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Теорія дуальності для не увігнутих функцій корисності за умов невизначеності моделі

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Duality theory under model uncertainty for non-concave utility functions

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Основна мета даної роботи полягає у вивченні функціоналу максимізації функції корисності,
$\sup_{X\in\mathcal{X}(x)}\inf_{Q\in\mathcal{Q}} E_Q[U(X_T)]$, *за умов повної ринкової моделі, коли інвестор не впевнений в ймовірнісній моделі і протистоїть як ризику, так і невизначеності моделі. У попередній літературі ця проблема була вивчена для строго увігнутих функцій корисності, і ми розширили існуючі результати для не увігнутих функцій корисності, розглядаючи їхнє увігнення.*

Ключові слова: функціонал максимізації функції корисності, мінімакс проблема, увігнення.

The main goal for this paper is to study the robust utility maximization functional, i.e. $\sup_{X\in\mathcal{X}(x)}\inf_{Q\in\mathcal{Q}} E_Q[U(X_T)]$, *of the terminal wealth in complete market models, when the investor is uncertain about the underlying probabilistic model and averse against both risk and model uncertainty. In the previous literature, this problem was studied for strictly concave utility functions and we extended existing results for non-concave utility functions by considering their concavization.*

Key Words: robust utility maximization functional, minimax problem, concavization.

1 Introduction

One of the most popular and challenging investment problems concerns utility-maximizing investments strategies. There is a lot of aspects which can be considered in this problem such as completeness of the market, properties of utility function, modeling of the payoff, probability measures etc. The main interest is in the very general setup, with an incomplete market, general sets of prior models and non-concave utility functions.

2 Model setup

We will consider the model as in [5] with the additional assumption that the discounted price process is locally bounded.

We assume that investor has utility function $U(x)$, the conditions on $U(x)$ will be stated later.

We consider the discounted price process with d assets which modeled by a stochastic process $S = (S_t)_{0\leqslant t\leqslant T}$. We assume that S is a d-dimensional locally bounded semimartingale on $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to a filtration $(\mathcal{F}_t)_{0\leqslant t\leqslant T}$. The pair (x, ξ) is a self-financing trading strategy, where $x \in \mathbb{R}$ is the initial wealth and $\xi = (\xi_t)_{0\leqslant t\leqslant T}$ is a d-dimensional predictable and S is an integrable process. The corresponding value process X satisfies

$$X_t = X_0 + \int_0^t \xi_r dS_r, \quad 0 \leqslant t \leqslant T.$$

By $\mathcal{X}(x)$, $x > 0$ we denote the set of all such processes X, with $X_0 \leqslant x$ which are also admissible in the sense that $X_t \geqslant 0$, for $0 \leqslant t \leqslant T$.

Additionally, the authors of [5] impose the assumptions on the set of probability measures $\mathcal{Q}$ on $(\Omega, \mathcal{F})$.

Assumption 1. *(i)* $\mathcal{Q}$ *is convex;*

(ii) $\mathbb{P}[A] = 0$ *if and only if* $Q[A] = 0$ *for all* $Q \in \mathcal{Q}$;

(iii) The set $\mathcal{Z} := \{dQ/dP | Q \in \mathcal{Q}\}$ *is closed in* $L^0(\mathbb{P})$.

Also, to the Assumption 1 we add

(iv) The set $\mathcal{Z}_e := \{dQ/dP | Q \in \mathcal{Q}_e\}$ *is closed in* $L^0(\mathbb{P})$,

where $\mathcal{Q}_e$ denotes the set of measures in $\mathcal{Q}$ that are equivalent to $\mathbb{P}$.

Assumption 2. *There is a unique equivalent local martingale measure, which we denote as* Q^e.

We consider the next value function of the robust utility problem

$$u(x) := \sup_{X \in \mathcal{X}(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(X_T)].$$

Also, we denote

$$u_Q(x) := \sup_{X \in \mathcal{X}(x)} E_Q[U(X_T)],$$

the value function of the optimal investment problem.

