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TOPOLOGICAL STRUCTURE OF SIMPLE HAMILTONIAN FLOWS ON THE PROJECTIVE PLANE AND THE KLEIN BOTTLE

For every non-orientable surface, there exists an oriented double cover. We consider flows whose lift to the double cover are Hamiltonian flows with a Hamiltonian being a Morse function. By analogy with oriented manifolds, a flow is simple if there are no saddle connections between different saddles within it.

To describe the topological properties of functions on two-dimensional manifolds, an invariant known as the Reeb graph is used. In the case of simple Morse functions on closed oriented two-dimensional manifolds, this is a complete topological invariant for stratified equivalence and topological equivalence of Hamiltonian flows. However, in our case, on a non-orientable manifold, there is no function that generates a Hamiltonian flow. Therefore, the concept of the Reeb graph requires redefinition.

To investigate the topological properties of simple Hamiltonian flows on non-orientable surfaces, we describe the structure of all possible atoms, which are analogous to the Bolsinov-Fomenko atoms for Morse functions and Hamiltonian flows on oriented surfaces. In particular, there are five types of such atoms on non-orientable surfaces. We construct a complete topological invariant, which is the Reeb graph with marked vertices and half-edges incident to the vertices, describe its properties, and prove a criterion for the topological equivalence of flows. For the projective plane and the Klein bottle, we prove theorems about the existence of flows with a given Reeb graph. For cyclic Reeb graphs on the Klein bottle, we describe an algorithm for their enumeration. For all other types of Reeb graphs on the projective plane and the Klein bottle, we obtain recursive formulas for determining their numbers. The results provide calculations of the number of topological structures of flows with no more than eight saddles. The obtained results open up the possibility of studying the structures of Hamiltonian flows of greater complexity on compact non-orientable surfaces.

K e y w o r d s: *topological equivalence, Reeb graph, classification, non-orientable surface.*

AMS 2020 classification: 37C10, 37C15, 37C20.

Introduction

For oriented closed surfaces, the trajectories of Hamiltonian flows coincide with the level sets (components of the level curve) of the Hamiltonian function, which means that the topological structure of Hamiltonian flows is determined by the topological structure of the functions. The functions in general position are simple Morse functions, where each critical level contains one critical point, and this point is non-degenerate. For flows, simplicity means the absence of trajectories connecting saddle points.

For every non-oriented surface, there exists a unique (up to homeomorphism) oriented double cover. Therefore, functions and flows on non-oriented surfaces correspond to $\mathbb{Z}_2$-invariant functions and flows on oriented surfaces. For example, functions on the projective plane $\mathbb{R}P^2$ (the Klein bottle Kl) can be defined as even functions on the sphere S^2 (the torus T^2). However, such functions do not define a flow on a non-oriented surface, as they do not allow for the alignment of movement directions along the level sets on the non-oriented surface. Conversely, if we use odd functions instead of even ones, then the directions of movement along the trajectories of their Hamiltonian flows are consistent with the symmetry with respect to the origin. It is precisely such flows, defined by odd Morse functions, that we consider in this paper.

If we collapse each component of the function level on a closed 2-manifold to a point, we obtain a graph known as the Reeb graph (Reeb, 1946) or the Liapunov graph for the Liapunov function (de Rezende, & Franzosa, 1993). In the case of simple Morse functions on closed oriented two-dimensional manifolds, this is a complete topological invariant for fibre equivalence. If a linear order is given on the set of vertices, it becomes a complete topological invariant with respect to topological equivalence. Automorphisms of Reeb graphs was described in (Kravchenko, & Maksymenko, 2019).

In the case of non-orientable two-dimensional manifolds, as well as manifolds with boundaries, additional information is needed to construct a complete topological invariant. Another way to determine the topological structure of Morse functions is to use Morse vector fields with specified function values at critical points. Therefore, the topological classification of Morse vector fields is closely related to the classification of functions and Hamiltonian flows.

The topological classification of Morse vector fields for closed surfaces was obtained in (Cobo, Gutierrez, & Llibre, 2010; Oshemkov, & Sharko, 1998; Maksymenko, 2009; Neumann, & O'Brien, 1976; Nikolaev, & Zhuzhoma, 1999; Prishlyak, & Loseva, 2020; Wang, 1990; Yokoyama, 2022).

Moreover, topological invariants of Morse functions on surfaces were constructed in (Bolsinov, & Fomenko, 2004; Hladysh, & Prishlyak, 2019; Khohliyk, & Maksymenko, 2020; Maksymenko, 2006, 2020; Prishlyak, 2000; Sharko, 2003). The object of research is flows on non-orientable surfaces that do not have connections between different saddles, and whose preimages on a two-sided orientable cover are Hamiltonian flows for Morse functions.

The aim and objectives of the research is to investigate the topological properties of Reeb graphs of simple Hamiltonian flows on the projective plane and the Klein bottle, and to count the number of their different structures for flows with no more than 8 saddle points.

The methods of the research are following: 1) Reeb and Liapunov graph methods, 2) atoms and molecules metod according to Bolsinov-Fomenko, 3) topological graph theory, 4) combinatorial methods for enumerating trees.

The structure of the paper. In Section 1, we describe methods for defining flows on the projective plane and the Klein bottle. In Section 2, we study the local structure of simple Hamiltonian flows on non-orientable surfaces. To do this, we describe atoms that are analogous to the Bolsinov-Fomenko atoms for Morse functions. All atoms are divided into five types, and the structure of flows on each type is described. In Section 3, we investigate a global invariant that is analogous to the Reeb graphs of simple Morse functions. The properties of this graph are outlined in Theorem 1, while Theorem 2 provides a criterion for the topological equivalence of flows using Reeb graphs. Section 4 describes all possible structures of simple Hamiltonian flows on the projective plane (Theorem 3) and derives a formula for the number of Reeb trees (Lemma 1), which is used to prove a recursive formula for the number of flows on the projective plane (Theorem 4). In Section 5, a recursive formula for Reeb trees with two fixed leaves is derived. Section 6 describes all possible structures of Reeb graphs on the Klein bottle. Theorem 5 proves the realization of the Reeb graph by a flow, and Theorem 6 addresses the number of flow structures with a given number of saddles on the Klein bottle. Section 7 outlines a method for counting the number of Reeb graphs with a cycle (the last type of flows on the Klein bottle). Section 8 presents examples and computational results using the derived formulas.

1. Flows on the Projective Plane and the Klein Bottle

The real projective plane $\mathbb{R}P^2$ is the set of one-dimensional subspaces of three-dimensional Euclidean space. A one-dimensional subspace is a line that passes through the origin. The distance between points in projective space is defined as the sector between the lines that represent them. The projective plane can also be viewed as a quotient space of the sphere S^2 under an equivalence relation where each equivalence class consists of a pair of antipodal points. In this case, the sphere serves as a double-sheeted oriented cover over the projective plane. The projection of the sphere onto the projective plane allows us to construct a flow on the sphere from a flow on the projective plane. The corresponding vector field at symmetric points on the sphere is defined by opposite vectors.

If we cut out a regular neighborhood of any point from the projective plane, we obtain a set that is homeomorphic to the Möbius strip. Additionally, the projective plane can conveniently be represented as a two-dimensional disk D^2 where diametrically opposite points on the boundary are identified. In this case, the flow V at opposite points must satisfy the property in polar coordinates (ρ, φ) such that $V_\rho(1, \varphi) = -V_\rho(1, \varphi + \pi)$, $V_\varphi(1, \varphi) = V_\varphi(1, \varphi + \pi)$. In particular, if the flow is tangent to the boundary, then at opposite points it defines the same orientation of the boundary.

