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ALGORITHMIC APPROACHES TO MODELING AND ANALYZING ENDGAME POSITIONS IN BACKGAMMON

The final phase of the backgammon game, known as "races", is considered. This phase is characterized by minimizing the interaction of opponent's checkers and reducing the game process mainly to assessing the rate of moving checkers to the "home" and their subsequent withdrawal from the board. The delvered analysis covers the specifics of this stage in various backgammon variants, in particular in classic backgammon, using mathematical models and probability theory. The paper proposes methods for optimizing strategies in the final phase of the game, which are based on a comprehensive analysis of possible game positions and calculating the probabilities of the dices values. Special attention is paid to algorithmic approaches to decision-making, which allow calculating the most effective game actions to achieve the optimal result.

The results of the study are valuable both for improving the game strategies of practicing players and beginners, and for developing theoretical approaches to game analysis based on modeling and assessing the effectiveness of decision-making in conditions of limited interaction between participants.

K e y w o r d s : *backgammon, "races" stage, probability theory, total probability formula, simulation, optimal strategies, doubling cube, pip-count, game optimization.*

AMS 2020 classification: 62H30, 34C60.

Introduction

Backgammon is one of the oldest board games, gaining popularity both in Europe and the Middle East. Despite its centuries-old history, this game remains relevant due to the variety of its variations, among which the most common are long nardy, backgammon, tavli, etc. A special place among them is occupied by backgammon, which is the subject of numerous studies in the field of game theory and probability theory, and also serves as the basis for holding international championships and online tournaments. The main strategy of the game is to move checkers to the "home", or final area, with their subsequent withdrawal from the board. However, the most intensive mathematical and algorithmic analysis concerns the final stage of the game, which is known as "races" or "speed game". In this phase, the opponent's checkers cease to interact, and success is determined by the player's ability to bring checkers correctly into the house for their subsequent withdrawal as quickly as possible.

The complexity of analyzing strategies in backgammon has repeatedly attracted the attention of researchers. In particular, in the work (Witter, 2021) it was proven that the problem of determining the possibility of winning from an arbitrary position belongs to the class of computationally complex ones, which explains the relevance of the search for approximate methods. A significant contribution to the understanding of the stochastic aspects of the game was made by models of optimal strategies for doubling bets, presented in the work (Blath, & Morters, 2001), which showed the effectiveness of decision-making under uncertainty. A separate area of research is devoted to the "races" phase: in the article (Ross et al., 2007) mathematical models for estimating the probabilities of winning were built and compared with the results of simulations. The work of Tesauro played a key role in the development of the algorithmic approach, who used temporal difference learning methods and neural networks to evaluate positions and improve strategies in the TD-Gammon project (Tesauro, 1995). In addition, an analysis of the effectiveness of search algorithms, such as minimax and its variations, was presented in the works (Bell, 1997; Olsen, 2019), which confirms the promising application of optimization techniques in strategy simulation.

Thus, the analysis of "races", as a key phase of the backgammon game, has not only mathematical but also applied significance, as it allows to construct highly efficient strategies. This paper examines in detail the methods of optimizing strategies for this stage of the game with an emphasis on the use of modern analytical tools based on probability theory and decision-making algorithms.

1. Mathematical model. The main concept

Pip-count (PC). Let the fields of the board be numbered from 1 to 24, so that the nearest field in the house is numbered 1, etc., and the field farthest from the house is numbered 24. Then the sum of the products of the number of checkers by the numbers of the fields they occupy is called the pip-count or score. The opponent, to calculate his score, numbers the fields in reverse order. Thus, in backgammon, before the start of the game, each player's score equals to

$$2 \cdot 24 + 5 \cdot 13 + 3 \cdot 8 + 5 \cdot 6 = 167.$$

In fact, the score is the minimum number of points that a player must roll (over a number of moves) to bring the checkers into the house and then to withdraw them. It is said that a player who has a lower/higher score than the opponent is ahead/behind in speed, respectively. The skill of quickly calculating the score is an important component of a player's skill, the technique of such calculations can be found in (Kissane, 1992).