Consider the "abstract version"of $\mathcal{X}(x)$, as in [6, 7]:

$$C(x) := \{g \in L^0_+(\Omega, \mathcal{F}_T, \mathbb{P}) | 0 \leqslant g \leqslant X_T \text{ for some } X \in \mathcal{X}(x)\}.$$

It is easy to see that

$$u(x) = \sup_{X \in \mathcal{X}(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(X_T)] = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(g)];$$
$$u_Q(x) = \sup_{X \in \mathcal{X}(x)} E_Q[U(X_T)] = \sup_{g \in C(x)} E_Q[U(g)].$$

Remark 3. *It is known from Delbaen and Schachermayer (see [2] for the case of a locally bounded semimartingale* S, *[3] for the general case and [4] for more detailed version) that for* $g \geqslant 0$, *it holds that*

$$g \in C(x) \Longleftrightarrow \sup_{Q \in \mathcal{M}_e} E_Q(g) \leqslant x \Longleftrightarrow \sup_{Q \in \mathcal{M}_a} E_Q(g) \leqslant x,$$

where $\mathcal{M}_e$ *is the set of equivalent local martingale measures and* $\mathcal{M}_a$ *is the set of absolutely continuous local martingale measures.*

In our model setup $\mathcal{M}_e = \{Q^e\}$, and hence

$$g \in C(x) \Longleftrightarrow E_{Q^e}(g) \leqslant x.$$

3 Main result

We want to investigate the minimax identity for the robust non-concave utility functional in a complete market model, i.e.

$$\sup_{X \in \mathcal{X}(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(X_T)] = \inf_{Q \in \mathcal{Q}} \sup_{X \in \mathcal{X}(x)} E_Q[U(X_T)].$$

We consider utility function $U : [0, \infty) \to \mathbb{R}$ with $U(\infty) > 0$, which is non-constant, non-decreasing, upper semi-continuous and satisfies the mild growth condition:

$$\lim_{x \to \infty} \frac{U(x)}{x} = 0.$$

We set $U(x) = -\infty$ for $x < 0$.

It follows from [8, Proposition 3.1] that $U(x)$ has a non-decreasing and continuous concave envelope $U_c(x)$ that is the smallest concave function such that $U_c(x) \geqslant U(x)$ for all $x \in \mathbb{R}$.

Denote by

$$u^c(x) := \sup_{X \in \mathcal{X}(x)} \inf_{Q \in \mathcal{Q}} E_Q[U_c(X_T)] = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U_c(g)];$$
$$u^c_Q(x) := \sup_{X \in \mathcal{X}(x)} E_Q[U_c(X_T)] = \sup_{g \in C(x)} E_Q[U_c(g)].$$

Also, we need the finiteness of the value functions of the optimal investment problem, which we can write as

Assumption 4.

For all $x > 0$ *exists a measure* $Q_0 \in \mathcal{Q}_e$ *such that* $u_{Q_0}(x) < \infty$.

Assumption 5.

$u^c_{Q_0}(x) < \infty$ *for some, and hence for all* $x > 0$ *and some* $Q_0 \in \mathcal{Q}_e$.

Theorem 1. *Additionally to Assumption 1, 2, 3 and 4 we assume that the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is atomless. Then the following holds*

$$\begin{array}{ccccc}
\sup\limits_{g \in C(x)} \inf\limits_{Q \in \mathcal{Q}_e} E_Q[U_c(g)] & \overset{(1\star)}{=} & \sup\limits_{g \in C(x)} \inf\limits_{Q \in \mathcal{Q}} E_Q[U_c(g)] & \overset{(2\star)}{=} & \inf\limits_{Q \in \mathcal{Q}} \sup\limits_{g \in C(x)} E_Q[U_c(g)] \\
\verteq\!\!\!\!\!\geqslant (4\star) & & & & \| (3\star) \\
\sup\limits_{g \in C(x)} \inf\limits_{Q \in \mathcal{Q}_e} E_Q[U(g)] & & & & \inf\limits_{Q \in \mathcal{Q}_e} \sup\limits_{g \in C(x)} E_Q[U_c(g)] \\
\| (6\star) & & & & \| (5\star) \\
\sup\limits_{g \in C(x)} \inf\limits_{Q \in \mathcal{Q}} E_Q[U(g)] & \overset{(7\star)}{\leqslant} & \inf\limits_{Q \in \mathcal{Q}} \sup\limits_{g \in C(x)} E_Q[U(g)] & \overset{(8\star)}{\leqslant} & \inf\limits_{Q \in \mathcal{Q}_e} \sup\limits_{g \in C(x)} E_Q[U(g)]
\end{array}$$

The proof of this theorem will be divided into several parts.