The Klein bottle Kl can conveniently be defined as a quotient space of the square $[0, 1] \times [0, 1]$ under an equivalence relation generated by the following relations: $(0, v) \sim (1, v), v \in [0, 1]$, $(u, 0) \sim (1 - u, 1), u \in [0, 1]$. Then, the vector field $V = \{V_1, V_2\}$ on the Klein bottle is defined by a vector field on the square with coordinates (u, v) under the conditions $V(0, v) = V(1, v)$, $V_1(u, 0) = -V_1(1 - u, 1)$, $V_2(u, 0) = V_2(1 - u, 1)$, $u, v \in [0, 1]$.

If we consider the torus as a surface of revolution in $\mathbb{R}^3$, then by identifying points that are symmetric with respect to the center of symmetry of the torus, we obtain a Klein bottle. This allows us to view the torus as a double cover over the Klein bottle. Therefore, vector fields on the Klein bottle correspond to vector fields on the torus that are symmetric with respect to the center of symmetry of the torus.

Note that oriented double covers over non-oriented surfaces are uniquely defined up to homeomorphism.

By analogy with flows on the projective plane and the Klein bottle, we can define flows on other non-oriented closed surfaces. For a non-oriented surface of odd genus, we attach empty handles to a sphere (taking the connected sum with an even number of tori at pairs of symmetric points). On the resulting surface, we define a flow that is symmetric with respect to its center of symmetry. After factoring by the action of the group $\mathbb{Z}/2\mathbb{Z}$ as central symmetry, we obtain a flow on a non-oriented surface of odd genus. Flows on non-oriented surfaces of even genus can be defined using a similar construction, starting with a torus instead of a sphere.

2. Atoms of Simple Hamiltonian Flows on Non-Oriented Surfaces

Definition. A flow on a non-orientable surface is called *Hamiltonian* if its lift to the orientable double cover is a Hamiltonian flow. A Hamiltonian flow on a non-orientable surface is called *simple* if each of its separatrices forms a loop, and the Hamiltonian of the lift to the orientable double cover is a Morse function.

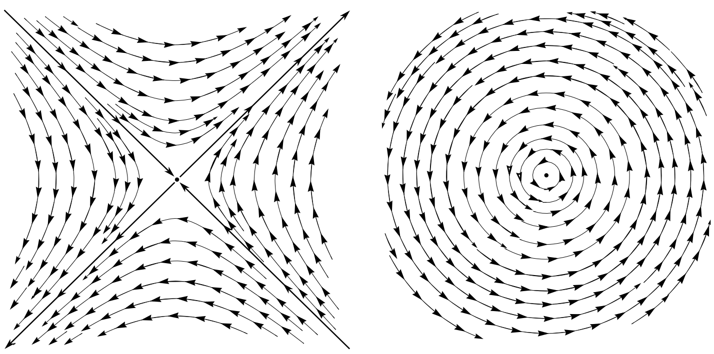

Fig. 1. Singular points of simple Hamiltonian flows

Since in a Morse function the critical points are non-degenerate, they can be of three types: local minima (there exists a coordinate system x, y such that in the neighborhood of this point the function $f(x, y) = x^2 + y^2$), saddles ($f(x, y) = x^2 - y^2$), and local maxima ($f(x, y) = -x^2 - y^2$). For a saddle, the singular point of the Hamiltonian vector field is also a saddle, $X(x, y) = \{y, x\}$, while for local extrema it is a center, $X(x, y) = \{-y, x\}$ (see Fig. 1).

Under the condition of simplicity of the flow, the trajectories entering and exiting a saddle point form loops. Therefore, they create a graph that is homeomorphic to 8. The neighborhood of this graph is called the atom of the saddle point if its boundary consists of closed trajectories of the flow and the neighborhood does not contain other singular points (except for the point belonging to 8). If this neighborhood is oriented, then it is homeomorphic to a three-holed sphere (a sphere with three holes), and in it, the flow is a Hamiltonian flow of a Morse function with one saddle point and constant on the boundary components. We denote this atom as A_0 (Fig. 2).

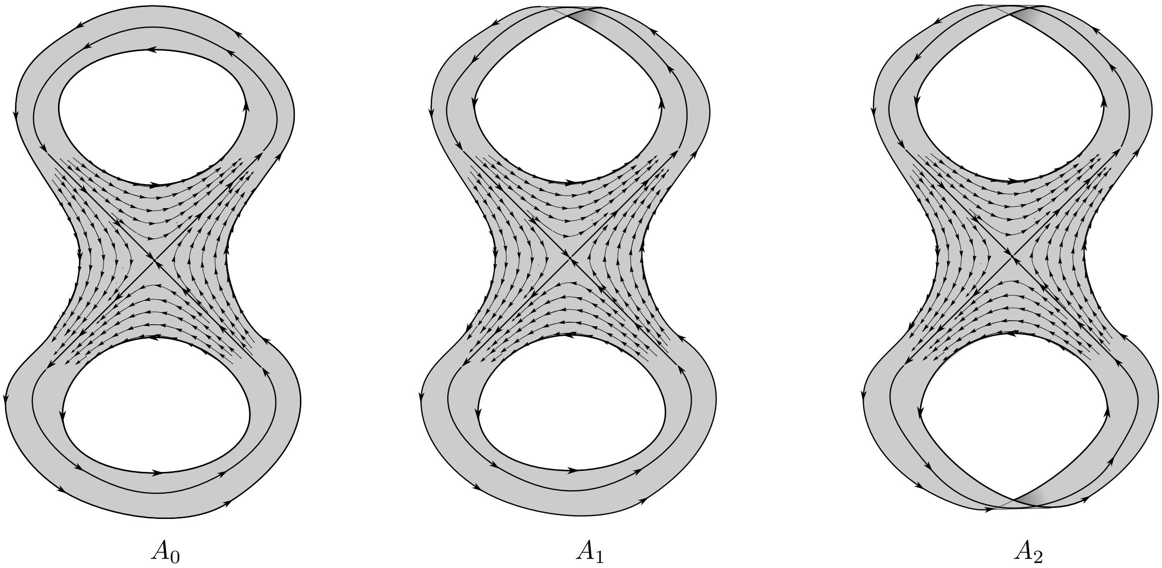

Fig. 2. Atoms of Saddle Points

Example 1. On the torus, as a surface of revolution ($x = (2 + \cos u) \cos v$, $y = (2 + \cos u) \sin v$, $z = \sin u$), we consider a Hamiltonian flow generated by the function $f(x, y, z) = x$. Then, the atom A_0 can be taken as the set $A_0 = f^{-1}([-3/2, -1/2])$. If, on the torus, we identify points that are symmetric with respect to the origin of coordinates, we obtain the atom A_0 on the Klein bottle.

In other cases, we have unoriented atoms. There are two possible cases depending on the number of boundary components of the atom. Each of these cases can be obtained from an oriented atom through the following procedure. Consider a regular point on one of the loops. We draw a transversal to the flow that intersects the boundary of the atom at two points. We cut the atom along this transversal and glue it back together, using a reflection that reverses the orientation and is constant at the point of the loop. In the first variant, we apply this operation to one loop, and in the second – to each of the loops.

Atom A_1. If the neighborhood of one of the two loops of the atom is homeomorphic to a Möbius strip, and the other loop is homeomorphic to a cylinder, then the atom has two boundary components. Thus, the atom is homeomorphic to the projective plane with two holes (or, equivalently, a Möbius strip with a hole). The double cover of this atom is a sphere with four holes.