The roll. Each player's move is preceded by the roll of two dice. In this case, if the number of points on the dice matches (for example, 3-3), then it is said that a "double" has been rolled, and the number of points rolled is doubled (i. e. in this case the player has not two, but four moves of three points). The probability distribution of the number of points in one roll, taking into account doubles, is given in Tab. 1.

Table 1

Probability distribution of the number of points in one roll (two dice)

j	3	4	5	6	7	8	9	10	11	12	16	20	24
π_j	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{4}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$

This probability distribution has the following characteristics:

$$M(j) = \mu = \frac{49}{6} \approx 8.167, \sigma(j) = \sigma = \frac{\sqrt{665}}{6} \approx 4.297.$$

Doubling cube. In backgammon there is a rule of doubling bets. Let the game be played at a single bet at start. A player who has sufficient (in his subjective opinion) advantages in the position can say "double" before his roll. In this case, his opponent must make one of two decisions - to surrender without further play (and pay a single bet) or to continue the game at a doubled bet. There can be several doublings during the game. A double cube is a cube with degrees of 2 on its sides - from 2 to 64 and which is an indicator of the doubling status. At the beginning, when the game is played at a single bet, the cube is in the middle and this situation is called "our cube". If the player has accepted the offer to double, then the cube moves closer to him on the board, and the upper face shows the current (doubled) bet value. In this case, he is called the "owner of the cube", and the situation that has arisen is "my cube" (for the one who accepted the doubling) and "his cube" (for the opponent), and only the "owner of the cube" has the right to further double.

2. Optimal doubling strategies

It turns out that the correct managing of the double cube (i. e., an adequate evaluation of situations in which the player needs to double the bet himself, and either to accept to surrender as opponent doubles) is no less important than the ability to make the right moves. This problem was first considered in (Keeler, & Spencer, 1975). Let $W(t)$ be a standard Wiener process and $W(0) = 0.5$, the observation of the process ends at time t_0, when the value of the Wiener process trajectory takes the value 1 (then it is considered that the 1st player won) or 0 (then the 2nd player won). Then at each time t the conditional probability of the 1st player winning is equal to the value $W(t)$, and the conditional probability of the 2nd player winning is equal to the additional probability. For this situation, it is proved that the 1st player should double at the moment when $W(t) = 0.8$. The second player should accept the double if $W(t) < 0.8$ and surrender if $W(t) > 0.8$. If the first player doubled at $W(t) = 0.8$, then the second player is indifferent to whether to accept the double or surrender, in any case his average loss is equal to the current bet. Due to the symmetry of the model, the critical doubling point for the second player is obviously 0.2. But at the end of the game, situations may arise that are not adequate to the continuous game model. The situations where this inadequacy is most evident are the so-called one-step and two-step games considered in (Tuck, 1980).

In subsequent works, it was shown that the "continuous game" model, although it plays a significant role in estimating the threshold values of doubling and doubling acceptance, is not fully adequate, since the players' chances can change dramatically during a single roll (Tuck, 1980).

One step game. Let the first player after his throw win with probability p and lose with additional probability. Such a model is fully adequate to the real game situation in which the 2nd player has one or two checkers, which he will definitely throw away if it comes to his turn, i. e. one checker in the cell $1,2$ or 3 or two checkers $1-1$ or $1-2$. It is worth doubling for the 1st player when his expected gain at doubling is greater than the expected gain if he does not double, i. e. $4p - 2 >> 2p - 1$. Then the first player in the situations "my cube" and "our cube" should double if $p > 0.5$. In this case, the 2nd player should surrender if, having accepted the doubling, he loses more than by surrendering, i. e. $4p - 2 > 1$ or $p > 0.75$.