Lemma 1. *Suppose that Assumptions 1 and Assumption 4 hold.*

Then, we have

$$u^c(x)$$
$$= \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U_c(g)] = \inf_{Q \in \mathcal{Q}} \sup_{g \in C(x)} E_Q[U_c(g)]$$
$$= \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)] = \inf_{Q \in \mathcal{Q}_e} \sup_{g \in C(x)} E_Q[U_c(g)]$$

Proof. *The proof is similar to the proof of [5, Lemma 3.4].*

We do not prove the minimax identity in the general case of non-concave functions but we establish some conditions under which it will hold.

Lemma 2. *If Assumptions 1 and Assumption 3 hold, then for all* $g \in C(x)$

$$\inf_{Q \in \mathcal{Q}} E_Q[U(g)] = \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)]. \tag{1}$$

Proof. *The idea of the proof is the same as in [5, Lemma 3.4].*

To prove equality ($5\star$), we follow the paper of Christian Reichlin [1]. He considered non-constant, increasing and upper semi-continuous function $U : (0, \infty) \to \mathbb{R}$, with $U(\infty) > 0$ satisfying a mild growth condition:

$$\lim_{x \to \infty} \frac{U(x)}{x} = 0.$$

We take the function

$$u(x, U) = \sup\{\mathbb{E}[U(f)] \,|\, f \in L^0_+(\Omega, \mathcal{F}, \mathbb{P}) \text{ with } E_Q(f) \leqslant x\},$$

for a (pricing) measure $Q \approx P$.

For convenience, introduce one more notation. Namely, the set

$$C'(x) := \{f \in L^0_+(\Omega, \mathcal{F}, \mathbb{P}) \text{ with } E_Q(f) \leqslant x\} \tag{2}$$

Note that $u(x, U) = \sup\limits_{f \in C'(x)} \mathbb{E}[U(f)]$.

One of the main results of the Reichlin's paper is [1, Theorem 5.1].

Theorem 2 (Theorem 5.1 by [1])**.** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be atomless. Then it holds that*

$$u(x, U) = u(x, U_c) \textit{ for all } x > 0.$$

The non-concave problem $u(x, U)$ *admits a maximizer if and only if the concavized problem* $u(x, U_c)$ *admits a maximizer. Every maximizer for the non-concave problem* $u(x, U)$ *also maximizes the concavized problem* $u(x, U_c)$*.*

Remark. *By our assumptions, there exists at least one equivalent martingale measure, which is also equivalent local martingale measure. Noting that* Q^e *is a unique equivalent local martingale measure it is also unique equivalent martingale measure.*

Noting Remark 3, we can use the Q^e as a pricing measure, in the Reichlin's paper and obtain

$$f \in C(x) \Longleftrightarrow f \in C'(x).$$

Then it follows from Theorem 2 that

$$\sup_{g \in C(x)} \mathbb{E}[U(g)] = \sup_{g \in C(x)} \mathbb{E}[U_c(g)], \text{ for all } x > 0.$$

Lemma 3. *Suppose Assumption 2 holds and assume that* $(\Omega, \mathcal{F}, \mathbb{P})$ *is atomless.*

Then it holds that

$$\inf_{Q \in \mathcal{Q}_e} \sup_{g \in C(x)} E_Q[U(g)] = \inf_{Q \in \mathcal{Q}_e} \sup_{g \in C(x)} E_Q[U_c(g)].$$

