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ENTROPIC RISK MEASURE EVAR REAL DATA TESTING

Investigating risk measures is crucial for understanding and managing the uncertainties inherent in various financial and economic activities. Risk measures provide quantitative assessments of potential losses, enabling businesses and financial institutions to make informed decisions and mitigate adverse outcomes. The background of this research lies in the increasing complexity and interconnectedness of financial markets, which have increased the impact of risks and the need for robust risk management frameworks. Traditional measures, such as Value at Risk (VaR), have been widely used, but their limitations, particularly during financial crises, have highlighted the necessity for more comprehensive approaches like Expected Shortfall (ES) and Entropic Value at Risk (EVaR).

The relevance of investigating risk measures is underscored by the financial turmoil experienced during global crises, such as the 2008 financial meltdown and the COVID-19 pandemic. These events exposed the vulnerabilities of existing risk management practices and underscored the need for more resilient and accurate risk assessment tools. The purpose of this research is to evaluate and compare different risk measures, providing insight into their strengths and weaknesses, and to develop more effective strategies for risk management.

In the paper, we investigate the performance of the Entropic Value-at-Risk (EVaR) risk measure on real financial data, comparing it with traditional risk measures such as Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR). We consider historical return data from the S&P 500 index and Bank of America Corp. shares (2009–2019) and apply various parametric models, including normal and Laplace distributions. We also employ standard backtesting methods (coverage and independence tests) to assess the empirical adequacy of these measures. Our analysis highlights the mathematical properties of EVaR and its ability to detect heavy-tailed behavior more effectively than VaR, while providing coherent and potentially more robust estimates in volatile market conditions.

Future research perspectives include the development of new risk measures that can better capture systemic risks in financial markets. In addition, integrating machine learning and artificial intelligence with traditional risk measures presents an avenue to improve predictive accuracy and robustness. This ongoing research is important for building more resilient financial systems and for protecting economic stability.

K e y w o r d s: ***Entropic Value-at-Risk, EVaR, coverage tests, heavy tails, real data analysis, normal distribution, Laplace distribution.***

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Introduction

Using risk measures is crucial for quantifying potential losses in investments or business activities. This allows companies and investors to better understand the risks they are taking on and how these risks can impact their financial results. For example, statistical metrics such as the standard deviation or value-at-risk (VaR) help assess asset volatility and predict possible fluctuations in their value.

In addition, risk measures are essential for effective risk management. They enable organizations to identify which actions are necessary to minimize potential losses. This could include diversifying an investment portfolio, hedging risks, or implementing internal controls and procedures. By taking these steps, organizations can reduce the impact of adverse events and increase their resilience to financial shocks.

Finally, risk measures are necessary for regulatory compliance. Many financial institutions must adhere to specific standards and regulations that require regular risk assessment and reporting. For example, banks must maintain certain capital levels according to the Basel Agreements, which require the use of various risk measures to evaluate credit, market, and operational risks. This ensures the stability of the financial system and protects the interests of depositors and investors.

The simplest and best-known measure is VaR (Value-at-Risk). Provided that a stock's fluctuation over time has no peaks, VaR is useful for predicting the risk associated with a portfolio. But when we face financial crises of different types, the chance of VaR to give a complete overview of possible risks decreases, as it is not sensitive to anything beyond its loss threshold. In addition, VaR is not coherent. This problem is partially solved by CVaR (Conditional Value-at-Risk), but not well enough. An important disadvantage of CVaR is that it cannot be computed efficiently, even for a sum of arbitrary independent random variables.

Furthermore, financial markets may face high volatility and instability. In such circumstances, traders and managers use more stringent and conservative measures to manage risk rather than VaR and CVaR. In this case a more complicated measure can be used, namely a coherent measure of risk called Entropic Value-at-Risk (EVaR), introduced in (Ahmadi-Javid, 2012). The basic properties of this measure are described in the papers (Ahmadi-Javid, 2012; Ahmadi-Javid & Fallah-Tafti, 2019; Ahmadi-Javid & Pichler, 2017). A quantitative analysis of various risk measures, including EVaR, is performed in (Pichler, 2017). Delbaen provides a relation between EVaR and other commonotone risk measures in (Delbaen, 2019).

The properties of the entropic risk measure EVaR in relation to selected distributions are considered in the paper (Mishura et al., 2024). In continuation of the specified work, in this paper we consider EVaR real data testing.

Considering Entropic Value at Risk in real data testing is important for several reasons:

1. Enhanced Risk Measurement: EVaR provides a more comprehensive risk assessment by capturing tail risk more effectively than traditional measures like VaR or CVaR. This is particularly useful in scenarios with extreme events or heavy-tailed distributions, where conventional methods might underestimate the risk.
2. Robustness to Model Assumptions: EVaR is less sensitive to specific assumptions about the distribution of returns. This makes it a more robust risk measure when dealing with real-world data, which often deviates from theoretical models.
3. Improved Decision-Making: By incorporating EVaR into risk management practices, organizations can make better informed decisions. This can lead to more effective risk mitigation strategies, optimized capital allocation, and improved financial stability, especially in volatile or uncertain market conditions.

EVaR testing with real data ensures that the risk measure is practical and reliable, providing valuable information on the actual risk profile of investments or business activities.

1. Key definitions

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and let $\mathcal{L}_0 = \mathcal{L}_0(\Omega, \mathcal{F}, \mathbf{P})$ be the space of random variables $X\colon \Omega \to \mathbb{R}$.

Among various risk measures, the following risk measure is often used in modern risk management practice (Föllmer & Schied, 2016; Holton, 2003).

Definition 1. The *Value-at-Risk* (VaR) with confidence level $\alpha \in (0,1)$ of a random variable X is the smallest number y such that X does not exceed it with a minimum probability of $1-\alpha$, i.e.

$$\mathrm{VaR}_\alpha(X) := \inf\{x \in \mathbb{R} : F_X(x) > 1-\alpha\}.$$

Definition 2. The *Conditional Value-at-Risk* (CVaR) at level α is defined as:

$$CVaR_\alpha(X) := \frac{1}{1-\alpha}\,\mathbb{E}\left[X \,|X \geqslant \mathrm{VaR}_\alpha(X)\right].$$

However, in this paper, we focus on another risk measure, namely EVaR, introduced in (Ahmadi-Javid, 2012) according to the following definition.