Two steps game. Let the first player after his roll win with probability p and with additional probability the move goes to the second player. If the move has reached the 2nd player, he also makes a roll, after which he wins with (conditional) probability q, otherwise the 1st player wins. Then the following result occurs. Consider the square $[0; 1] \times [0; 1]$, where the values of p and q are plotted along the abscissa/ordinate axes, respectively. Let the square be divided into 4 regions by such curves.

$$P_D(q) = \begin{cases} 0.5, q \geq 0.75 \,; \\ \dfrac{4q-2}{4q-1}, 0.5 \leq q \leq 0.75 \,; \\ 0, q \leq 0.5. \end{cases} \tag{1}$$

$$P_R(q) = \begin{cases} \dfrac{3-2q}{4-2q}, q \geq 0.75 \,; \\ \dfrac{6q-3}{6q-2}, 0.5 \leq q \leq 0.75 \,; \\ 0, q \leq 0.5. \end{cases} \tag{2}$$

$$P_Q(q) = \begin{cases} 0.75, q \geq 0.75\,; \\ \dfrac{8q-3}{8q-2}, 0.5 \leq q \leq 0.75\,; \\ 1-\dfrac{1}{4q}, 0.25 \leq q \leq 0.5\,; \\ 0, q \leq 0.25. \end{cases} \tag{3}$$

Then the correct behavior of the 1st player will be as follows. In the region "left and above" $P_D(q)$ it should not be double in any case. In the region enclosed between$P_D(q)$ and $P_R(q)$, it should be double in the situation "our cube" and not double in the situation "my cube". In the region "right and below" $P_R(q)$ it should be doubled in any case (see Fig. 1).

The correct behavior of the 2nd player is to accept doubling if the point (p, q) lies "to the left and above" and to surrender if "to the right and below" the curve $P_Q(q)$. In more detail, how the inequalities, with respect to expected winnings, give rise to the above-described curves $P_D(q)$, $P_R(q)$, $P_Q(q)$, was considered in (Tuck, 1980).

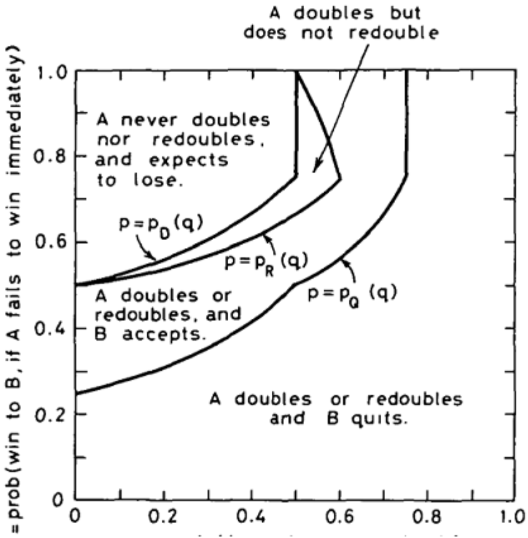

Fig. 1. Diagram of reasonable doubling and accepting doubling zones

Applying the results to backgammon gives the following practical rules. In a one-step game, player 1 should double if he has one checker left in any position or two checkers in positions $2-5$ or better. Player 2 should surrender if player 1 has one checker left in any position or two checkers in positions $1-4$ or better. Otherwise, player 2 should accept the double. For example, in a one-step game in position $2-5$, although player 1's probability of winning is $\frac{19}{36} \approx 0.528$, player 1 should double, and player 2 should accept the double. This "increased aggressiveness" of player 1 is explained only by the fact that his chances of winning, although only slightly exceed $\frac{1}{2}$, are in a "double now or never" situation, i. e. he will not have a chance to wait for a more favorable situation, as in the continuous game model (Keeler, & Spencer, 1975).

3. Single checker model

In (Ross et al., 2007) the "one-checker model" in the tempo game was considered. This means the following. Let the players' checkers not interfere with each other, the players have scores S_A and S_B, respectively, and after the roll, each player substracts the number of points that were rolled from his score. The player who first withdraw all his points wins. Such a model would be adequate if the players had one checker each, and these checkers moved to their homes on a "large board". But, as noted above, the score is the minimum number of points that a player must throw to remove the checkers from the game. In reality, it may be necessary to throw more points. For example, let a player have two checkers in positions 2 and 3 (totally 5 points), and throws $1-6$ (7 points). Although the number of points thrown out formally exceeds the score, it is not enough to throw both checkers off the board.