Proof. *First, note that for his theorem, C.Reichlin used only the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and pricing measure* $Q^p \approx \mathbb{P}$*. Moreover, the condition* $E_{Q^p}(f) \leqslant x$ *does not depend on* $\mathbb{P}$*. If* $(\Omega, \mathcal{F}, \mathbb{P})$ *is atomless then* $(\Omega, \mathcal{F}, Q)$ *is atomless for all* $Q \approx \mathbb{P}$*. Moreover, the set* $L^0_+(\Omega, \mathcal{F}, Q)$ *is the same for all* $Q \approx \mathbb{P}$*, and it means that everything remains true if we substitute* $\mathbb{P}$ *with another measure* $\widetilde{Q} \approx \mathbb{P}$*. And equivalent local martingale measure* Q^e *remains the same and unique. So, we use* Q^e *as a pricing measure in Reichlin's paper.*

Hence,

$$\sup_{g \in C(x)} E_Q[U(g)] = \sup_{g \in C(x)} E_Q[U_c(g)], \textit{ for all } Q \in \mathcal{Q}_e. \tag{3}$$

Consequently,

$$\inf_{Q \in \mathcal{Q}_e} \sup_{g \in C(x)} E_Q[U(g)] = \inf_{Q \in \mathcal{Q}_e} \sup_{g \in C(x)} E_Q[U_c(g)].$$

Proof of Theorem 1.

- ($1\star$) - ($3\star$) follows from Lemma 1;
- ($4\star$) follows from the fact that $U_c \geqslant U$;
- ($5\star$) follows from Lemma 3;
- To obtain ($6\star$) we need to take the $\sup\limits_{g \in C(x)}$ of the both sides in the equality (1);
- The inequality ($7\star$) follows from the fact that for all $Q \in \mathcal{Q}$ and all $g \in C(x)$ holds
  $$\inf_{Q \in \mathcal{Q}} E_Q[U_c(g)] \leqslant \sup_{g \in C(x)} E_Q[U_c(g)].$$
- Since $\mathcal{Q}_e \subseteq \mathcal{Q}$, the inequality ($8\star$) is clear.

$\square$

Lemma 4. *Suppose that Assumption 1 holds and that for all* $Q \in Q_e : u^c_Q(x) < \infty$ *for some* $x > 0$.

Additionally, assume that $\lim\limits_{x\to\infty} \frac{u^c_Q(x)}{x} = 0$, *for each* $Q \in \mathcal{Q}_e$. *Then, for any* $x > 0$, *there exist some* $\widehat{g} \in C(x)$ *and* $\widehat{Q} \in \mathcal{Q}$ *such that*

$$u^c(x) = \inf_{Q\in\mathcal{Q}} E_Q[U_c(\widehat{g})] = E_{\widehat{Q}}[U_c(\widehat{g})] = u^c_{\widehat{Q}}(x)$$

Proof. *The proof follows from the first part of the proof of [5, Theorem 2.6 and Lemma 4.1 (a)]. Note that we have* $U : [0,\infty) \to \mathbb{R}$, *hence we do not have to consider* $U(g+\varepsilon)$ *and proof of [5, Lemma 4.1 (a)] in our case is easier.*

The complete proof of the relation $u^c(x) = u^c_{\widehat{Q}}(x)$ *can be done similarly to the proof of Theorem 3 part* $(ii) \Rightarrow (i)$ *noting that minimax identity for concavized objective function* U_c *holds.*

Remark 6. *Actually, under assumptions of the lemma above for any* $x > 0$, *there exist some* $\widehat{g} \in C(x)$ *and* $\widehat{Q}_0 \in \mathcal{Q}_e$ *(instead of* $\mathcal{Q}$*) such that*

$$u^c(x) = \inf_{Q\in\mathcal{Q}} E_Q[U_c(\widehat{g})] = E_{\widehat{Q}_0}[U_c(\widehat{g})] = u^c_{\widehat{Q}_0}(x)$$

This is true because we add additional assumption on the set $\mathcal{Q}_e$ *to Assumption 1 which is that the set* $\mathcal{Z}_e$ *is closed in the* $L^0(\mathbb{P})$.

From now on by $\widehat{Q}_0 \in \mathcal{Q}_e$ we will denote such a measure for which the following holds

$$u^c(x) = u^c_{\widehat{Q}_0}(x).$$

Now we are going to present conditions under which the minimax identity for a non-concave utility function U holds.

