ДИФЕРЕНЦІАЛЬНІ РІВНЯННЯ, МАТЕМАТИЧНА ФІЗИКА ТА МЕХАНІКА

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AXISYMMETRIC POROELASTICITY PROBLEM FOR A MULTILAYERED CYLINDER

Poroelastic materials are the object of close attention of researchers due to their wide representation both in the natural environment (geological formations, biological tissues) and in technical applications (engineering structures, filtration systems). Among them, cylindrical structures are of particular importance, often characterised by radial layering and heterogeneity. The study of the stress-strain state of such objects is of significant practical importance for calculating their strength, stability, and efficient operation. Despite the considerable amount of scientific work, most of it is based on numerical modelling. At the same time, it is analytical methods that allow us to gain a deeper understanding of physical laws, identify limit regimes, and verify numerical approaches.

In this paper, we consider an analytical solution of an axisymmetric problem for a finite-length poroelastic cylinder with a radially layered structure. A load is applied to the cylindrical surface, which can be either a mechanical or a pressure load. The top and bottom edges of the cylinder are in smooth contact and are impermeable. Perfect mechanical contact is maintained between the layers. The application of the Fourier integral transform method allows us to reduce the original problem to a one-dimensional vector boundary value problem, the general solution of which is found using the matrix differential calculation. The method of recurrence relations is used to find the unknown constants of each layer of the cylinder. As a result, an exact solution was derived, which allows us to study the distribution of normal stress and pore pressure depending on the applied load, geometric characteristics, and physical and mechanical properties of the layers. The obtained results are important for the further development of the analytical mechanics of poroelastic media

K e y w o r d s : ***axisymmetric problem, poroelasticity, multilayered cylinder, integral Fourier transform, matrix differential calculation, recurrent correspondences.***

AMS 2020 classification: 35Q74, 42A38, 74G05.

Introduction

Relevance of research. The investigation of the stress-strain state of layered finite poroelastic cylinders is an important scientific and practical task due to the growing requirements for the reliability and efficiency of structures made of poroelastic materials. Such cylindrical elements are widely used in engineering practice. The presence of a layered structure along the radius allows for structural optimisation of the distribution of mechanical characteristics, which provides increased resistance to external loads. Despite the existence of a significant number of studies on the mechanics of poroelastic bodies, the issue of stress and strain distribution in finite layered poroelastic cylinders, taking into account boundary conditions and interlayer interaction, remains insufficiently studied. The construction of mathematical models for such objects, as well as the analytical solution of the corresponding boundary value problems, is important for the development of the theory of poroelasticity and the practical application of its results in the design of advanced materials and structures.

The object of research is a poroelastic layered solid cylinder with ideal contact conditions between the layers under the action of a load applied along a cylindrical surface.

The aim and objectives of the research are to construct an analytical solution of the poroelasticity problem for a multi-layered solid cylinder with ideal contact conditions between the layers under the action of a load applied along a cylindrical surface, as well as to construct and analyse graphs of the distribution of normal stress and fluid pressure

In recent years, many scientific papers have been published on numerical methods for finding solutions and studying important characteristics of multilayer poroelastic cylinders (Eldeeb, Shabana, & Elsawaf, 2021; Bociu et al., 2021). Some studies combine analytical and numerical approaches (Naccache et al., 2022; Gnadjro, d'Almeida, & Franklin, 2024; Grigorenko, Y., Grigorenko, O., & Rozhok, 2022). At the same time, only analytical methods allow us to study important qualitative properties of solutions (Gohari, Zarastvand, & Talebitooti, 2020; Kubenko et al., 2023; Zhupanska, & Ulitko, 2005; Meleshko, & Tokovyy, 2012). In this paper, we present an analytical approach that allows us to find an exact solution of the axisymmetric poroelasticity problem for a multilayered solid cylinder.

