгоджується з властивостями самого дробового броунівського руху. Дослідження надає явні формули, де це можливо, та велику кількість таблиць із чисельно обрахованими значеннями коефіцієнтів проєкції для різних значень індексу Хюрста, пропонуючи розуміння властивостей і прогнозу ідБР, що закладає основу для подальших досліджень.

Ключові слова: ***дробовий броунівський рух, інтегрований дробовий броунівський рух, коваріаційна функція, задача проєкції, коефіцієнти проєкції.***

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