Theorem 3. *Suppose that all assumptions of Theorem 1 hold.*

Then, the next two equalities are equivalent

(i) $\sup\limits_{g\in C(x)} \inf\limits_{Q\in\mathcal{Q}_e} E_Q[U(g)] = \sup\limits_{g\in C(x)} E_{\widehat{Q}}[U(g)]$, *for* $\widehat{Q} \in \mathcal{Q}_e$.

(ii) $\sup\limits_{g\in C(x)} \inf\limits_{Q\in\mathcal{Q}_e} E_Q[U(g)] = \sup\limits_{g\in C(x)} \inf\limits_{Q\in\mathcal{Q}_e} E_Q[U_c(g)]$

Proof. *Suppose that* (i) *holds, then from* (3) *it follows that*

$$\sup_{g\in C(x)} \inf_{Q\in\mathcal{Q}_e} E_Q[U(g)] = \sup_{g\in C(x)} E_{\widehat{Q}}[U(g)]$$
$$= \sup_{g\in C(x)} E_{\widehat{Q}}[U_c(g)] \geqslant \sup_{g\in C(x)} \inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g)].$$

Noting that for all $x \geqslant 0 : U_c(x) \geqslant U(x)$, *we have* (ii).

Assume (ii) *holds, then from Theorem 1 we obtain that*

$$\sup_{g\in C(x)} \inf_{Q\in\mathcal{Q}_e} E_Q[U(g)] = \inf_{Q\in\mathcal{Q}_e} \sup_{g\in C(x)} E_Q[U(g)]$$
$$= \inf_{Z\in\mathcal{Z}_e} \sup_{g\in C(x)} \mathbb{E}[ZU(g)].$$

From the definition of infimum it follows that exists such a sequence $(Z_n)_{n\in\mathbb{N}} \in \mathcal{Z}_e$ *that*

$$\lim_{n\to\infty} \sup_{g\in C(x)} \mathbb{E}[Z_nU(g)] = \inf_{Z\in\mathcal{Z}_e} \sup_{g\in C(x)} \mathbb{E}[ZU(g)].$$

Using [2, Lemma A1.1], we conclude tht there exists a sequence $(\widehat{Z}_n)_{n\in\mathbb{N}} \in conv(Z_n, Z_{n+1}, \dots)$ *such that* $\widehat{Z}_n$ *converges a.s. to some* $\widehat{Z}$.

Moreover, $(\widehat{Z}_n)_{n\in\mathbb{N}} \subset \mathcal{Z}_e$, *because* $(Z_n)_{n\in\mathbb{N}} \subset \mathcal{Z}_e \subset \mathcal{Z}$. *It follows that* $(\widehat{Z}_n)_{n\in\mathbb{N}} \subset \mathcal{Z}$ *and also for all integer* $n : \widehat{Z}_n \approx \mathbb{P}$.

From Assumption 1 (iv) it follows that $\widehat{Z} \in \mathcal{Z}_e$.

For each $n \in \mathbb{N}$ *there exists a sequence* $\alpha^n_i \in \mathbb{R}^+$ *such that* $\sum\limits_{i\in\mathbb{N}} \alpha^n_i = 1$ *and* $\widehat{Z}_n = \sum\limits_{i\in\mathbb{N}} \alpha^n_i Z_{n+i-1}$. *Hence,*

$$\sup_{g\in C(x)} \mathbb{E}[\widehat{Z}_nU(g)] = \sup_{g\in C(x)} \mathbb{E}\left[U(g)\sum_{i\in\mathbb{N}} \alpha^n_i Z_{n+i-1}\right]$$
$$\leqslant \sum_{i\in\mathbb{N}} \alpha^n_i \sup_{g\in C(x)} \mathbb{E}\left[Z_{n+i-1}U(g)\right].$$

Thus, by applying $\limsup\limits_{n\to\infty}$, *we obtain*

$$\limsup_{n\to\infty} \sup_{g\in C(x)} \mathbb{E}[\widehat{Z}_nU(g)] \leqslant \inf_{Z\in\mathcal{Z}_e} \sup_{g\in C(x)} \mathbb{E}[ZU(g)].$$