1. Statement of the problem

The finite solid poroelastic cylinder $0 < r < 1, 0 < z < h, -\pi < \varphi < \pi$, which consists of N layers $a_{i-1} < r < a_i, i = \underline{= 1, N}, a_0 = 0, a_N = 1$ is considered. The ideal contact conditions are fulfilled between the layers (Cheng, 2016)

$$u_l|_{r=a_l-0} = u_{l+1}|_{r=a_l+0}, w_l|_{r=a_l-0} = w_{l+1}|_{r=a_l+0}, p_l|_{r=a_l-0} = p_{l+1}|_{r=a_l+0}, \sigma_r^l|_{r=a_l-0} = \sigma_r^{l+1}|_{r=a_l+0},$$

$$\tau_{rz}^l|_{r=a_l-0} = \tau_{rz}^{l+1}|_{r=a_l+0}, k_l \frac{\partial p_l}{\partial r}|_{r=a_l-0} = k_{l+1} \frac{\partial p_{l+1}}{\partial r}|_{r=a_l+0}, l = \underline{1, N-1},$$

where $u_l(r,z), w_l(r,z)$ are dimensionless displacements of *l*-th layer of solid skeleton regarding axes r, z respectively, $p_l(r,z)$ is dimensionless pore pressure of *l*-th layer, $\sigma_r^l(r,z), \tau_{rz}^l(r,z)$ are dimensionaless normal and tangential stress of *l*-th layer correspondingly. At the cylindrical surface $r = 1$ the loading is applied $\sigma_r^N|_{r=1} = -L(z) - \alpha_N P(z)$, $\tau_{rz}^N|_{r=1} = T(z)$, $p_N|_{r=1} = P(z)$, where $L(z), T(z), P(z)$ are known functions, α_l is Biot's coefficient of *l*-th layer. The edges $z = 0, z = h$ are under slide contact conditions, and they are impermeable $w_l|_{z=0,h} = 0, \tau_{rz}^l|_{z=0,h} = 0, \frac{\partial p_l}{\partial z}|_{z=0,h} = 0$. The displacements and pore pressure should satisfy the system of equations (Verruijt, 2010). The stress-deformable state of the multilayered cylinder with indicated boundary and transmission conditions should be found.

2. The solving of the boundary value problem

The original problem is reduced to the one-dimensional boundary value problem with the help of finite sin-, cos- Fourier transform regarding variable z (Vaysfeld, & Zhuravlova, 2023). The derived one-dimensional problem in transform domain is written in a vector form

$$\{L_2\vec{y}_{l,\beta}(r)=0, a_{l-1}<r<a_l, A_\beta\vec{y}'_{N,\beta}(1)+B_\beta\vec{y}_{N,\beta}(1)=\vec{g}_\beta, \vec{y}_{l,\beta}|_{r=a_l-0}==\vec{y}_{l+1,\beta}|_{r=a_l+0}, \left(S_{l,\beta}\vec{y}'_{l,\beta}+T_{l,\beta}\vec{y}_{l,\beta}\right)|_{r=a_l-0} = \left(S_{l+1,\beta}\vec{y}'_{l+1,\beta}+T_{l+1,\beta}\vec{y}_{l+1,\beta}\right)|_{r=a_l+0}, \tag{1}$$

where differential operator

$$L_2=\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)-\frac{1}{r^2}-\frac{\kappa_l-1}{\kappa_l+1}\beta^2\ \frac{2\beta}{\kappa_l+1}\frac{d}{dr}-\alpha_l\frac{\kappa_l-1}{\kappa_l+1}\frac{d}{dr}\ -\frac{2\beta}{\kappa_l-1}\frac{1}{r}\frac{d}{dr}(r)\ \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)-\frac{\kappa_l+1}{\kappa_l-1}\beta^2\ \alpha_l\beta - -\frac{\alpha_l}{K_l}\frac{1}{r}\frac{d}{dr}(r)\ -\frac{\alpha_l\beta}{K_l}\ \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)-\beta^2-\frac{S_{Pl}}{K_l}\right),$$

vectors $\vec{y}_{l,\beta}(r)=\left(u_{l,\beta}(r),w_{l,\beta}(r),p_{l,\beta}(r)\right)^T$, $\vec{g}_\beta=\left(\frac{(\kappa_N-1)}{2}(\alpha_N P_\beta-L_\beta),T_\beta,P_\beta\right)^T$, matrices