Example 2. An example of such a cover is the neighborhood U of the critical level (preimage of 0) of the function $f(x, y, z) = x^3 - y^3 + xyz$ on the unit sphere $x^2 + y^2 + z^2 = 1$, $U = f^{-1}([-0.01, 0.01])$ (Fig. 3). If this function is extended to the sphere so that each component of the complement to the atom contains one critical point that is a local extremum, then it serves as a Hamiltonian for the Hamiltonian flow on the sphere. Upon identifying symmetric points, we obtain a Hamiltonian flow on the projective plane with one saddle and two centers.

Atom A_2. If the neighborhood of each of the two loops is homeomorphic to a Möbius strip, then the atom has one boundary component. Therefore, the atom is an unoriented surface of genus 2 with one boundary component – the Klein bottle with a hole. The double cover of this atom is a torus with two holes.

Example 3. In local coordinates u, v on the torus, the double cover is the neighborhood of the preimage of 0 of the function $f(u, v) = \sin u + \sin v$, $u \in [0, 2\pi], v \in [0, 2\pi]$. The atom A_2 is obtained from $[0, 2\pi] \times [0, \pi]/\sim$, where $(u, 0) \sim (2\pi - u, \pi), u \in [0, 2\pi]$, by removing a neighborhood of the point $(\pi/2, \pi/2)$. The field of the skew gradient of the function f defines a Hamiltonian flow on A_2. By extending it to the flow on the removed neighborhood, we obtain a Hamiltonian flow on the Klein bottle with one saddle and one center.

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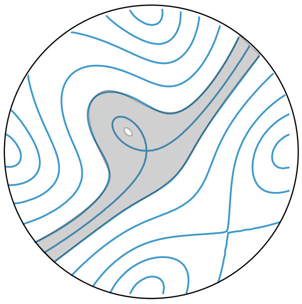

Fig. 3. Atom A_1. The gray region is the image of U by the orthogonal projection of the hemisphere

Definition 1. A closed trajectory is called *regular* if it has a neighborhood homeomorphic to $S^1 \times [0,1]$. A closed trajectory is called *singular* if its regular neighborhood is homeomorphic to a Möbius strip.

The atom of the center is called its regular neighborhood, the boundary of which is a closed trajectory. We denote it as C. Thus, C is homeomorphic to D^2 (Fig. 4).

An atom B (Fig. 4) be called a regular neighborhood of a singular closed trajectory, the boundary of which is a regular closed trajectory of the flow. Thus, B is homeomorphic to a Möbius strip.

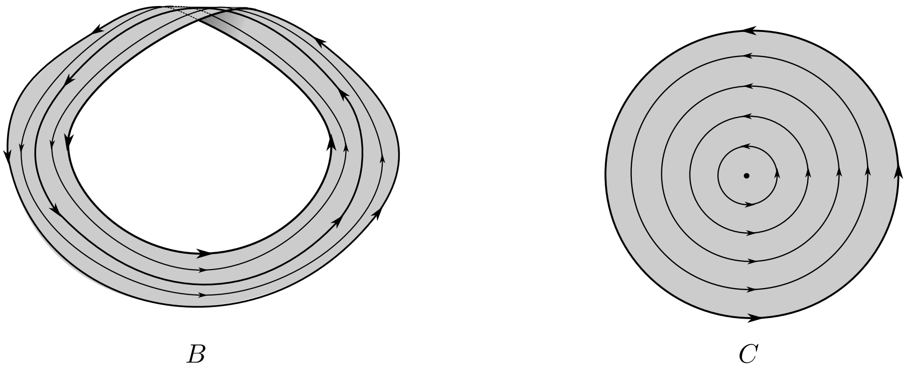

Fig. 4. Atoms B and C

Example 4. If we consider the function $f(x,y,z) = z$ on the sphere $S^2 = \{(x,y,z) : x^2 + y^2 + z^2 = 1\}$, then its Hamiltonian vector field is symmetric with respect to the origin. After identifying diametrically opposite points on the sphere, we can distinguish two atoms:

$$B = \{(x,y,z) \in S^2 : |z| \leqslant \frac{1}{3}\}/\sim, \qquad C = \{(x,y,z) \in S^2 : |z| \geqslant \frac{2}{3}\}/\sim .$$

As the last example shows, there exists a Hamiltonian flow on the projective plane without saddles. From the Poincaré-Hopf theorem, it follows that on the projective plane, the number of centers is one more than the number of saddles, while on the Klein bottle, the number of centers equals the number of saddles. Since every Morse function on a closed manifold has extremum points, the simple Hamiltonian flow contains at least one center. Therefore, on the Klein bottle, every simple Hamiltonian flow contains a saddle.

3. Reeb Graph

Definition 2. A *fiber* of a simple Hamiltonian flow be one of the following sets: 1) a *regular fiber* is a regular closed trajectory; 2) a *singular fiber* is either a singular closed trajectory or the union of a singular point along with its separatrices.

Definition 3. The factor space $M/\sim$ is called a *Reeb graph*, where $x_1 \sim x_2$ if x_1 and x_2 belong to the same fiber. In this graph, for each vertex of degree 1, the type of the corresponding atom (A_2, B, or C, meaning that vertices of degree 1 are colored in three colors) is indicated, and for each vertex of degree greater than 1, one of the incident edges is marked. *Marked edge* for a vertex v of the Reeb graph is the edge, which corresponds to two or three hyperbolic sectors of a singular point, that corresponds to v. Reeb graphs are equivalent if there exists an isomorphism between them that preserves the type of vertices of degree 1 and the marked incident edges for vertices of degree greater than 1.

Let $N(A_i)$ ($N(B)$, $N(C)$) denote the number of vertices in the Reeb graph corresponding to A_i (B, C) atoms, V be the total number of vertices, and E be the total number of edges.

Theorem 1. *The Reeb graph of a simple Hamiltonian flow on a connected non-orientable surface has the following properties:*

1) the graph is connected, and the degree of the vertices does not exceed 3;
2) the non-orientable genus g of the surface can be computed using the formula:

$$g = N(A_1) + 2N(A_2) + N(B) + 2(E - V + 1). \tag{1}$$

Proof. 1) The Reeb graph is a factor space of a connected manifold, and therefore it is connected. Since the vertices of the Reeb graph correspond to singular fibers, their degrees are the same as those of the corresponding atoms in the Reeb graphs: one for A_2, B, C, two for A_1, and three for A_0.

2) If all atoms are removed from the surface, we obtain a disconnected union of cylinders. Therefore, the surface can be viewed as a union of these sets glued together. Each Möbius strip increases the genus by 1. Thus, gluing each of the atoms A_1 or B increases the genus by 1. The atom A_2 is a union of two Möbius strips, so its gluing increases the genus by 2. Hence, if the Reeb graph is a tree, the genus of the surface is determined by the number of atoms A_1, A_2, B. Any graph can be obtained from a tree by connecting pairs of vertices with edges. Adding such an edge to the Reeb graph is equivalent to adding a blank handle to the surface (removing two disks and gluing a cylinder to the boundaries of the holes). This operation increases the oriented genus of the surface by 1 or the non-oriented genus by 2. This accounts for the presence of the last term in the sum. □

Remark 1. For Morse-Smale fields and Lyapunov graphs, analog of Theorems 1 was proven in (de Rezende, & Franzosa, 1993).

Theorem 2. *Two simple Hamiltonian flows on non-oriented closed connected surfaces are topologically equivalent if and only if their Reeb graphs are equivalent.*

Proof. The proof scheme is analogous to the proof for the oriented case (Bolsinov, & Fomenko, 2004). First, we prove the topological equivalence of flows on the atoms, and then we extend the obtained homeomorphisms to the complements of the atoms (cylinders).