As a result, one checker remains on the board, which will be withdrawn on the next move. If the player has a large number of checkers in positions 1 and 2, this will also lead to the fact that the real amount of points that will be needed to withdraw all the checkers will exceed the score, since in one throw you can remove at best 2 or 4 checkers (if a double did not fall out/fell out, respectively). For example, let there be 3 checkers left on 1 and 2 on 2 (i. e. the total score is 7). Then it is not important for the player whether large numbers will be present in the throw. For example, let $6-5$ be thrown out, but only 2 checkers can be removed. The next throw $5-4$ can remove 2 more checkers. Then $5-3$ the last checker is removed. So for throwing 5 checkers with a score of 7, throwing $11+9=20$ points turned out to be not enough. The thrown points are obviously used inefficiently. Another factor that leads to inefficient use of discarded points at the stage of discarding checkers is "holes in positions". For example, let a player have 5 checkers in positions 5, 3 and 1: formally the score is 45, but there is a hole in position 4 and, having discarded 4, the player can only move checkers from 5 to 1 (see Fig. 2).

A clear explanation of the inefficiency of using the points on the rolled dices can be the following situation. Suppose you have a sum of money in notes of different denominations. Suppose you periodically need to buy goods for different amounts, but the seller never gives change. If at first you may have no problems finding the required amount without change, then later you have to overpay, and the actual amount of money spent exceeds the total cost of the purchased goods.

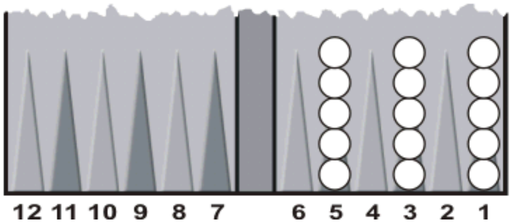

Fig. 2. Possible situation on the final stage of the game

In (Lamford, 2002) it was delivered a set of heuristic rules for calculating the "add-on" to the formal value of a count. Although these rules are not strictly substantiated, simulations show that they are adequate.

Let's return to the one-checker model. Its main advantage is the significant reduction in the state space (from $1.7 \cdot 10^{17}$ to 180) and the fact that, taking into account the above-mentioned corrections, the model adequately estimates the real game in terms of tempo.

Let $p_i(s)$ –the probability to withdraw s points in exactly i moves, $i \geq 0$. Let $p_0(0) = 1, p_i(0) = 0, i \geq 1$. Let $\vec{p}(s)$ be a vector with components $p_i(s)$, starting from zero, $sh(\vec{a})$ be a vector shift operator that adds a zero component to the left, so that що $sh((a_0, \dots, a_n)^T) = (0, a_0, \dots, a_n)^T$. Let the vectors $\vec{p}(0), \dots, \vec{p}(s-1)$ have already been calculated. Then, based on law of total probability and the principle of dynamic programming

$$\vec{p}(s) = sh\left(\sum_{j=3}^{24} \pi_j p(s-j)_+\right), \tag{4}$$

where π_i – the probability of rolling a sum i on the dice. Let $P(S)$ be a vector whose components are equal to $P_i(s) = \sum_{k=1}^{i} p_k(s)$ be the probability that it will take no more than i moves to withdraw s points, $F(k, s)$ be the probability that A wins, provided that A has k, and B has s points, and the move belongs to A. Since at least 3 points are withdrawn in one move, the maximum index of the $\vec{p}(s)$ non-zero component does not exceed $\lceil s/3 \rceil = ceil(s/3)$. Then, according to the law of total probability

$$F(k, s) = \sum_{j=0}^{\left\lceil \frac{s}{3} \right\rceil} p_j(s) P_j(k). \tag{5}$$