$$A_\beta=(\kappa_N+1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0), B_\beta=(3-\kappa_N\ (3-\kappa_N)\beta\ 0\ -\beta\ 0\ 0\ 0\ 0\ 1), S_{l,\beta}=\left(\frac{\kappa_l+1}{\kappa_l-1}G_l\ 0\ 0\ 0\ G_l\ 0\ 0\ 0\ K_l\right), T_{l,\beta}= =\left(\frac{3-\kappa_l}{\kappa_l-1}\frac{G_l}{a_l}\ \frac{3-\kappa_l}{\kappa_l-1}\beta G_l\ 0\ -\beta G_l\ 0\ 0\ 0\ 0\ 0\right),$$

$\kappa_l=3-4\mu_l$ is Muskhelishvili's constant, μ_l is Poisson ratio, S_{Pl}, K_l – are dimensionless values of storativity of the pore space S_{pl} and permeability coefficient k_l of *l*-th layer respectively.

The apparatus of matrix differential calculation (Gantmacher, 1959) is applied for the find of general solution of the boundary value problem (1). According to it, the corresponding matrix differential equation $L_2Y_{l,\beta}(r)=0$ is considered. The following correspondences take place (Popov, & Protserov, 2016) $L_2Q_i(r,\xi)=-Q_i(r,\xi)M_{l,\beta}(\xi), i=1,2$, where

$$Q_1(r,\xi)=(J_1(\xi r)\ 0\ 0\ 0\ J_0(\xi r)\ 0\ 0\ 0\ J_0(\xi r)), Q_2(r,\xi)=\left(N_1(\xi r)\ 0\ 0\ 0\ N_0(\xi r)\ 0\ 0\ 0\ N_0(\xi r)\right),$$

$J_1(\xi r), J_0(\xi r)$ are Bessel functions, $N_1(\xi r), N_0(\xi r)$ are Neumann functions, $M_{l,\beta}(\xi)$ is known matrix. The solution of the matrix equation is found by the formula $Y_{l,\beta,j}(r)=\frac{1}{2\pi i}\oint_C\ Q_j(r,\xi)M^{-1}_{l,\beta}(\xi)d\xi, j=1,2$, where closed contour C covers all singularity points of the integrand. The genral solution of the vector boundary value problem (1) has the form $\vec{y}_{l,\beta}(r)=Y_{l,\beta}(r)\vec{C}_l$, where

$$Y_{l,\beta}(r)=\{Y_{1,\beta,1}(r), l=1, \left(Y_{l,\beta,1}(r),Y_{l,\beta,2}(r)\right), 2\le l\le N\ , \vec{C}_l=\{\left(c_{1,1},c_{1,2},c_{1,3}\right)^T, l=1, \left(c_{l,1},c_{l,2},c_{l,3},c_{l,4},c_{l,5},c_{l,6}\right)^T, 2\le l\le \le N\ , H_{l,\beta}(r)=\left(Y_{l,\beta}(r)\ S_{l,\beta}Y'_{l,\beta}(r)+T_{l,\beta}Y_{l,\beta}(r)\right).$$

The transmission conditions in (1) can be written in the following matrix form $H_{l,\beta}(a_l)\vec{C}_l=H_{l+1,\beta}(a_l)\vec{C}_{l+1}, l=\underline{1,N-1}$, from which, according to (Popov, 2013) the constants of each layer $\vec{C}_l$ are recurrendly expressed through the constants of the first layer $\vec{C}_1$:

$$\vec{C}_l=R_{l,\beta}\vec{C}_1=H^{-1}_{l,\beta}(a_{l-1})H_{l-1,\beta}(a_{l-1})\dots H^{-1}_{2,\beta}(a_1)H_{1,\beta}(a_1)\vec{C}_1, l=\underline{2,N}. \tag{2}$$

Substituting (2) in the boundary condition in (1), we derive $\vec{C}_1=\left[\left(A_\beta Y'_{N,\beta}(1)+B_\beta Y_{N,\beta}(r)(1)\right)R_{N,\beta}\vec{C}_1\right]^{-1}\vec{g}_\beta$. The case when $\beta=0$ is solved analogically.