Definition 3. Let $X \in \mathcal{L}_0$. Assume that its moment-generating function $m_X(t) = \mathbf{E}e^{tX}$ is well defined for all $t \geqslant 0$. An *entropic risk measure* EVaR with a confidence level $\alpha \in [0,1)$ (or a risk level $1-\alpha$) is defined as follows:

$$\mathrm{EVaR}_\alpha(X) := \inf_{t>0} t^{-1} \log\left(\frac{1}{1-\alpha} m_X(t)\right), \quad \alpha \in [0,1). \tag{1}$$

There is also a related risk measure Entropic Drawdown at Risk (EDaR) described in detail in (Cajas, 2021).

Definition 4. Assume that the moment-generating function of its drawdown series $DD(X,j)$, defined as

$$DD(X,j) = \max_{0\leqslant t\leqslant j}\left(\sum_{i=0}^{t} X_i\right) - \sum_{i=0}^{j} X_i,$$

exists. The *Entropic Drawdown at Risk* (EDaR) at confidence level $\alpha \in (0,1)$ is defined as:

$$EDaR_\alpha(X) = \mathrm{EVaR}_\alpha(DD(X)).$$

Normal distribution is one of the most frequently used in various models. The formula derived in this theorem is given in (Ahmadi-Javid, 2012).

Theorem 1. *Let $X \sim N(\mu, \sigma^2)$ be a normal distribution. Then* $\forall \alpha \in (0,1)$:

$$\mathrm{EVaR}_\alpha(X) = \mu + \sigma\sqrt{2\log\frac{1}{1-\alpha}}.$$

We will also need the following result. The proof can be found in (Mishura et al., 2024). Let us now consider losses which have a Laplace distribution.

Theorem 2. *Let $X \sim L(\mu, b)$ have a Laplace distribution. Then for all $\alpha \in (0,1)$,*

$$\mathrm{EVaR}_\alpha(X) = \mu - bW_{-1}(\gamma)\left(1 + \frac{2}{W_{-1}(\gamma)}\right)^{\frac{1}{2}}, \tag{2}$$

where $\gamma = -2e^{-2}(1-\alpha)$, W - Lambert function. The expression for $\mathrm{EVaR}_\alpha(X)$ *is continuous in α.*

2. EVaR real data testing

The main task of risk measures is to estimate possible losses for real processes, whose distribution is not known in advance. There are many problems with the analysis of real data. First of all, we need to build a mathematical probability model, and only then apply risk measures. In this part, we will look at certain models for working with real data and compare the most well-known probabilistic measures. After each model, we will perform a back-testing procedure, which will also show the advantages and disadvantages of risk measures. We will consider three risk measures: VaR, ES (CVaR), and EVaR. All subsequent actions will be performed only for two thresholds or confidence levels: 95 % and 99 %. All computer modeling will be done using the R programming language.

Consider historical data on the value of the $S\&P500$ index and the value of Bank of America Corp. shares from 2009 to 2019. An important feature of these data is their significant correlation. Their behavior is very similar, although the spread of returns is different. The first model we will consider is based on the assumption of a normal distribution of returns $r(t)$, where $r(t) = \frac{p_t}{p_{t-1}} - 1$ for price p_t in time t.

$$r(t) \sim N(\mu, \sigma^2).$$

Our goal is to determine the daily risk estimate for all of these measures. To calculate the value in time t, we will use the last 250 observations. Based on this sample, we estimate the distribution parameters with the method of moments. That is

$$\hat{\mu} = \frac{1}{n}\sum_{t=1}^{n} r(t),$$

$$\hat{\sigma^2} = \frac{1}{n-1}\sum_{t=1}^{n}(r(t) - \hat{\mu})^2.$$

To calculate EVaR, we apply the following formula:

$$EVaR_\alpha(X) = \mu + \sigma\sqrt{2\log\frac{1}{1-\alpha}}.$$

For the obtained estimates, we will calculate the risk value using the following formulas:

$$VaR_\alpha(X) = \hat{\mu} + \sqrt{\hat{\sigma^2}}F^{-1}_{N(0,1)}(1-\alpha),$$

$$CVaR_\alpha(X) = \hat{\mu} + \sqrt{\hat{\sigma^2}}\frac{\phi(F^{-1}_{N(0,1)}(\alpha))}{1-\alpha}.$$

The previous history of 250 days is used to calculate the risk value. The construction process is as follows: we start from the 250th day, calculate a one-day measure using the last 250 observations, and so on until the last day of the period under consideration. The results are shown in Fig. 1.

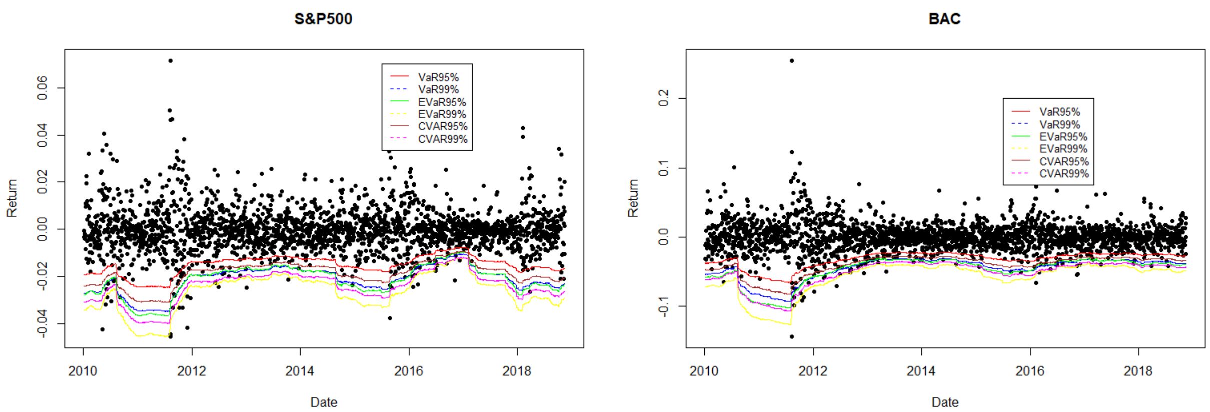

Fig. 1. Normal model

As can be seen from the return charts, the BAC has higher volatility, although both charts look quite similar. Coverage tests are often used to assess the quality of risk measures models. They assess whether the frequency of outliers corresponds to the loss quantile that the risk measure should reflect. Let us apply the standard test, which considers the hypothesis of a binomial distribution of exceedances, i.e.

$$H_0 : \sum_{t=0}^{T} 1(VaR_\alpha(X) < X) \sim Bin(n, \alpha),$$

where n is the number of observations, α is the theoretical probability of excess, which is equal to the confidence level for

VaR. We obtained the following confidence intervals for any VaR model in Tab. 1 since they depend only on the number of observations.