The isomorphism of Reeb graphs defines a bijection between atoms and their boundary components. We fix a Riemannian metric on the manifold, which in the neighborhoods of singular points coincides with the Euclidean metric of the coordinate system in which the Hamiltonian $H(x, y) = \pm x^2 \pm y^2$. Then, the trajectories orthogonal to the flow, along with the flow trajectories, can be used to construct a local coordinate system, and the topological equivalence of flows can be defined through the convergence of coordinates.

Let π be a double covering, H be a Hamiltonian, p be a saddle point, x, y be local coordinates in a neighborhood of the point p, in which $H(x, y) = x^2 - y^2$, the atom A is a component of $\pi(H^{-1}(-\varepsilon, \varepsilon))$, $\varepsilon > 0$, containing $p = (0, 0)$; $U = \{(x, y) : |x^2 - y^2| \leqslant \varepsilon, |xy| \leqslant \varepsilon\}$ is a neighborhood of the point p; $b \subset A$ is the singular fiber; β_i are the components of $|x^2 - y^2| \leqslant \varepsilon$, $|xy| = \varepsilon$; α_i are the components of $b \setminus \mathrm{Int}(U)$ (in Fig. 2, U is the region filled with trajectories, β_i is an arc perpendicular to the flow at the boundary of this region, and α_i is an arc of the critical level in the region not filled with trajectories). Note that the curves $xy =$ const are orthogonal to the trajectories of the Hamiltonian field lying on the curves $x^2 - y^2 =$ const. In U, the homeomorphism is defined by the equality of the corresponding coordinates of points. We then extend this homeomorphism to the atom A. For this, we define homeomorphisms for α_i. Since the components $A \setminus U$ are homeomorphic to $\alpha_i \times \beta_i$, and homeomorphisms are constructed on each of α_i and β_i, they define a homeomorphism for $\alpha_i \times \beta_i$ and for A. Similarly, the homeomorphism from the atoms extends to the cylinders. The consistency of the constructed homeomorphisms on each of the segments follows from the isomorphisms of Reeb graphs. □

4. Simple Hamiltonian Flows on the Projective Plane

To obtain the number of topologically non-equivalent Hamiltonian flows, we calculate the number of Reeb graphs with a given number of vertices that correspond to saddle critical points.

We begin calculations for graphs that are trees, having vertices of degree one (type C) and three (type A_0), with one of the degree-one vertices serving as the root. Such graphs are Reeb graphs of Morse functions (and their Hamiltonian flows) on the 2-disk (the root is correspond to the boundary fiber). We orient all edges according to the following rules: 1) if the edge is incident to the root, then it is directed away from it, 2) if the marked edge for a saddle vertex is entering it, then two other edges are outgoing from it, 3) if the marked edge is exiting from a saddle, then two other edges are incoming to the saddle.

Since a vertex of degree 2 is not a source or sink on the graph, the orientation of one of the edges incident to it determines the orientation of the second edge. Similarly, at a vertex of degree 3, the condition that the marked edge has the opposite orientation to the other two edges allows the orientation of one edge to determine the orientation of the other two incident edges. Since the tree is a connected graph, the orientation of one edge sequentially determines the orientations of the edges adjacent to it, then the orientations of the edges adjacent to the adjacent ones, and so on, until the orientation of all edges of the graph is determined.

Lemma 1. *The number K_k of Reeb graphs that are rooted trees with k saddle points (vertices of type A_0) and other vertices of type C can be computed using the following recursive formulas: $K_0 = 1$,*

$$K_{2n} = 3 \sum_{i=0}^{n-1} K_i K_{2n-1-i} \qquad \text{for even } k = 2n; \tag{2}$$

$$K_{2n+1} = \frac{3K_n^2 + K_n}{2} + 3\sum_{i=0}^{n-1} K_i K_{2n-i} \qquad \text{for odd } k = 2n+1. \tag{3}$$

Proof. If k=0, then the tree has no saddles. Such a graph is unique. It has two vertices and one edge connecting them. From now on, we assume that $k > 0$.

Consider the saddle v closest to the root of a rooted tree G with k saddle points (vertices of type A_0) and other vertices of type C. It is a vertex of degree 3. We cut the graph by v. We obtain three connected components. One of them is the edge e connecting v to the root. The other two components (A and B) can be viewed as rooted Reeb graphs, where the root is v. Let us assume that the number of vertices $|V_A|$ of no fewer than the number of vertices $|V_B|$ of B ($|V_A| \geqslant |V_B|$).

Then there are three possibilities:

1) the marked edge to a vertex v is e;

2) the marked edge belongs to A;

3) the marked edge belongs to B.

Multiplying by 3 in formulas (2) and (3) is to indicate which subgraph with the marked edge is connected to.

If $k = 2n$, then the number of graph vertices is different ($|V_A| > |V_B|$), and A cannot be isomorphic to B. The number of non-isomorphic trees with i saddles is equal to K_i, and with $2n-1-i$ saddles is equal to K_{2n-1-i}, so the total number of trees G of this type is equal to $K_i K_{2n-1-i}$. By considering all possible values of i, we obtain formula (2).

If $k = 2n+1$ and $|V_A| > |V_B|$, then we have similar products as in the case of $k = 2n$. Let's consider the case of $|V_A| = |V_B| = n$ in more detail. If edge e is not marked, then we relabel the graphs so that the marked edge belongs to graph A. Since each of the graphs can be chosen in K_n ways, there are K_n^2 graphs G of this type. If edge e is marked, then two cases are possible: 1) graph A is not isomorphic to graph B, 2) graphs A and B are isomorphic. In the first case, we have $K_n(K_n-1)/2$ non-isomorphic graphs, and in the second case, K_n graphs. Summing everything together, we obtain formula (3). □

The results of the calculations of the number of Reeb rooted trees on $\mathbb{R}P^2$ are presented in Tab. 1.

From theorem 1 it follows:

Corollary 1. *The Reeb graph on the projective plane has the following properties:*

1) the graph is a tree;

2) the graph has no vertices of type A_2 and only one vertex of type A_1 or B.

Proof. In formula (1), all four summands are non-negative, therefore, since the genus of the projective plane is $g = 1$, it follows that $E - V + 1 = 0$ (which is a criterion for a connected graph to be a tree), $N(A_2) = 0$, and one of the following holds: 1) $N(A_1) = 1$, $N(B) = 0$ or 2) $N(A_1) = 0$, $N(B) = 1$. □

Theorem 3. *Let a graph G be given, which is a connected tree with vertices of degree 1 colored in two colors (B, C), and has one of two types:*

1) it has a single vertex of degree 2 (type A_1), while the remaining vertices have degrees 1 and 3, all vertices of degree 1 are of type C, and for the vertex of degree 2 and the vertices of degree 3, one incident edge is marked to each;

2) all vertices have degrees 1 or 3, one of the vertices of degree 1 is of type B, while the remaining vertices of degree 1 are of type C, and for the vertex of degree 3, one incident edge is marked to it.

Then there exists a simple Hamiltonian flow on the projective plane, whose Reeb graph is equivalent to G.

Proof. In the first case, we construct the atom A_1 and the flow on it as in Example 3. The vertex of degree 2 splits the graph into two rooted trees – the Reeb graphs of simple Morse functions on a two-dimensional disks, constant on the boundary. We consider the Hamiltonian flows generated by these functions. By attaching disks to the boundary of A_1, we obtain an undirected surface with the desired flow on it.