3. The results of the numerical analysis

The derived exact solution of the original problem allows to conduct the numerical investigation of normal stress and pore pressure change inside the cylinder. This analysis was carried out for the applied concentrated mechanical loading $L(z)=\delta\left(z-\frac{h}{2}\right), T(z)=0, P(z)=0$ ($\delta(z)$ is Dirac delta function). Geometrical sizes of the cylinder were chosen as: $a_1=0.5\ m, h=1\ m$. The two-layered cylinder with the materials Ohio sandstone for the inner layer and Boise sandstone for the outer layer (Cheng, 2016) was studied.

At Fig. 1, the normal stress and pore pressure are symmetrical, which is explained by the same boundary conditions at the edges $z=0, z=h$ and the fact that the load is applied in the middle of the cylinder. The normal stress is negative, and no tensile stress is observed. The maximum is reached at the point of load application. The highest values are seen closer to the loaded face $r=1$. The pore pressure is nonzero, which is due to fluid leakage on the cylindrical surface, which also affects the compression of the cylinder.

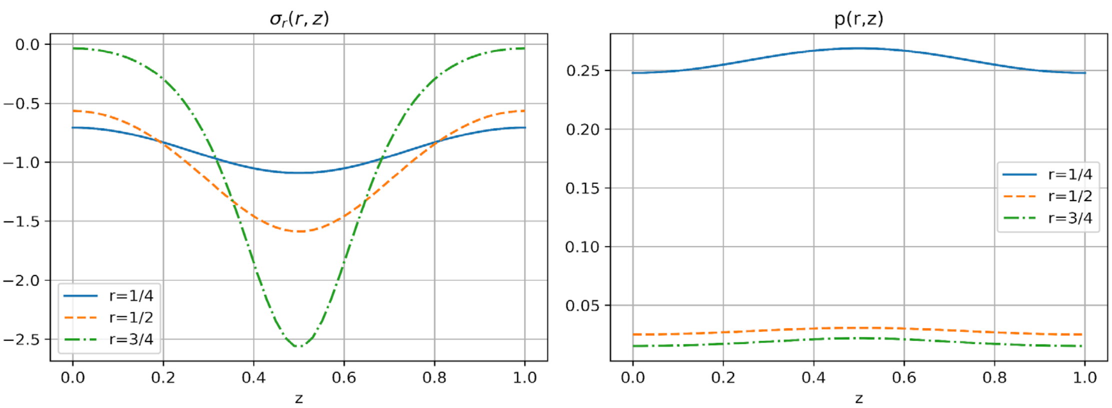

Fig. 1. Normal stress and pore pressure inside the cylinder

As can be seen at Fig. 2, an increase in the thickness of the inner layer of the cylinder increases the pore pressure inside this layer. This is due to the fact that the inner layer of the cylinder (Ohio sandstone) has a higher permeability coefficient. Changing the height of the cylinder also has a significant effect on the pore pressure. Thus, the highest pore pressure values are observed for a longer cylinder. This can be explained by the fact that the outer surface area through which the fluid can flow increases.