Table 1

Confidence intervals for VaR for binomial coverage test, S&P500 and BAC

Window size / Accuracy	5%	1%
100	[99,140]	[12,37]
250	[92,132]	[11,35]
500	[81,119]	[9,32]
1000	[58,91]	[6,26]

This test cannot be applied directly to CVaR and EVaR. For them, we will estimate the theoretical probability of $\hat{\alpha}$ of outperformance using the simulation method. That is, we simulate 100,000 price trajectories, each step of which has a theoretical distribution with parameters that we determined using the method of moments to calculate CVaR/EVaR at that moment. For each model, the confidence intervals for these risk measures depend on the distribution. The results are shown in Tab. 2.

Table 2

Confidence intervals for CVaR and EVaR for binomial coverage test, S&P500 and BAC

Window size/Measure	CVaR95%	CVaR99%	EVaR95%	EVaR99%
100	[34,60]	[2,18]	[16,36]	[0,11]
250	[38,66]	[4,21]	[17,37]	[1,12]
500	[56,89]	[12,37]	[31,56]	[6,26]
1000	[78,116]	[21,52]	[45,74]	[11,35]

Window size/Measure	CVaR95%	CVaR99%	EVaR95%	EVaR99%
100	[34,60]	[2,18]	[13,31]	[0,10]
250	[41,70]	[5,24]	[16,36]	[1,13]
500	[60,94]	[15,42]	[32,58]	[8,29]
1000	[100,141]	[37,74]	[61,95]	[22,53]

The important point of this test is that, on the one hand, a failure to fall within the confidence interval should reject the model for the specified risk measure. However, the measure may be too conservative and the data may not quite fit the model. In this case, it is worth paying attention to whether the number of outputs for the risk measure exceeds the right-hand side of the confidence interval. The confidence intervals show that the risk models will perform differently depending on the amount of data used for parametric estimates of the distribution of the daily risk measure. The second test that we will use is the independence test in Christophersen format (Holton, 2003). This is a likelihood ratio test that looks for unusually frequent consecutive outliers.

$$q_0^* = P(I_t = 0|I_{t-1} = 0),$$
$$q_1^* = P(I_t = 0|I_{t-1} = 1).$$

We test the independence hypothesis, i.e. $H_0 : q_0^* = q_1^*$. To do this, we calculate the likelihood ratio

$$\Lambda = (\frac{\hat{q}}{\hat{q}_0})^{\alpha_{00}}(\frac{1-\hat{q}}{1-\hat{q}_0})^{\alpha_{01}}(\frac{\hat{q}}{\hat{q}_1})^{\alpha_{10}}(\frac{1-\hat{q}}{1-\hat{q}_0})^{\alpha_{11}}.$$

Where α_{ij} is the number of pairs such that $(I_t, I_{t-1}) = (i, j)$,

$$\hat{q} = \frac{\alpha_{00}+\alpha_{01}}{\alpha_{00}+\alpha_{01}+\alpha_{10}+\alpha_{11}}, \hat{q}_0 = \frac{\alpha_{00}}{\alpha_{00}+\alpha_{01}}, \hat{q}_1 = \frac{\alpha_{10}}{\alpha_{10}+\alpha_{11}}.$$

Then $H_0 : -2\log(\Lambda) \sim \chi^2(1, 0)$. That is, for our confidence levels, the resulting statistics should not exceed 3.841 for 95 % and 6.635 for 99 %. An important point to note when applying this test is that it is meaningless if there are no outliers. Now, let us calculate the number of excesses of the estimates obtained using the risk measures for this model. Here and in the following, we will mark in green the hits in the confidence interval, in red the outliers, and in yellow the boundary values. The results are shown in Tab. 3 and 4.

Table 3

Coverage and independence tests, S&P500

Window size/Measure	VaR95%	VaR99%	CVaR95%	CvaR99%	EVaR95%	EvaR99%
100	140	43	69	25	51	22
250	105	44	57	25	45	22
500	105	41	61	25	42	20
1000	80	31	48	22	27	15

Window size/Measure	VaR95%	VaR99%	CVaR95%	CvaR99%	EVaR95%	EvaR99%
100	1.7406851598	1.4115963038	5.6867828597	0.5301201947	0.0002306584	0.4100016978
250	1.790612	3.398643	12.063101	1.149089	5.338749	1.548824
500	6.4600779	6.6212141	9.1131985	0.9781833	3.0056490	1.6708185
1000	4.4697770	1.9275618	0.1262367	0.9348360	0.1366628	0.3065448

Table 4

Coverage and independence tests, BAC

Window size/Measure	VaR95%	VaR99%	CVaR95%	CvaR99%	EVaR95%	EvaR99%
100	136	46	74	27	48	12
250	127	46	73	24	44	11
500	119	50	72	31	49	22
1000	98	39	52	27	36	22

Window size/Measure	VaR95%	VaR99%	CVaR95%	CvaR99%	EVaR95%	EvaR99%
100	1.36460684	0.01414058	2.57964038	0.61885905	0.19177997	0.12146825
250	1.9476527	5.8437422	0.9803172	1.2730132	0.8568735	0.1089113
500	4.5024847	4.0761754	3.5401206	6.3371277	0.3974163	5.1948436
1000	3.0779306	2.7028451	4.1526883	2.7729393	0.0614779	0.9348360

As can be seen from the results for the first dataset, the VaR at the low confidence level only satisfies the tests at the medium 250 and short 100 data horizons used for parameter estimation. CVaR and EVaR perform well for the high confidence level in a medium window of size 500 and a large window of size 1000. EVaR also performed well for a low confidence level for a medium window of 500. For the second dataset, the situation is almost identical, except that for the large window of size 1000, only EVaR performed well. Moreover, EVaR again proves the result that it is the upper bound of the two previous measures by Chernow's inequality. As can be seen from the results, the independence test failed CVaR and VaR for a low confidence level for a less volatile asset. Combining the results of the tests, it can be determined that the normal model can be applied to these historical data only for a window of 500, using EVaR for both data sets and CVaR for the second set.

When building a model, it is worth looking at how the distribution of the model matches the actual distribution of the data. This is called a goodness-of-fit test. The null hypothesis here is obviously that the predicted distribution is identical to the true distribution. Conventional statistical analysis cannot be applied here because the distribution changes every day. This can be circumvented as follows. Let $U_t = \Phi_{t-1|t}(L_t)$ be the probability of loss of L_t at time t according to the distribution that we assume for time t at time t-1. Then our hypothesis becomes $H_0 : U_t \sim Unif(0,1)$. There is a more extended test by Berkowitz (Holton, 2003), which consists of testing compliance with the standard normal distribution of $N_j' = \Phi(U_t)$. The results are shown in Fig. 2.