In the second case, we start with the atom B in Example 4, and then, as in the first case, we construct the flow on the two-dimensional disk and attach this disk to the boundary of B. We smooth the obtained flows as it is done in (Gutierrez, 1986; Neumann, 1978). Since the field is smoothed at the gluing points and, according to these works, the topological structure of the flow is preserved during smoothing, the smoothing process actually consists of smoothing the Hamiltonian function. Consider a neighborhood of the edge as a collar $S^1 \times [0, 1)$ with a Hamiltonian on it $H_1(s,t) = H_1(t)$ for the first of the glued surfaces and $S^1 \times (-1, 0]$ with a Hamiltonian $H_2(s,t) = H_2(t)$. Here, $S^1 \times \{0\}$ is a common boundary for both surfaces. Let's define the Hamiltonian function on the glued surface as $H(s,t) = H_1(s,t)$ for the points of the first surface. And for the points of the second surface, $H(s,t) = H_2(s,t) + H_1(0) - H_2(0)$ if $H_1'(0)H_2'(0) > 0$ and $(s,t) = -H_2(s,t) + H_1(0) + H_2(0)$ if $H_1'(0)H_2'(0) < 0$. On the set $S^1 \times (-1, 1)$ we have $H(s,t) = H(t)$ and this function is monotonic and smooth except at point $t = 0$. Smoothing it at 0, we obtain the desired Hamiltonian function. □

Remark. Theorem 3 also holds in the other direction: if G is the Reeb graph of a Hamiltonian flow on the projective plane, then it is of type 1) or 2). This follows directly from Theorem 1.

Theorem 4. *The number of topologically non-equivalent simple Hamiltonian flows with k saddles on the projective plane $\mathbb{R}P^2$ can be calculated using the formula*

$$N(\mathbb{R}P^2)_k = K_k + \sum_{i=0}^{k-1} K_i K_{k-i-1}.$$

Proof. Since the projective plane is a non-orientable surface of genus 1, the flow must contain either 1) one singular closed trajectory, or 2) one atom A_1. In the case of a singular closed trajectory, it corresponds to a vertex B on the Reeb graph, which we consider as the root. Therefore, the number of such flows is equal to K_k.

In the second case, the complement to the atom A_1 is a disconnected union of two two-dimensional disks. On each of them, we have a Hamiltonian flow. The complete topological invariant of such flows is the rooted Reeb graphs (the root corresponds to the boundary of the 2-disk). We order these two 2-disks: the first disk lies in one sector of the saddle point of the atom A_1, while the second disk lies in three sectord of this point. Since the flow on the second disk does not depend on the flow on the first disk, and the total number of saddle points on these two 2-disks is equal to $k-1$, the total number of flows satisfies the formula from the theorem. □

The results of the calculations of the number of simple Hamiltonian flows on $\mathbb{R}P^2$ are presented in Tab. 1.

Table 1

Numbers of rooted trees and simple Hamiltonian flows on $\mathbb{R}P^2$

k	0	1	2	3	4	5	6	7	8
K_k	1	2	6	25	111	540	2736	14396	77649
$N(\mathbb{R}P^2)_k$	1	3	10	41	185	898	4560	23985	129415

5. Reeb Graphs with Two Fixed Leaves

For a tree with two fixed leaves and with one and three degree vertices, we orient all edges according to the following rules: 1) if the edge is incident to the first leaf, then it is directed away from it, 2) if the marked edge for a saddle vertex is entering it, then two other edges are outgoing from it, 3) if the marked edge is exiting from a saddle, then two other edges are incoming to the saddle. We say that the isomorphism of the graph preserves the order of leaves if it maps the first fixed leaf to first fixed leaf an second fixed leaf to the second one.

Lemma 2. *Up to isomorphism that preserves the order of leaves, the number L_k of Reeb graphs that are trees with two ordered fixed leaves and k saddle points (saddles have degree three, and all other vertices have degree one) can be computed using the following recursive formula: $L_0 = 1$,*

$$L_k = 3(L_0K_{k-1} + L_1K_{k-2} + \ldots + L_{k-1}K_0).$$

Proof. If $k = 0$, there is one graph with one edge: $L_0 = 1$. Suppose $k > 0$. Just as in the previous theorem, consider the saddle point v closest to the first fixed leaf. It is a vertex of degree 3, since $k > 0$. We cut the graph at v. This results in three connected components. One of them is the edge connecting v to the first leaf. The other two components can be viewed as the rooted Reeb graph G_1 and a graph G_2 with two ordered fixed leaves, where the root (the first leaf) is v.

Again, we have three options for selecting the marked edge in v, which explains the presence of the multiplier 3 in the formula:

1) in both graphs, the edges incident to v are outgoing (the marked edge does not belong to either G_1 or G_2);

2) this edge is outgoing in G_1, while it is incoming in G_2 (the marked edge belongs to G_1);

3) it is incoming in G_2, while it is outgoing in G_1 (the marked edge belongs to G_2).

The first and second graphs are not isomorphic, as the first has two fixed leaves, while the second has only one (the root). Therefore, the number of non-isomorphic original graphs with such subgraphs equal the number of non-isomorphic subgraphs multiplied together. This gives the desired formula. □

Example. In the case $k = 1$ the graph has the shape of the letter Y. At the vertex of degree 3 there are three options for choosing the incident edge corresponding to the vertices of degree 1: 1) the first fixed vertex, 2) the second fixed vertex, 3) a non-fixed vertex. Therefore $L_1 = 3$.

Furthermore, we need the concept of a monotonic graph. A *monotonic graph* is a Reeb graph of a simple Morse function with two fixed leaves $v_1,\ v_2$ and each saddle vertex of the simple path from v_1 to v_2 has the marked edges belonging to the path. This means that an orientation can be assigned to the graph that is consistent with the fixed edges, and the path from v_1 to v_2 is directed (the restriction of the Hamiltonian to them is monotonic). Let M_k^S denote the number of non-isomorphic monotonic graphs with k saddles (up to isomorphism that can preserve or change the order of the fixed pair of vertices) and let M_k denote the number of non-isomorphic monotonic graphs with k saddles up to isomorphism, which preserves the order of the fixed pair of vertices (the orientation of the path from the first to the second vertex).

Let's find a recursive formula for M_k similar to Lemma 2. There exists a unique graph without saddles – a segment. Therefore, $M_0 = M_0^S = 1$. Let k>0 and v_1v be an edge incident to v_1. Then v is a vertex of degree 3 belonging to the path from v_1 to v_2. It divides the graph into three parts: 1) the edge v_1v, 2) a monotone graph with two fixed leaves v and v_2, 3) a tree with root v. The monotone graph and the rooted tree have a total of $k-1$ saddle vertices. Therefore, the following formula holds:

$$M_k = 2(M_0K_{k-1} + M_1K_{k-2} + \ldots + M_{k-1}K_0).$$

Let's find the number of monotone graphs with k saddles for which there exists a graph isomorphism onto itself that swaps fixed leaves and preserves the distinguished edges. This isomorphism has an invariant vertex or edge. If it is a vertex, then it has degree 3 and one of the edges is distinguished, and therefore its image under the isomorphism is also a distinguished edge, which is impossible, because each saddle vertex has only one distinguished edge. Therefore, the invariant is an edge. It divides the graph into two isomorphic subgraphs, which are monotone. Therefore, the number of such graphs is equal to $M_{k/2}$. Taking this into account, we have $M_k^S = \frac{1}{2}M_k$ for odd k, and $M_k^S = \frac{M_k+M_{k/2}}{2}$ for even k. The results of calculations using these formulas are shown in Tab. 2.