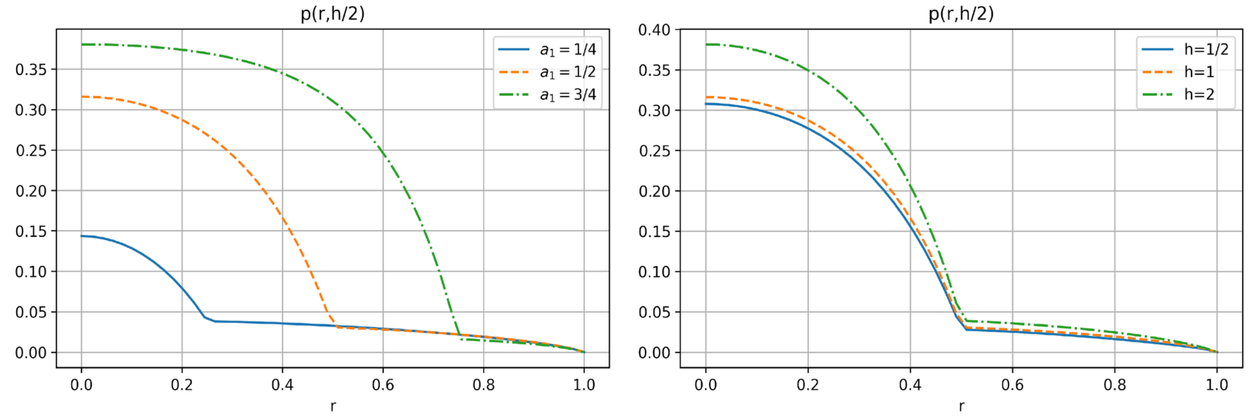

Fig. 2. The change of pore pressure regarding the change of thickness of the inner layer (left) and the change of the cylinder's height (right)

Discussion and conclusions

In the present study, an exact analytical solution of the axisymmetric problem for a finite multilayered poroelastic cylinder was obtained using the methods of integral transforms, matrix differential calculation, and recurrence relations. The numerical analysis revealed the nature of changes in normal stress and pore pressure depending on geometric parameters.

The highest absolute values of normal stress and pore pressure were recorded at the point of concentrated mechanical loading's application, which indicates a localised increase in the interaction between the solid matrix and the pore medium. It is established that an increase in the thickness of the inner layer causes an increase in the pore pressure in this area, while an increase in the height of the cylinder generally contributes to an increase in pore pressure along its entire length.

The proposed methodology demonstrates high versatility and can be adapted to analyse cylindrical structures with more complex conditions of interlayer communication, which opens up prospects for its further application in engineering practice and biomechanics.

Authors' contribution: Natalya Vaysfeld – preparation of theoretical foundations of research; conceptualization; Zinaida Zhuravlova – methodology; software.

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ОСЕСИМЕТРИЧНА ЗАДАЧА ПОРОПРУЖНОСТІ ДЛЯ ШАРУВАТОГО ЦИЛІНДРУ

Пороружні матеріали є об'єктом пильної уваги дослідників завдяки широкій представленості як у природному середовищі (геологічні формації, біологічні тканини), так і в технічних застосуваннях (інженерні конструкції, фільтраційні системи). Серед них особливої ваги набувають циліндричні структури, що нерідко характеризуються радіальною шаруватістю та неоднорідністю. Вивчення напружено-деформівного стану таких об'єктів має суттєве практичне значення для розрахунку їхньої міцності, стійкості й ефективного функціонування. Незважаючи на значний обсяг наукових праць, більшість з них базується на чисельному моделюванні. Водночас саме аналітичні методи дають змогу отримати глибше розуміння фізичних закономірностей, виявити граничні режими та забезпечити перевірку чисельних підходів.

У пропонованій роботі розглянуто аналітичне розв'язання осесиметричної задачі для скінченного за довжиною поропружного циліндра з радіально-шаруватою структурою. До циліндричної поверхні прикладено навантаження, що може бути як механічним, так і навантаженням тиску. Верхній і нижній торці циліндру перебувають в умовах гладкого контакту та є непроникними. Між шарами виконуються умови ідеального механічного контакту. Застосування методу інтегрального перетворення Фур'є надає можливість звести вихідну задачу до одновимірної векторної крайової задачі, загальний розв'язок якої отримано за допомогою апарату матричного диференціального числення. Для знаходження невідомих сталих кожного шару циліндру використано метод рекурентних співвідношень. У підсумку отримано точний розв'язок, що дає змогу дослідити розподіл нормальних напружень і тиску рідини залежно від прикладеного навантаження, геометричних характеристик і фізико-механічних властивостей шарів. Одержані результати є важливими для подальшого розвитку аналітичної механіки поропружних середовищ.

К л ю ч о в і   с л о в а : ***осесиметрична задача, поропружність, шаруватий циліндр, інтегральне перетворення Фур'є, матричне диференціальне числення, рекуренті співвідношення.***

Автори заявляють про відсутність конфлікту інтересів. Спонсори не брали участі в розробленні дослідження; у зборі, аналізі чи інтерпретації даних; у написанні рукопису; в рішенні про публікацію результатів.

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