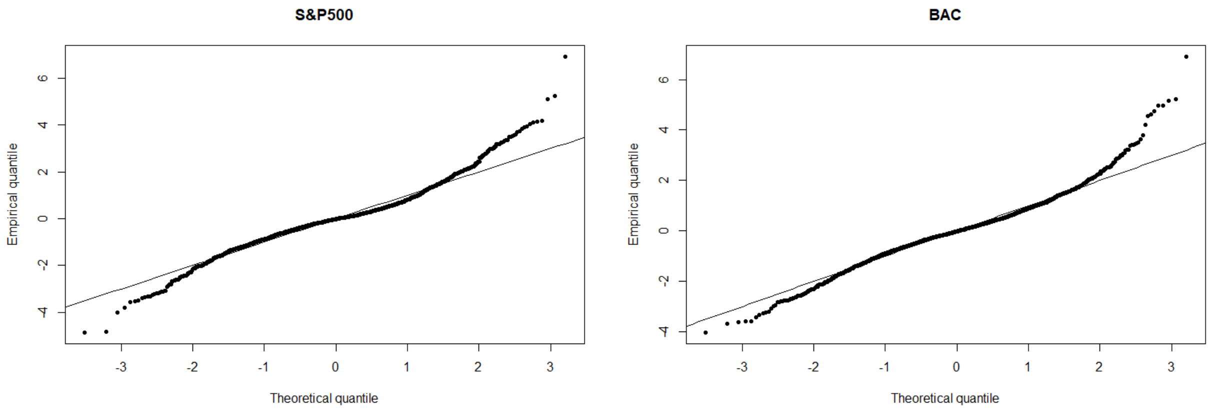

Fig. 2. Distribution test for Normal model

Consider the following model with the basic assumption: $r(t) \sim L(\mu, b))$ – that is, the return has a Laplace distribution. This distribution has heavier tails than the normal distribution. Then, for this model, the corresponding risk measures will be calculated as follows:

$$VaR_\alpha(X) = \mu - b\log(2-2\alpha),$$
$$CVaR_\alpha(X) = \mu + \lambda(1-\log(2-2\alpha)) = \lambda + VaR_\alpha(X).$$

Let us use the result from the paper (Mishura et al., 2024):

$$EVaR_\alpha(X) = \mu - bW_{-1}(-2e^{-2}(1-\alpha))\sqrt{1+\frac{2}{W_{-1}(-2e^{-2}(1-\alpha))}}.$$

The estimates of the method of moments for the Laplace distribution are as follows:

$$\hat{\mu} = med(\{X_i\}),$$

$$\hat{b} = \frac{1}{n}\sum_{i=1}^{T}|X_i - \hat{\mu}|.$$

We get the following result in Fig. 3.

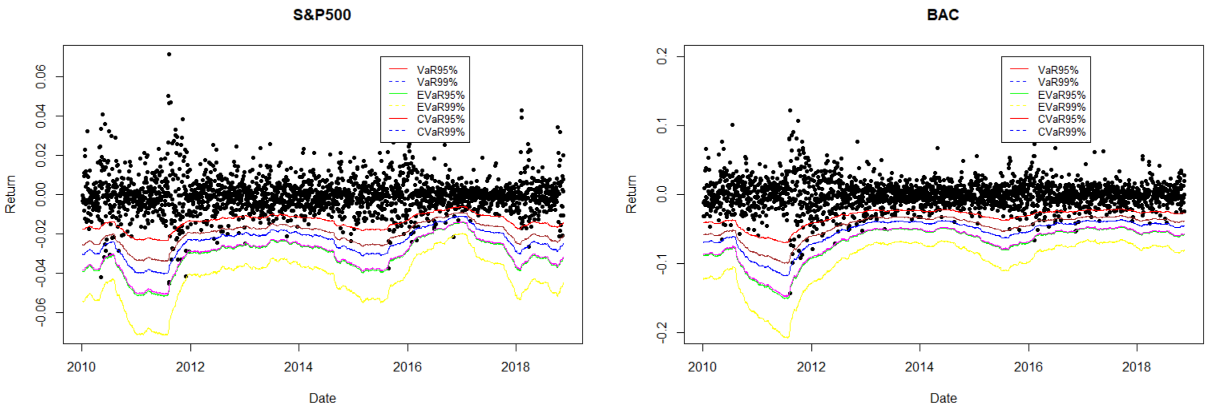

Fig. 3. Laplace model

All risk measures have become smoother and there are fewer jumps. The distribution test has become much better than before. We can see this in Fig. 4.

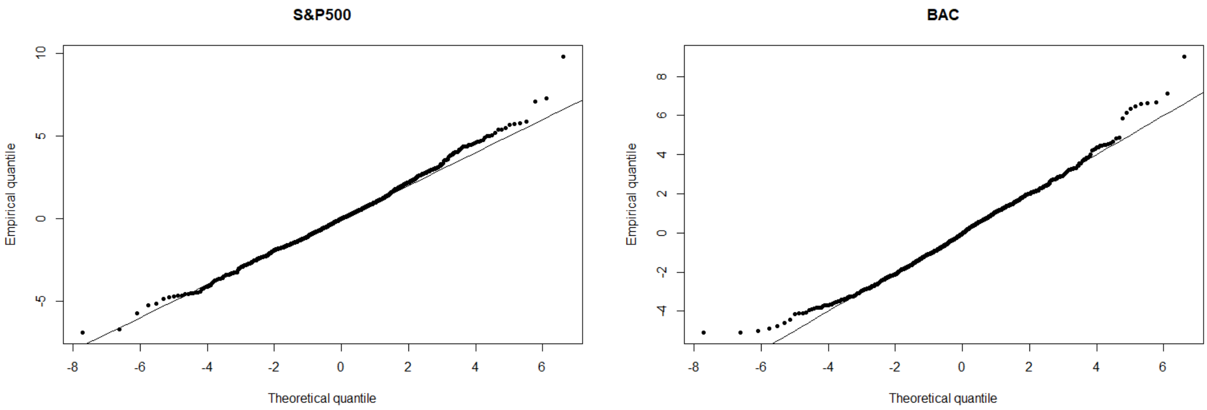

Fig. 4. Distribution test for Laplace model

Since our theoretical distribution has changed, the confidence intervals for the coverage test have also changed. The new confidence intervals are shown in the Tab. 5 for $S\&P500$ and BAC, respectively.