Noting that

$$\liminf_{n\to\infty} \sup_{g\in C(x)} \mathbb{E}[\widehat{Z}_nU(g)] \geqslant \inf_{Z\in\mathcal{Z}_e} \sup_{g\in C(x)} \mathbb{E}[ZU(g)],$$

we get that

$$\inf_{Z\in\mathcal{Z}_e} \sup_{g\in C(x)} \mathbb{E}\left[ZU(g)\right] = \lim_{n\to\infty} \sup_{g\in C(x)} \mathbb{E}[\widehat{Z}_nU(g)] \tag{4}$$

Since $U(g)$ *is bounded from below, it follows that there exists* $c > 0$ *such that* $U(g) + c > 0$. *From Fatou's lemma it follows that for all* $g \in C(x)$ *it holds that*

$$\mathbb{E}\left[\liminf_{n\to\infty} \widehat{Z}_nU(g)\right] \leqslant \liminf_{n\to\infty} \mathbb{E}\left[\widehat{Z}_nU(g)\right].$$

This gives us the relations

$$\sup_{g\in C(x)} \mathbb{E}\left[\liminf_{n\to\infty} \widehat{Z}_n U(g)\right] \leqslant \sup_{g\in C(x)} \liminf_{n\to\infty} \mathbb{E}\left[\widehat{Z}_n U(g)\right]. \tag{5}$$

Noting that for all $Z_n \in \mathcal{Z}_e$ *and all* $g \in C(x)$ *holds* $\mathbb{E}\left[\widehat{Z}_n U(g)\right] \leqslant \sup\limits_{g\in C(x)} \mathbb{E}\left[\widehat{Z}_n U(g)\right]$, *we have*

$$\liminf_{n\to\infty} \mathbb{E}\left[\widehat{Z}_n U(g)\right] \leqslant \liminf_{n\to\infty} \sup_{g\in C(x)} \mathbb{E}\left[\widehat{Z}_n U(g)\right]$$

Since g *is arbitrary, we obtain*

$$\sup_{g\in C(x)} \liminf_{n\to\infty} \mathbb{E}\left[\widehat{Z}_n U(g)\right] \leqslant \liminf_{n\to\infty} \sup_{g\in C(x)} \mathbb{E}\left[\widehat{Z}_n U(g)\right] \tag{6}$$

Combining (4), (5) *and* (6) *we get*

$$\sup_{g\in C(x)} \mathbb{E}[\widehat{Z}U(g)] \leqslant \inf_{Z\in\mathcal{Z}_e} \sup_{g\in C(x)} \mathbb{E}\left[ZU(g)\right]$$

Also, it is true that

$$\inf_{Z\in\mathcal{Z}_e} \sup_{g\in C(x)} \mathbb{E}\left[ZU(g)\right] \leqslant \sup_{g\in C(x)} \mathbb{E}[\widehat{Z}U(g)]$$

This relation concludes the proof.

Recall that it follows from Remark 6 that the measure $\widehat{Q}_0 \in \mathcal{Q}_e$.

Theorem 4. *Suppose that all assumptions from Lemma 4 and from Theorem 1 hold. Assume that at least one of the items below holds*

- *(i) There exists such a measure* $\widehat{Q} \in \mathcal{Q}_e$ *that for all* $g \in C(x) : \inf\limits_{Q\in\mathcal{Q}_e} E_Q[U_c(g)] = E_{\widehat{Q}}[U_c(g)]$;
- *(ii) For any sequence* $g_n \in C(x)$ *such that* $\lim\limits_{n\to\infty} E_{\widehat{Q}_0}[U_c(g_n)] = \sup\limits_{g\in C(x)} \inf\limits_{Q\in\mathcal{Q}_e} E_Q[U_c(g)]$ *it holds that*
  $$\lim_{n\to\infty} E_{\widehat{Q}_0}[U_c(g_n)] = \lim_{n\to\infty} \inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g_n)]$$

Then, we have

$$\sup_{g\in C(x)} \inf_{Q\in\mathcal{Q}_e} E_Q[U(g)] = \sup_{g\in C(x)} \inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g)].$$

Hence, all inequalities in Theorem 1 turn to equalities.