Example. If k=1, then the monotone graph with two fixed leaves has the form of the letter Y. Then, the marked edge for vertex 3 can be incident to either the first leaf or the second. If the isomorphism preserves the order of the leaves, then these two options are different, and if it can change the order of the leaves (the orientation of the path from one leaf to the other), then they are isomorphic. Therefore, $M_1 = 2$ and $M_1^S = 1$.

Table 2

Numbers of trees with two fixed leaves and monotonic graphs

k	0	1	2	3	4	5	6	7
L_k	1	3	15	81	462	2700	16074	96876
M_k	1	2	8	36	178	918	4904	26824
M_k^S	1	1	5	18	93	459	2470	13412

6. Simple Hamiltonian Flows on the Klein Bottle

A vertex of a graph is called an *R-vertex* for a cycle in the graph if the marked edge for this vertex does not belong to the cycle.

Since the non-orientable genus of the Klein bottle is 2, it follows from Theorem 1 that:

Corollary 2. *The Reeb graph on the Klein bottle has one of the following types:*

a) *the graph is a tree containing two vertices of degree 2, and the vertices of degree 1 are of type C;*
b) *the graph is a tree containing one vertex of degree 2, one vertex of type B, and the remaining vertices of degree 1 are of type C;*
c) *the graph is a tree that does not contain vertices of degree 2, has two vertices of type B, and the remaining vertices of degree 1 are of type C;*
d) *the graph is a tree that does not contain vertices of degree 2, has one vertex of type A_2, and the remaining vertices of degree 1 are of type C;*
e) *the graph has one simple cycle, this cycle contains an odd number of R-vertices, and all vertices of the graph are either of type C or of degree three.*

Proof. As noted earlier, the condition $E - V + 1 = 0$ for a connected graph is equivalent to it being a tree. Therefore, we have the first four options. In the fifth option, $E = V$, which is equivalent to the existence of a cycle in the graph. In this case, the first three summands in the formula (1) are zero. If the cycle is cut at an interior point of an edge, we obtain a Reeb graph that is a tree and defines a Hamiltonian flow on the cylinder. We fix one of the boundary components of the cylinder. Projecting one base of the cylinder onto the other allows us to compare the orientations on them. If the orientations are the same, then after gluing the bases, we obtain a torus; if they are different, we obtain the Klein bottle. Moving a point along the cycle on the Reeb graph, we compare the orientations of the corresponding closed trajectories with the orientation of the initial one. Passing through R-vertices changes the orientation to the opposite, while other vertices do not change it. Therefore, to obtain a Klein bottle after gluing, there must be an odd number of R-vertices on the cycle. □

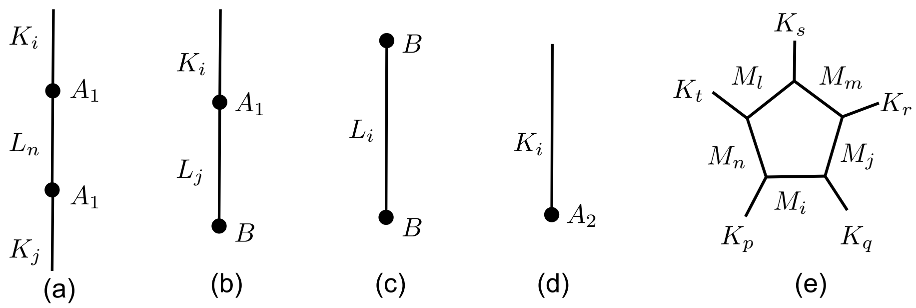

Fig. 5. Five types of Reeb graphs on Kl (each edge denotes a tree without A_1, A_2, B-vertices)

Theorem 5. *Let a graph G be given, which has one of the five types from the previous theorem. Then there exists a simple Hamiltonian flow on the Klein bottle, the Reeb graph of which is equivalent to the graph G.*

Proof. For each vertex of the graph, we construct an atom corresponding to the neighborhood of this vertex and a flow on it as was done in the examples of atoms. Each edge of the Reeb graph corresponds to a cylinder $S^1 \times [0, 1]$, on which a Hamiltonian flow is defined, generated by the projection onto the second factor. We attach the cylinders to the corresponding boundary components of the atoms, ensuring that the directions of movement along the trajectories on the boundary component of the cylinder and the boundary component of the atom coincide. We smooth the obtained flow as it is done in (Gutierrez, 1986; Neumann, 1978). The Hamiltonian is smoothed like proof of Theorem 3. As a result, we obtain a closed surface that is non-orientable for the first four types of graphs due to the presence of a non-orientable atom. In the last type, the non-orientability of the surface is justified in the proof of the corollary. According to formula (1), the genus of the surface is equal to 2. Thus, we have a flow on the Klein bottle. On the non-orientable atoms, this flow lifts to a Hamiltonian flow on the double cover as in the examples, while on the orientable atoms and disks, it is duplicated twice for each of the lift copies. Therefore, the constructed flow is Hamiltonian with the given Reeb graph. □

Theorem 6. *The number $N(Kl)_k$ of topologically non-equivalent simple Hamiltonian flows on the Klein bottle can be computed using the following formula for even $k = 2m$ or odd $k = 2m + 1$ numbers of saddles:*

$$N(Kl)_k = N(Kl)_k^a + N(Kl)_k^b + N(Kl)_k^c + N(Kl)_k^d + N(Kl)_k^e,$$

where the number $N(Kl)_k^a$ of Reeb graphs containing two A_1-vertices (degree 2) is computed using the formula

$$N(Kl)_k^a = 2 \sum_{i+j+n=k-2} K_i K_j L_n + \sum_{2(i+j+1)=k} K_i L_j + \sum_{2i+2j+n=k-3} K_i L_j K_n;$$

the number $N(Kl)_k^b$ of Reeb graphs having one A_1-vertex and one B-vertex is computed using the formula

$$N(Kl)_k^b = 2 \sum_{i+j=k-1} K_i L_j;$$

the number $N(Kl)_k^c$ of Reeb graphs having two vertices of type B is computed using the formula

$$N(Kl)_k^c = \frac{1}{2}(L_k + L_m + \sum_{2i+j+1=k} L_i K_j), \quad k = 2m,$$

$$N(Kl)_k^c = \frac{1}{2}(L_k + \sum_{2i+j+1=k} L_i K_j), \quad k = 2m + 1;$$

the number $N(Kl)_k^d$ of Reeb graphs that have one vertex of type A_2 is calculated using the formula

$$N(Kl)_k^d = K_{k-1};$$

and $N(Kl)_k^e$ is the number of Reeb graphs that contain a cycle.

Here, in all sums, $i, j, n \geqslant 0$ and the sums are taken over these indices.

Note that the cases (a)–(e) of the flows from the theorem correspond to Fig. 5 (a)–(e).

Proof. Case a. (Fig. 5 (a)). Two atoms A_1 divide the surface into three parts: two 2-disks and one cylinder. If the cylinder lies in one hyperbolic sector of one of the atoms A_1 (which we call the first), and in three hyperbolic sectors of the second atom A_1, then we have a natural ordering of the atoms. Therefore, the number of possible flows in this case is equal to

$$S_{KKL}^{k-2} = \sum_{i+j+n=k-2} K_i K_j L_n.$$

The cases where the cylinder lies in one hyperbolic sector of each atom and where it lies in three hyperbolic sectors of each of the A_1 are identical, so we only consider the first case and multiply the resulting number of flows by 2. We fix the order of the A_1 atoms. The number N of Reeb graphs up to isomorphism that preserve an unordered pair of vertices is equal to the sum of the number N_1 of such graphs for which there is no isomorphism that swaps the vertices, and the number N_2 of such graphs for which the specified isomorphism exists. If we fix the order of the pair of vertices and consider graph isomorphisms that preserve this order, we obtain $K = 2N_1 + N_2$ graphs. Thus, $N = (K + N_2)/2$. Considering the doubling of the total number, we have that the number of non-isomorphic graphs equals the sum of the number of graphs with an ordered pair of fixed vertices (A_1) up to an isomorphism that preserve order of the vertexes and the number of self-symmetric graphs up to an isomorphism. As before, up to an isomorphism that preserves the given order of A_1-vertices, the total number of Reeb graphs is equal to S_{KKL}^{k-2}. Changing the order of the A_1 atoms leads to a rotation of the orientation on the cylinder and the symmetry of the Reeb graph.