Table 5

Confidence intervals of the coverage test for the Laplace model CVaR/EVaR, $S\&P500$ and BAC, respectively

Window size/Measure	CVaR95%	CVaR99%	EVaR95%	EVaR99%
100	[42,70]	[4,21]	[4,17]	[0,5]
250	[37,64]	[3,19]	[4,15]	[0,5]
500	[31,57]	[2,17]	[3,14]	[0,5]
1000	[22,43]	[1,14]	[2,11]	[0,4]

Window size/Measure	CVaR95%	CVaR99%	EVaR95%	EVaR99%
100	[35,62]	[3,18]	[3,15]	[0,5]
250	[32,57]	[2,17]	[3,14]	[0,5]
500	[27,51]	[2,16]	[2,13]	[0,4]
1000	[19,40]	[1,13]	[1,10]	[0,4]

Having performed the coverage and independence tests for this model, we obtained the following results, which are summarized in the following Tab. 6 and 7.

Table 6

Coverage and independence tests for the Laplace model, S&P500

Window size/Measure	VaR95%	VaR99%	CVaR95%	CvaR99%	EVaR95%	EvaR99%
100	163	34	65	14	13	1
250	138	39	56	14	13	3
500	126	35	54	16	14	3
1000	111	32	44	11	11	5

Window size/Measure	VaR95%	VaR99%	CVaR95%	CvaR99%	EVaR95%	EvaR99%
100	6.7628639458	0.4330115887	4.2535132480	0.1654716259	1.8239232235	0.0008396306
250	6.098507073	1.763444269	5.997583414	3.164232475	1.626287602	0.008071751
500	7.313792547	13.324441000	15.760158615	2.450338730	0.018257308	0.009090913
1000	0.93885240	8.16754568	6.70819398	3.33485255	0.04613822	0.03382956

Table 7

Coverage and independence tests for the Laplace model, BAC

Window size/Measure	VaR95%	VaR99%	CVaR95%	CvaR99%	EVaR95%	EvaR99%
100	126	23	40	12	10	2
250	103	25	39	9	9	2
500	103	28	50	11	10	3
1000	91	29	41	13	12	3

Window size/Measure	VaR95%	VaR99%	CVaR95%	CvaR99%	EVaR95%	EvaR99%
100	15.391634812	5.523522253	1.800417036	3.885751074	0.373859483	0.003359933
250	6.803391148	1.149089240	0.657218515	0.003585836	1.763444269	4.921353239
500	5.242324821	3.466750716	4.076175449	3.881659915	0.386112086	0.009090913
1000	8.50802462	5.56215684	4.78312724	15.25862751	1.66610288	0.0121621

For the first dataset, EVaR and CVaR for the high confidence level passed the tests for all windows. For the second, more volatile dataset, EVaR and CVaR for all levels worked well for all windows except the large one, while VaR worked well only for the high level. In the large window, as in the previous model, almost no one did well. This is most likely due to too much smoothing of the parameters of the theoretical distribution.

As we have already mentioned, the data distribution changes every day, so it is also worth considering models when the variance also changes over time. The generalized autoregressive conditional heteroscedasticity model (GARCH) will help us with this. Recently, models of this type have been used very often for EVaR. Such models are well suited for volatile markets(Ahmed, Soleymani, Ullah, & Hasan, 2021). The first model of this class is GARCH(1,1) with a mean. It has the following form:

$$r(t) = \mu + z_t = \mu + \sigma_t \epsilon_t,$$
$$\sigma_t^2 = \omega + \alpha_1 z_{t-1}^2 + \beta_1 \sigma_{t-1}^2.$$

where ϵ_t is the error, which can have different distributions. For this model, we first assume that $\epsilon_t \sim L(0,1)$. In other words, we actually have the same Laplace model, but with a completely different approach to estimating the variance. The formulas remain the same as in the previous model, but we calculate the variance as a forecast of the previous values. The result is shown in the Fig. 5.

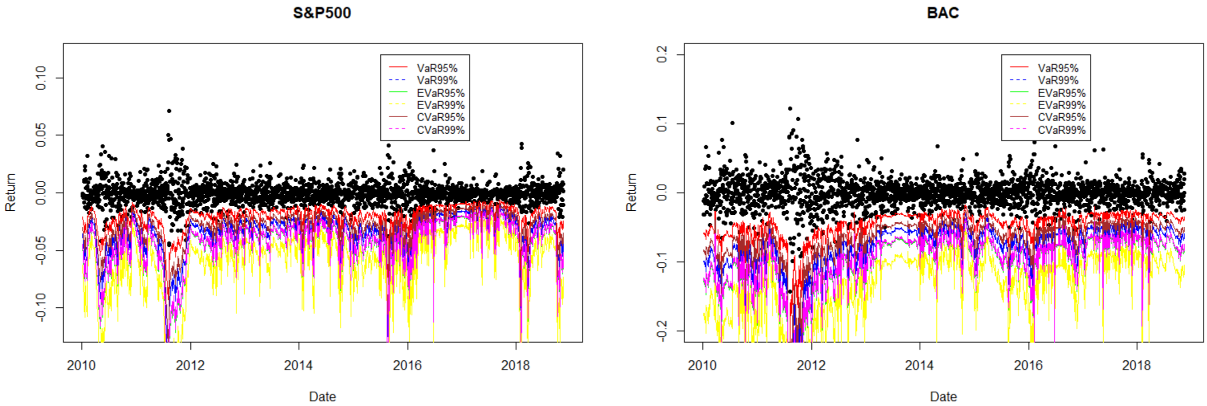

Fig. 5. GARCH(1,1) - Laplace model

The distribution test has become worse than before. This is shown in Fig. 6.

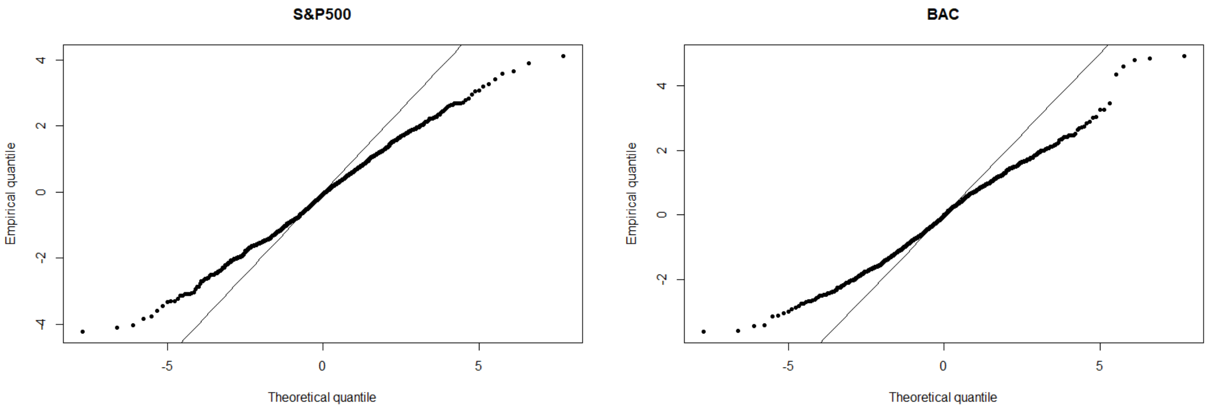

Fig. 6. Distribution test for GARCH(1,1)-Laplace model

As we can see, all the measures start to behave more conservatively, they are more sensitive to fluctuations. Since our theoretical distribution has changed, the confidence intervals for the coverage test have also changed. The new confidence intervals are shown in Tab. 8 for $S\&P500$ and BAC, respectively.