To make a decision about doubling or accepting/not accepting the offer to double, it would be useful for the player to know the value of this function. It is clear that directly calculating in the head the sums of the products of the position numbers by the number of checkers is hardly acceptable. For such calculations, there are various alternative methods that can be divided into two main types. One method is similar to the method of products applied to calculating the sample mean in statistics. Let us divide the player's positions into zones of 3 positions in the zone, so that positions $1-3$ are considered the minus first zone, $4-6$ are zero, $7-9$ are the second, etc., and the maximum zone number is 6, although in a tempo game, as a rule, the player's checkers are no further than the 3rd zone. Let k_1 be the sum of the products of the number of checkers in each zone by the zone number (this sum is much easier to calculate than k). In each zone, from the number of checkers that are further away, we subtract the number of checkers that are closer than the "average" position of the zone, and denote the resulting sum for all zones by k_2 Then

$$k = 3k_1 + k_2 + 75.$$

Another method is based on dividing the set of checkers into clusters (if possible, symmetrical) and further calculating the number of checkers in a cluster on the "coordinates of the centers of mass" (Kissane, 1992). The use of these methods will not only considerably simplify the procedure for calculating the current pip count, but you can also use it as a training tool for improving oral calculation skills.

The authors found three different methods for calculating $F(k, s)$ in the literature that are of theoretical interest but are inconvenient or extremely inconvenient in a real game.

In (Keeler, & Spencer, 1975) the following table of probabilities of a player's winning (in percentages) is given, constructed based on Monte Carlo simulations (Tab. 2).

Table 2

Player's winning probability (in percents)

$s-k$ \ k	**0**	**5**	**10**	**15**	**20**	**25**	**30**
20	70	81	91	96	98	99	99
40	65	76	85	92	95	97	98
60	62	72	80	87	92	95	97
80	60	68	77	86	91	94	95
100	58	67	76	83	87	91	94
120	57	67	76	82	86	89	93

A more complete table was given in (Zadeh, & Kobliska, 1977). Here the calculation of elements was based on the principles of dynamic programming. The authors performed simpler calculations using formula (5) and the similar results are obtained.

In article (Lamford, 2002) the following formula was delivered

$$P(k, s) = 50\% + \frac{900 + k + 400L}{k + 7L + 25}\%, \text{ де } L = s - k.$$

This formula is based on the fact that the function $P(k, s)$ is convex in both of its parameters, and the dependence is sought in the form of a hyperboloid, and the parameters were selected by the least squares method, and turns out to be quite accurate (gives an error within 2 % for $k \leq 40$ and less than 1 % for $100 \leq k \leq 150$).

In (Ross et al., 2007) a method for estimating $F(k, s)$ was proposed, based on the central limit theorem for recovery processes, formulated as follows: if independent identically distributed random variables with mean μ and variance σ^2 are summed up until a sum equal to T is reached, then the number of terms has an asymptotically normal distribution with mean $\frac{T}{\mu}$ and variance $\frac{\sigma^2 T}{\mu^3}$.

Thus, to withdraw k points, the following number of throws will be needed:

$$n_k \sim N\left(\frac{k}{\mu}, \frac{\sigma^2 k}{\mu^3}\right),$$

$$F(k,s) = P(n_s - n_k \geq 0) \sim \Phi\left(\frac{s-k+\mu/2}{\sqrt{s+k}} \cdot \frac{\sqrt{\mu}}{\sigma}\right). \quad (6)$$

However, even at the assumption, that a player has phenomenal computational abilities and can calculate the value of $F(k,s)$ using one of the formulas (or study Table 1 and perform linear interpolation), the main question still remains open – to double or not to double?

4. Tips for doubling

Instead of calculating the probabilities of winning, the authors calculated the "advice to double" and "advice to accept doubling" matrices using dynamic programming and presented the data in a form convenient for practical use.

In (Buro, 1999) the concept of "equity" (E) is considered, which means a dimensionless quantity equal to the ratio of the expectation of the win to the current bet, which varies in the range from 1 (if the player is sure to win) to -1 (if he loses). Let us assume that both players behave rationally, that is, they double their bets, fold, or accept the doubling in such a way as to maximize the mathematical expectation of their winning.