With an odd number of saddles, the symmetric graph can have one fixed edge (Fig. 6 (a1)). This edge divides the graph into two identical parts, each consisting of one root tree A and one tree with two fixed leaves B. Their total possible number is

$$\sum_{2(i+j+1)=k} K_i L_j.$$

If instead of the edge, C is invariant under symmetry subgraph (Fig. 6 (a2)), then there is a vertex v, which devide the Reeb graph onto 3 part, one of which is C. Other parts are isomorhic. Each of them consists of a root tree A and tree with two fixed leaves B. The number of Reeb graphs of such a type is equal $\sum_{2i+2j+n=k-3} K_i L_j K_n$. We add this to the total number of flows. Thus, we obtain the first formula.

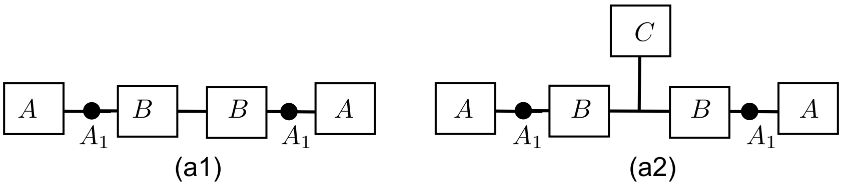

Fig. 6. Symmetric graphs with two A_1 vertices ($V(A) = i$, $V(B) = j$, $V(C) = n$)

Case b. The Reeb graph in this case does not allow symmetries that change two fixed points: the vertices A_1 and B. The vertex A_1 divides the Reeb graph into two parts: a root tree and a tree with two fixed leaves. The tree with two fixed leaves has a canonical orientation, generated at vertex B by the movement from vertex B to A_1. Therefore, the total number of Reeb graphs is the sum of the products of K_i and L_j. The total number of saddles in these trees is $k-1$. Since at vertex A_1 there is the possibility to marking each of the two incident edges, we double the total sum.

Case c. We consider two vertices B as marked leaves. The symmetry of the graph that swaps them can occur in two cases: 1) when there is no fixed vertex under the symmetry (possible only with an even number of saddles), 2) when there is a fixed tree. In the first case, graphs with such symmetries are determined by their halves, so their number equals L_m. In the second case, the complement to the tree consists of two isomorphic graphs with two leaves. Their common vertex is also a saddle (and is an R-vertex). Therefore, the total number of such graphs equals

$$\sum_{2i+j+1=k} L_i K_j.$$

All together gives the desired formulas.

Case d. We consider vertex A_2 as the root, thus we have a rooted tree with $k-1$ saddles, and the number of Reeb graphs equals K_{k-1}. □

7. Reeb graphs with a cycle

For Reeb graphs that have a cycle, the number of vertices in the cycle, with the marked edge not belonging to the cycle, is odd. If the cycle is a loop (Fig.!7.1), then the vertex of the loop is the root of the complement to the loop. In the Reeb graph, the loop corresponds to a monotonic graph, and the complement to the loop are rooted trees. Since the graph can have symmetry that change orientation of the loop, the number of monotonic trees is calculated as M_i^S. Thus, the number of such graphs is

$$\sum_{i=0}^{k-1} M_i^S K_{k-1-i}.$$

In other cases, when calculating the number of graphs, one should consider the possible symmetries of the graph relative to the cycle (Fig. 5e). An algorithm for finding all cyclic Reeb graphs with n saddles can be constructed according to the following scheme:

1) We find this number separately for each odd m ($m \leqslant n$) of R-vertices on the cycle. We represent the cycle as a regular m-gon. The symmetries of the graph correspond to the symmetries of the m-gon. They are divided into two types: a) axial symmetries – the axis passes through a vertex and the center of the opposite side (Fig. 7.11 and 14), b) rotations – cyclic permutations of the vertices (Fig. 7.10 and 15).

2) R-vertices (vertices of the m-gon) partition the Reeb graph into rooted trees and monotonic graphs (Fig. 5e). Thus, each vertex of the m-gon corresponds to a rooted tree, and each edge corresponds to a monotonic graph. The total number of saddle vertices in the rooted trees and monotonic graphs equals $n-m$. We count based on the maximum number k of saddle vertices in each rooted tree and monotonic graph. We first consider the case when one vertex of the m-gon has k saddles. Then only the symmetry relative to the axis passing through this vertex is possible. If $n-m-k$ is odd, then there are no self-symmetric graphs, so the total number is half the sum of the products K_i and M_j, corresponding to the sides and vertices of the m-gon:

$$S(n-m-k) = \frac{1}{2} \sum_{i_2+\ldots+i_{m-1}+j_1+\ldots+j_{m-1}=n-m-k} K_{i_1} \ldots K_{i_m} M_{j_1} \ldots M_{j_m}.$$

Here $i_1 = k$. If $n-m-k$ is even, then there are graphs with an axis of symmetry, which are determined by their half, that is, a graph with $\frac{n-m-k}{2}$ vertices. In this case, the number of non-isomorphic Reeb graphs equals

$$S(n-m-k) + S(\frac{n-m-k}{2}).$$

If we have two vertices with the same number k, then there is a unique axial symmetry that maps one vertex to the other. Therefore, the calculations are analogous to the case of one vertex with k. In the case of three vertices with the same k and m being a multiple of three, in addition to axial symmetries, cyclic permutations are also possible. In this case, the graphs that are invariant under such permutations are defined by graphs with $\frac{n-m-k}{3}$ saddles. Here we assume that each $0 \leqslant j_l \leqslant k$.

Separately, we consider the case when the maximum number of vertices on an edge (in a monotone graph) equals k, while on all root trees it is less than k. If there is only one such edge (monotone tree), then the number of Reeb graphs of this type equals $M_k S(n-m-k)$ for odd k, and taking into account the symmetries relative to this edge – $(M_k+M_{k/2})S(n-m-k)$ for even k. If we have two or more edges with a maximum number of k, then the possible symmetries are accounted for similarly to those in the case of vertices.

8. Examples

Let us illustrate how the number $N(Kl)_6^C$ is obtained using the example of graphs with 6 saddles.