Table 8

Confidence intervals of the coverage test for the GARCH model with Laplace distribution for errors, CVaR/EVaR, S&P500 and BAC, respectively

Window size/Measure	CVaR95%	CVaR99%	EVaR95%	EVaR99%
100	[39,67]	[3,20]	[4,16]	[0,5]
250	[36,63]	[3,19]	[4,15]	[0,5]
500	[31,56]	[2,17]	[3,14]	[0,5]
1000	[21,43]	[1,14]	[2,11]	[0,4]

Window size/Measure	CVaR95%	CVaR99%	EVaR95%	EVaR99%
100	[33,60]	[2,18]	[3,14]	[0,5]
250	[30,56]	[2,17]	[3,14]	[0,5]
500	[26,50]	[2,15]	[2,12]	[0,4]
1000	[19,39]	[1,13]	[1,10]	[0,4]

The coverage and independence tests yielded the following results, which are summarized in the Tab. 9 and 10.

Table 9

Coverage and independence tests for GARCH model with mean and Laplace distribution for errors, S&P500

Window size/Measure	VaR95%	VaR99%	CVaR95%	CvaR99%	EVaR95%	EvaR99%
100	43	3	9	0	0	0
250	30	2	6	0	0	0
500	30	1	3	0	0	0
1000	20	0	1	0	0	0

Window size/Measure	VaR95%	VaR99%	CVaR95%	CvaR99%	EVaR95%	EvaR99%
100	1.580430834	0.007563027	0.068239422	0	0	0
250	0.817092890	0.003585836	0.032330529	0	0	0
500	0.921695235	0.001009082	0.009090913	0	0	0
1000	0.546838633	0	0.001349528	0	0	0

Table 10

Coverage and independence tests for GARCH model with mean and Laplace distribution for errors, BAC

Window size/Measure	VaR95%	VaR99%	CVaR95%	CvaR99%	EVaR95%	EvaR99%
100	38	4	5	0	0	0
250	30	5	6	1	0	0
500	26	5	7	0	0	0
1000	18	3	4	0	0	0

Window size/Measure	VaR95%	VaR99%	CVaR95%	CvaR99%	EVaR95 %	EvaR99 %
100	1.23161041	0.01345104	0.02102609	0	0	0
250	0.6479114525	0.0224416705	0.0323305286	0.0008960574	0	0
500	0.69087367	0.02527809	0.04959525	0	0	0
1000	0.44233195	0.01216217	0.02163627	0	0	0

As can be seen from the tables, CVaR and EVaR have almost no excess. All measures showed fewer outliers than the lower bound of the confidence interval. We can say that the models do not work very well, although they protect against possible unlikely shocks. The results of the independence test can be disregarded because the number of outliers is too small. Formally, only the EVaR for high levels passes the coverage test, but since it yields 0 outliers, we should not rely entirely on these tests. It is obvious from the data that the model greatly overestimates the tails of the distribution. Therefore, the only argument for accepting the EVaR at a high confidence level is the need to have a very conservative measure, or the expectation of certain possible significant shocks. In general, it is better not to use this model for available data. Now let us consider the GARCH(1,1) model with mean and normal error distributions.

$$r(t) = \mu + z_t = \mu + \sigma_t \epsilon_t,$$

$$\sigma_t^2 = \omega + \alpha_1 z_{t-1}^2 + \beta_1 \sigma_{t-1}^2.$$

The result is shown in Fig. 7.

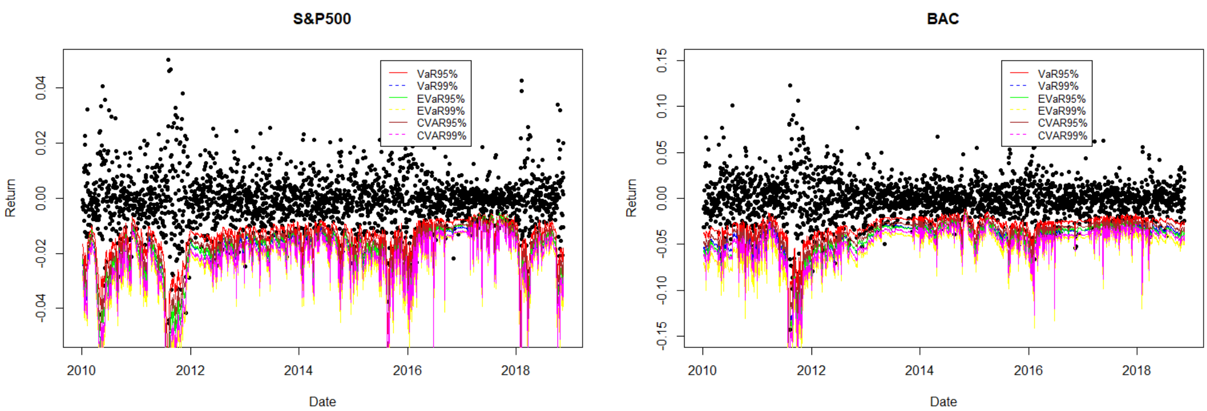

Fig. 7. GARCH(1,1) - Normal model

All risk measures have become smoother and there are fewer jumps. The distribution test has become much better than before. This we can see in Fig. 8.