Proof. *Let's prove the first part, i.e:*

$$(i) \Longrightarrow \sup_{g\in C(x)} \inf_{Q\in\mathcal{Q}_e} E_Q[U(g)] = \sup_{g\in C(x)} \inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g)].$$

Due to the arguments of Lemma 3 it follows from [1, Proposition 5.3] that for all $g \in C(x)$ *exists* $g^\star \in C(x)$ *satisfying* $\{g^\star \in \{U < U_c\}\} = \emptyset$ *and*

$$E_{\widehat{Q}}[U_c(g)] = E_{\widehat{Q}}[U_c(g^\star)] = E_{\widehat{Q}}[U(g^\star)]$$

Since $\{g^\star \in \{U < U_c\}\} = \emptyset$, *we have*

$$\inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g^\star)] = \inf_{Q\in\mathcal{Q}_e} E_Q[U(g^\star)]$$

Hence, (i) *gives us the following equalities*

$$\begin{aligned} \inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g)] &= E_{\widehat{Q}}[U_c(g)] = E_{\widehat{Q}}[U_c(g^\star)] \\ &\overset{(i)}{=} \inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g^\star)] \\ &= \inf_{Q\in\mathcal{Q}_e} E_Q[U(g^\star)] \leqslant \sup_{g\in C(x)} \inf_{Q\in\mathcal{Q}_e} E_Q[U(g)]. \end{aligned}$$

Because $g \in C(x)$ *was arbitrary, we obtain*

$$\sup_{g\in C(x)} \inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g)] \leqslant \sup_{g\in C(x)} \inf_{Q\in\mathcal{Q}_e} E_Q[U(g)].$$

Noting that $U_c \geqslant U$, *we have*

$$\sup_{g\in C(x)} \inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g)] = \sup_{g\in C(x)} \inf_{Q\in\mathcal{Q}_e} E_Q[U(g)].$$

Now, let us prove the second part, i.e

$$(ii) \Longrightarrow \sup_{g\in C(x)} \inf_{Q\in\mathcal{Q}_e} E_Q[U(g)] = \sup_{g\in C(x)} \inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g)].$$

Recall that it follows from Remark 6 that $\widehat{Q}_0 \in \mathcal{Q}_e$, *so we have*

$$\begin{aligned} u^c(x) &= \sup_{g\in C(x)} \inf_{Q\in\mathcal{Q}} E_Q[U_c(g)] \\ &= \sup_{g\in C(x)} E_{\widehat{Q}_0}[U_c(g)] = u^c_{\widehat{Q}_0}(x). \end{aligned} \tag{7}$$

Note that due to Lemma 1 it holds that

$$\sup_{g\in C(x)} \inf_{Q\in\mathcal{Q}} E_Q[U_c(g)] = \sup_{g\in C(x)} \inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g)]$$

By the definition of supremum exists such a sequence $g_n \in C(x)$ *that*

$$\lim_{n\to\infty} \inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g_n)] = \sup_{g\in C(x)} \inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g)] \tag{8}$$

Noting that

$$\inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g_n)] \leqslant E_{\widehat{Q}_0}[U_c(g_n)] \leqslant \sup_{g\in C(x)} E_{\widehat{Q}_0}[U_c(g)],$$

we have

$$\begin{aligned}&\lim_{n\to\infty}\inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g_n)]\\ &\leqslant \lim_{n\to\infty} E_{\widehat{Q}_0}[U_c(g_n)] \leqslant \sup_{g\in C(x)} E_{\widehat{Q}_0}[U_c(g)]\end{aligned}$$

From (7) *and* (8) *we obtain*

$$\begin{aligned}\lim_{n\to\infty}\inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g_n)] &= \lim_{n\to\infty} E_{\widehat{Q}_0}[U_c(g_n)]\\ &= \sup_{g\in C(x)}\inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g)] \qquad (9)\end{aligned}$$

Due to the arguments of Lemma 3 it follows from [1, Proposition 5.3] that for all $g_n \in C(x)$ exists $g_n^\star \in C(x)$ satisfying $\{g_n^\star \in \{U < U_c\}\} = \emptyset$ and

$$E_{\widehat{Q}_0}[U_c(g_n)] = E_{\widehat{Q}_0}[U_c(g_n^\star)] = E_{\widehat{Q}_0}[U(g_n^\star)]$$