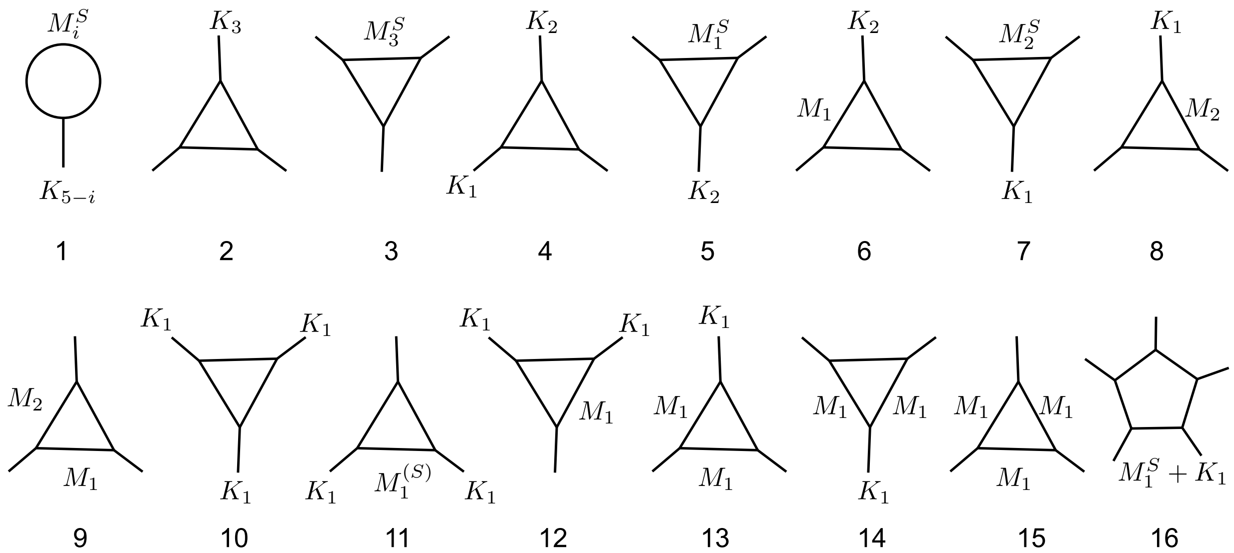

Fig. 7. Cyclic Reeb graphs with 6 saddles (the marked edges represent trees)

All possible types of cyclic graphs with no more than 6 saddles are shown in Fig. 7. Since R-vertices correspond to saddles, the following cases are possible: 1) one R-vertex, 2–15) three R-vertices, 16) five R-vertices. In the first case, the R-vertex divides the Reeb graph into a monotonic graph (loop) and a tree. Since the graph may have symmetry, we have M_i^S possible graphs for the loop, taking symmetry into account. For the tree, there are K_{5-i} graphs. In total, in the first variant, we obtain $\sum_{i=0}^{5} M_i^S K_{5-i} = 1529$ cyclic graphs. In the case of three R-vertices, graphs 4, 6, 8, 9, 12, and 13 have no symmetries. Therefore, the number of cyclic graphs is determined as the product of the corresponding numbers. Additionally, symmetries are considered for graphs 2, 3, 5, 7, and 11. In the case of 14, with a fixed K_1, we have two options when the graphs are symmetric to themselves, and two when they are symmetric to each other. Thus, taking K_1 into account, there are a total of 6 cyclic graphs of this type. Graph 10 can have both axial symmetries and symmetries with cyclic permutations of vertices (rotation by $2\pi/3$). To find the number of options, we denote the graphs with K_1 as G_1 and G_2. In a cyclic graph, there can be from 0 to 3 graphs G_1. Therefore, there are 4 cyclic graphs of type 10. For type 15, symmetries and cyclic permutations are also possible. We fix the direction of traversal of the cycle. In relation to it, there are two possible situations: 1) all monotonic graphs are the same, 2) two are the same, and the third is of a different type. Thus, there are only two cyclic graphs of type 15. In summary, we find 147 cyclic graphs with three R-vertices. In the case of type 16, we have 5 R-vertices. If the sixth (non-R) vertex lies on the monotonic graph, then there is exactly one variant up to symmetry. If it belongs to the tree, then there are 2 variants. In total, there are three cyclic graphs with 5 R-vertices. Overall, we have 1679 cyclic graphs with 6 saddles.

Results of Calculations

Using the obtained formulas and the described algorithm, the number of Reeb graphs of each type was found, as well as the total number of flow structures with no more than 8 saddles. The results of the calculations are presented in the Tab. 3.

Table 3

Numbers of simple Hamiltonian flows on the Klein bottle

k	1	2	3	4	5	6	7	8
a	0	3	16	89	530	3214	19575	120407
b	2	10	54	308	1800	13016	64584	392938
c	2	9	45	254	1422	8400	50058	303211
d	1	2	6	25	111	540	2736	14396
e	1	3	14	62	316	1679	8987	49348
$N(Kl)_k$	6	27	135	738	4179	26849	145940	880300

9. Discussion and conclusions

A topological classification of simple Hamiltonian flows on the projective plane and the Klein bottle has been obtained. All possible topological structures of flows with no more than 8 saddles have been described. A recursive formula has been derived for determining the number of topologically non-equivalent flows with a given number of critical points on the projective plane. Flows on the Klein bottle have been divided into five types; for the first four types, recursive formulas have been obtained to find the number of their possible structures, while for the last type, an algorithm for finding such structures has been described.

The results obtained may be useful for further research on the topological properties of Hamiltonian flows on various surfaces.

Authors' contribution: Alexandr Prishlyak — conceptualization; methodology; empirical research; Kateryna Butok — software; empirical data collection and validation; analysis of sources, preparation of literature review and theoretical foundations of research.

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ТОПОЛОГІЧНА СТРУКТУРА ПРОСТИХ ГАМІЛЬТОНОВИХ ПОТОКІВ НА ПРОЄКТИВНІЙ ПЛОЩИНІ ТА ПЛЯШЦІ КЛЕЙНА

Для кожної неорієнтованої поверхні існує орієнтоване дволисте накриття. Ми розглядаємо потоки, підняття яких на дволисте накриття є гамільтоновими потоками з гамільтоніаном – функцією Морса. За аналогією з орієнтованими многовидами, потік простий, якщо в ньому не існує сідлових зв'язок між різними сідлами. Для опису топологічних властивостей функцій на двовимірних многовидах використовуємо інваріант, що називається графом Ріба. У випадку простих функцій Морса на замкнутих орієнтованих двовимірних многовидах – це повний топологічний інваріант для пошарової еквівалентності і топологічної еквівалентності гамільтонових потоків.

Проте в нашому випадку, на неорієнтовному многовиді не існує функції, що породжує гамільтоновий потік. З огляду на це поняття графа Ріба потребує перевизначення.

Для дослідження топологічних властивостей простих гамільтонових потоків на неорієнтовних поверхнях описано структуру всіх можливих атомів, що є аналогом атомів Болсінова-Фоменка для функцій Морса та гамільтонових потоків на орієнтованих поверхнях. Зокрема, на неорієнтованих поверхнях є п'ять типів таких атомів. Побудовано повний топологічний інваріант, що є графом Ріба з маркуваннням його вершин та інцидентних до вершин півребер, описано його властивості та доводено критерій топологічної еквівалентності потоків. Для проєктивної площини та пляшки Клейна ми доводимо теореми про існування потоків із заданим графом Ріба. Для циклічних графів Ріба на пляшці Клейна описано алгоритм для їхнього перерахування. Для всіх інших типів графів Ріба на проєктивній площині та пляшці Клейна отримано рекурсивні формули для знаходження їхнього числа. Наведено результати обчислень числа топологічних структур потоків з не більш ніж восьма сідлами. Отримані результати відкривають можливість дослідження структур гамільтонових потоків більшої складності на компактних неорієнтованих поверхнях.

К л ю ч о в і с л о в а: ***топологічна еквівалентність, граф Ріба, класифікація, неорієнтовна поверхня.***

Автори заявляють про відсутність конфлікту інтересів. Спонсори не брали участі в розробленні дослідження (у зборі, аналізі чи інтерпретації даних, якщо це мало місце), у написанні рукопису та в рішенні про публікацію результатів.

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