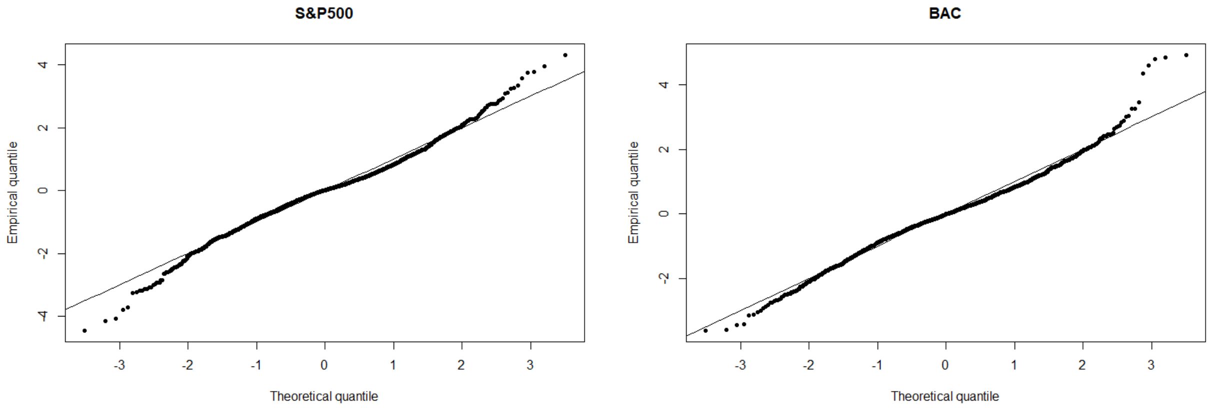

Fig. 8. Distribution test for GARCH(1,1) - Normal model

As we can see from the graphs, this model works quite well on our data, although it is more sensitive to disturbances, it fits the real data better. For this model we construct same Tab. 11, 12 and 13 with methodology as before.

Table 11

Confidence intervals of the coverage test for the GARCH model with mean and normal error distributions, CVaR/EVaR, S&P500 and BAC, respectively

Window size/Measure	CVaR95%	CVaR99%	EVaR95%	EVaR99%
100	[34,60]	[2,18]	[22,44]	[1,14]
250	[31,57]	[2,17]	[19,39]	[1,12]
500	[27,51]	[2,16]	[16,35]	[0,11]
1000	[19,40]	[1,13]	[10,26]	[0,9]

Window size/Measure	CVaR95%	CVaR99%	EVaR95%	EVaR99%
100	[34,60]	[2,18]	[14,33]	[0,10]
250	[31,57]	[2,17]	[11,29]	[0,9]
500	[27,51]	[2,16]	[10,26]	[0,8]
1000	[19,40]	[1,13]	[7,21]	[0,7]

Table 12

Coverage and independence tests for the GARCH model with mean and normal error distributions, S&P500

Window size/Measure	VaR95%	VaR99%	CVaR95%	CvaR99%	EVaR95%	EvaR99%
100	98	39	61	22	49	20
250	96	32	46	20	33	14
500	83	27	43	15	29	12
1000	62	20	36	13	22	8

Window size/Measure	VaR95%	VaR99%	CVaR95%	CvaR99%	EVaR95%	EvaR99%
100	3.4042251	1.2978415	3.2053644	0.4100017	4.1159108	0.3385567
250	1.4565400	0.9305189	1.9352136	0.3615051	2.9223692	0.1766573
500	0.7948890	0.7454224	1.9063417	0.2286608	2.4986172	0.1461196
1000	1.36563219	0.54683863	1.79147717	0.22993497	1.89345871	0.08678009

Table 13

Coverage and independence tests for the GARCH model with mean and normal error distributions, BAC

Window size/Measure	VaR95%	VaR99%	CVaR95%	CvaR99%	EVaR95%	EvaR99%
100	113	39	60	28	42	15
250	110	33	52	18	30	10
500	92	26	41	18	24	11
1000	69	23	34	12	20	6

Window size/Measure	VaR95%	VaR99%	CVaR95%	CvaR99%	EVaR95%	EvaR99%
100	0.5098184	0.1814266	0.2052620	0.6658331	0.6203913	0.1900351
250	0.47144742	0.42789397	0.04085206	0.29255401	0.18468272	0.08996881
500	0.13136629	0.69087367	2.26895347	0.12271869	0.02702392	0.32977560
1000	0.13136629	0.69087367	0.02702392	0.32977560	2.26895347	0.12271869

The result is quite different from the normal distribution model. VaR works well for medium- and long-observation horizons for both data sets and for both confidence levels. CVaR and EVaR work well only for a large window size. Almost all the measures on this model pass the independence test. Although the correlation in the distribution test is only 0.992 for 250 values for prediction and the overall test fails, this model is quite good for this data set.

In addition, the EDaR risk measure was considered. This measure of risk is quite often compared to EVaR. However, because for the process DD(X) does not have an elementary distribution for which it is possible to obtain the exact EVaR formula, we managed to obtain estimates only numerically for the sample. The calculation time is almost identical to that of EVaR if we use the nonparametric approach, which is much longer for the parametric approach we use. Actually in the Tab. 14 of time spent the advantage of parametric approach over non-parametric approach is clearly visible. In addition, this measure proved to be so conservative and distribution-insensitive that none of the tests presented can characterize this measure and provide results for backtesting. As can be seen in Fig. 9, the EDaR estimate strongly exceeds all values, making it impossible to perform an outlier analysis.

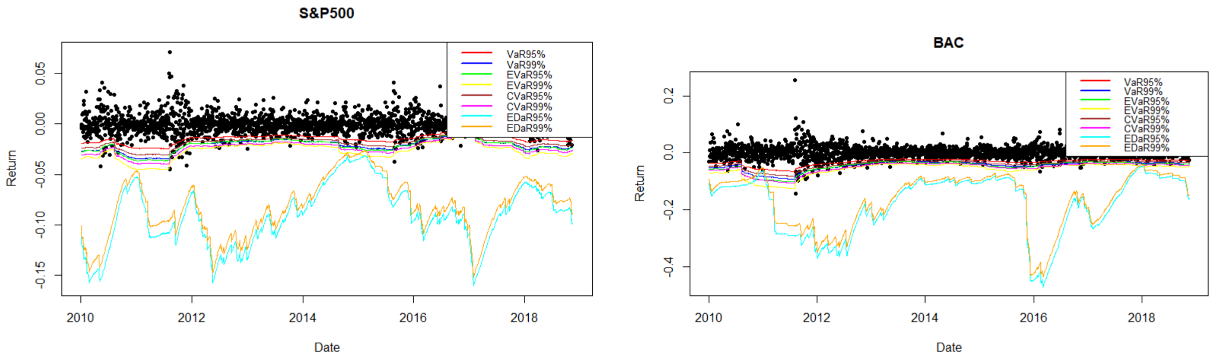

Fig. 9. Normal model in comparison to EDaR

Table 14

Time consumption in seconds to calculate a one-day risk value over the entire data interval

Window size	Parametric approach			Nonparametric approach			
	VaR	CVaR	EVaR	VaR	CVaR	EVaR	EDaR
100	0.0254	0.0271	0.0248	0.1712	0.4212	15.718	15.52
250	0.0358	0.0298	0.0295	0.1637	0.7374	21.174	22.461
500	0.0292	0.0352	0.0351	0.1607	1.0237	32.288	34.382
1000	0.0267	0.0258	0.0238	0.1159	1.4567	34.909	40.201

In summarizing the results of the models considered, the best fit to the data was shown by the Laplace model for EVaR at any confidence level and CVaR at a high confidence level. The GARCH model with mean and normal error distributions for VARs of all levels and medium and large windows, and the CVaR for the low confidence level, proved to be the second best. If a significant increase in volatility and unexpected shocks is expected, the EVaR model for GARCH with a Laplace error distribution for high confidence levels can be used.