Leads to

$$\begin{aligned}\lim_{n\to\infty} E_{\widehat{Q}_0}[U_c(g_n)] &= \lim_{n\to\infty} E_{\widehat{Q}_0}[U_c(g_n^\star)]\\ &\overset{(9)}{=} \sup_{g\in C(x)}\inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g)]\end{aligned}$$

Then, it follows from (*ii*) *that*

$$\lim_{n\to\infty} E_{\widehat{Q}_0}[U_c(g_n^\star)] = \lim_{n\to\infty}\inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g_n^\star)]$$

Noting that $\{g_n^\star \in \{U < U_c\}\} = \emptyset$, we have

$$\begin{aligned}&\sup_{g\in C(x)}\inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g)] = \lim_{n\to\infty}\inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g_n^\star)]\\ &= \lim_{n\to\infty}\inf_{Q\in\mathcal{Q}_e} E_Q[U(g_n^\star)] \leqslant \sup_{g\in C(x)}\inf_{Q\in\mathcal{Q}_e} E_Q[U(g)]\end{aligned}$$

Since $U_c \geqslant U$, we obtain

$$\sup_{g\in C(x)}\inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g)] = \sup_{g\in C(x)}\inf_{Q\in\mathcal{Q}_e} E_Q[U(g)].$$

Before presenting the next theorem let us introduce a notation. By $V(y)$ we denote the conjugate of U as

$$V(y) := \sup_{x>0}\{U(x) - xy\}.$$

Note that ∂V denotes the subdifferential of V; see more in [9, Chapter II].

Theorem 5. *Suppose that all assumptions of Theorem 1 hold. Additionally, assume that $\frac{dQ^e}{dP}$ has a continuous distribution. Suppose also that for some $\lambda(\geqslant 0)$ the maximizer for $u^c(x)$ satisfies $g^\star \in -\partial V(\lambda\cdot\frac{dQ^e}{dP})$.*

Then $g^\star$ is also a maximizer for $u(x)$. Moreover, we have

$$\sup_{g\in C(x)}\inf_{Q\in\mathcal{Q}_e} E_Q[U(g)] = \sup_{g\in C(x)}\inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g)].$$

Hence, all inequalities in Theorem 1 turn to equalities.

Proof. *Since $g^\star$ is maximizer for $u^c(x)$, we have*

$$\sup_{g\in C(x)}\inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g)] = \inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g^\star)].$$

If follows from the proof of [1, Lemma 5.7] that $\mathbb{P}(g^\star \in \{U < U_c\}) = 0$. Hence,

$$\inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g^\star)] = \inf_{Q\in\mathcal{Q}_e} E_Q[U(g^\star)].$$

Thus, we obtain

$$\begin{aligned}&\sup_{g\in C(x)}\inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g)] = \inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g^\star)]\\ &= \inf_{Q\in\mathcal{Q}_e} E_Q[U(g^\star)] \leqslant \sup_{g\in C(x)}\inf_{Q\in\mathcal{Q}_e} E_Q[U(g)]\end{aligned}$$

Since $U_c \geqslant U$, we have

$$\begin{aligned}&\sup_{g\in C(x)}\inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g)] = \inf_{Q\in\mathcal{Q}_e} E_Q[U_c(g^\star)]\\ &= \inf_{Q\in\mathcal{Q}_e} E_Q[U(g^\star)] = \sup_{g\in C(x)}\inf_{Q\in\mathcal{Q}_e} E_Q[U(g)].\end{aligned}$$

Now, from Lemma 2 it follows that $g^\star$ is a maximizer for $u(x)$.

4 Conclusions

In this paper, we have studied the minimax identity for non-concave utility functions with the help of their concavification in the complete market models. We have shown that under some assumptions the minimax identity for non-concave functions holds.

In the future, it will be really helpful to study further the minimax identity for the non-concave utility functions in the complete and incomplete market models. The proof of minimax identity for non-concave utility functions in the complete market models will lead to construction of the unique optimal investment strategy for the problem of maximizing the robust non-concave utility functional in complete market under natural assumptions.

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