To summarize the models reviewed, we can say that EVaR performs quite well on data with high volatility and heavy tails, when VaR does not work. EVaR performed on a par with CVaR, which it outperformed on data with heavy tails that were not taken into account by the theoretical model. In general, we confirmed that EVaR is an upper bound for VaR and CVaR. CVaR and EVaR were much more likely to pass the loss independence test, even if the model was not fully fit to the real data. However, if the model even slightly predicted heavier tails than the real data, the EVaR significantly exceeded the possible risk. All risk measures significantly and correctly increased their risk estimates when data volatility increased. This can be seen in the similarity of the results for the two data sets. To summarize the results, we can say that EVaR models perform well with these tests if the model underestimates the severity of the tails. But in case of overestimation, it is better to use other tests.

Discussion and conclusions

In this paper, we consider the Entropic Value-at-Risk (EVaR) risk measure on real data testing. Recently, this measure has been increasingly considered in scientific research and applied to solve practical problems. There are not many models for EVaR described in the current scientific literature. Most often, only the standard normal distribution model is considered. The lack of great results in this area is due to the complexity of the calculations for different models.

The result of this work gives the possibility of further studying the behavior of EVaR for the already defined models, as well as expanding the models for which EVaR can be analytically solved. In the future, one can either try to analytically approximate the EVaR value or develop a solution for working with mixtures of distributions.

The testing of some models on real data was performed using coverage tests, normality tests, and loss independence tests. These tests are standard in risk assessment practice and are widely used in model back-testing. As real data, we used publicly available data on the value of the $S\&P500$ index and the value of Bank of America Corp. shares for the period from 2009 to 2019. Each model was tested on a standard VaR task: to build a daily risk estimate for different available data volumes, for building estimates and risk models. The models for VaR, CVaR, and EVaR were compared for different data, different models, and different model parameters. The most commonly used GARCH models for financial forecasting were built and tested with different error distributions and the presence of a mean. The optimal models were selected from those proposed for the historical data. EVaR proved to be very sensitive to the heavy tails included in the model. It turned out that the GARCH models with Laplace errors, which are widely used today in practice, overestimated the real risk. The basic relationship between VaR, CVaR, and EVaR was confirmed in practice. For almost all models, the EVaR behavior was similar to CVaR and significantly outperformed the VaR behaviour. The comparison of EVaR and EDaR was also examined, and the difficulty of calculating the values was compared by comparing the time consumed to calculate the values.

From all of the above, we can conclude that the EVaR risk measure considered is relevant and has great promise. In our opinion, the study of EVaR is very promising, as this risk measure provides a better assessment of losses with heavy tails. Although, of course, this measure has disadvantages related to the complexity of numerical calculations, analysis of the impact of model parameters, and the use of complex models. In this view, there are not many methods for practical application of this measure. Some of them have been reviewed and elaborated in this paper. Given the superiority of this measure over others, we can conclude that further study of this measure and the development of methods for its effective application to solve various financial problems are needed.

Authors' contribution: Volodymyr Zubchenko – conceptualization, methodology, analysis of sources, preparation of literature review and theoretical foundations of research; Petro Zelenko – empirical data collection and validation, software, empirical research.

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ТЕСТУВАННЯ ЕНТРОПІЙНОЇ МІРИ РИЗИКУ EVAR НА РЕАЛЬНИХ ДАНИХ

Дослідження мір ризику є надзвичайно важливим для розуміння та управління невизначеностями, що властиві різним фінансовим та економічним активностям. Міри ризику надають кількісну оцінку потенційних втрат, що дає змогу бізнесу та фінансовим установам ухвалювати обґрунтовані рішення та зменшувати негативні наслідки. Основою пропонованого дослідження є дедалі більша складність і взаємопов'язаність фінансових ринків, що посилює вплив ризиків і потребує їхнього управління. Традиційні міри, такі як Value at Risk (VaR), широко використовуються, але їхні обмеження, особливо під час фінансових криз, підкреслюють необхідність більш комплексних підходів, таких як Expected Shortfall (ES) та Entropic Value at Risk (EVaR). Актуальність дослідження мір ризику підтверджується фінансовими потрясіннями, що спостерігалися під час глобальних криз, таких як фінансова криза 2008 р. та пандемія COVID-19. Ці події виявили вразливість наявних практик управління ризиками та підкреслили потребу в стійкіших і точніших інструментах оцінювання ризиків. Метою нашого дослідження є оцінка та порівняння різних мір ризику, надання уявлення про їхні переваги та недоліки, а також розроблення ефективніших стратегій управління ризиками. Проаналізовано ефективність ентропійної міри ризику Value-at-Risk (EVaR) на реальних фінансових даних, та виконано її порівняння з традиційними ризиковими мірами, такими як Value-at-Risk (VaR) та Conditional Value-at-Risk (CVaR). Розглянуто історичні дані щодо дохідності індексу S&P 500 та акцій Bank of America Corp. (2009–2019) та застосовано різні параметричні моделі, включно з моделями на основі нормального розподілу і розподілу Лапласа. Також використано стандартні методи бек-тестування (тести покриття та незалежності) для оцінювання емпіричної адекватності цих мір. Проведений аналіз підкреслює математичні властивості EVaR і здатність цієї міри ефективніше виявляти важкі хвости розподілу порівняно з VaR, забезпечуючи при цьому когерентні та потенційно надійніші оцінки в умовах волатильних ринків. Перспективи майбутніх досліджень передбачають розроблення нових мір ризику, які можуть краще враховувати системні ризики на фінансових ринках. Крім того, інтеграція машинного навчання та штучного інтелекту з традиційними мірами ризику відкриває можливості для підвищення точності та надійності прогнозів. Ці дослідження є важливими для побудови стійкіших фінансових систем і забезпечення економічної стабільності.

Ключові слова: ***ентропійна міра ризику, EVaR, тестування на реальних даних, нормальний розподіл, розподіл Лапласа.***

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