Let's take a look at these functions: $E^0_{my}(k,s), E^1_{my}(k,s), E_{my}(k,s), E^0_{our}(k,s), E^1_{our}(k,s), E_{our}(k,s)$.

Here the subscript my /our corresponds to the situation my cube/our cube respectively. The superscript is 0 if the player did not double, and 1 if he doubled in this situation (assuming that in all subsequent situations the players will be rational). Let

$$E_{my}(k,s) = max\left(E^0_{my}(k,s), E^1_{my}(k,s)\right), \quad (7)$$

$$E_{our}(k,s) = max\left(E^0_{our}(k,s), E^1_{our}(k,s)\right) \quad (8)$$

rational choice of the player at this step. For all introduced functions, we define initial values corresponding to the final (trivial) game situations:

$$E(0,j) = 1, \quad (9)$$

$$E(i,0) = -1, \quad (10)$$

$$E(1,j) = E(2,j) = E(3,j) = 1, \quad (11)$$

where $i,j > 0$.

$$E^*_*(i,j) = \begin{pmatrix} & 1 & 1 & \dots & 1 \\ & -1 & 1 & \dots & 1 \\ -1 & & 1 & \dots & 1 \\ -1 & & elements & that & are \\ -1 & & not & calculated i & yet \end{pmatrix}$$

First, let's calculate the elements of the matrices E^*_{my}, then E^*_{our}.

$$E^0_{my}(k,s) = \sum_{i=3}^{24}\sum_{j=3}^{24}\pi_i\,\pi_j\,E_{my}\left((k-i)_+,(s-j)_+\right), \quad (12)$$

$$E^1_{my}(k,s) = min\left(1, -2\sum_{i=3}^{24}\pi_i\,E_{my}\left(s,(k-i)_+\right)\right), \quad (13)$$

$$E^0_{our}(k,s) = -\sum_{j=3}^{24}\pi_j\,E_{our}\left(s,(k-j)_+\right), \quad (14)$$

$$E^1_{our}(k,s) = E^1_{my}(k,s). \quad (15)$$

These formulas may be explained as follows. Consider (12). Let in the situation (k,s) the 1st player not double. Then the first player withdraw some (random, according to his roll) number of points i from his k. The second player has no opportunity to double, he makes a throw and withdraw (random, according to his roll) number j, and so, the score becomes equal to $((s-i)_+,(k-j)_+)$, and the move again passes to the first player. By virtue of the assumption of optimality of all subsequent steps E becomes equal to $E_{my}\left((k-i)_+,(s-j)_+\right)$, averaged over the probabilities of possible throws of both players. If the 1st or 2nd arguments of the function become zero, this means that the 1st or 2nd player wins. If both arguments become zero, this means that the 1st player wins, since it was previously determined that, although formally, after the 1st argument becomes zero, the 2nd player makes a "useless" throw.

Consider (13). Let in the situation (k,s) "my cube" the 1st player doubled. If the 2nd player surrenders, then $E = 1$. If the 2nd player accepts the double, then the 1st player withdraw (random, according to his roll) points, the bet will double, the move and the right to double will pass to the 2nd player. Thus, for the 2nd player the situation will be $(s,(k-i)_+)$, my cube, i. e. E will be equal to $E_{my}\left((k-i)_+,(s-j)_+\right)$. Averaging over all possible throws and taking into account that the values of E are always equal in magnitude and opposite in sign for the opponents, we will obtain for the 1st player $E = -2E_{my}(s,(k-i)_+)$.

Based on the assumption of rationality of the players, the 2nd player will choose the one of the two actions (surrender, accept) that corresponds to the minimum of the two terms (13).

Consider (14). Let in the situation (k,s) and "our cube" the 1st player did not double. Then similarly to (12) the 1st player withdraw some (random, according to his roll) number of points i from his k, while the 2nd player gets into the situation, "our cube". Therefore, for the first player the value of E is taken with the opposite sign.

Consider (15). The consequences of doubling in the situations "my cube" and "our cube" are the same, all considerations are similar to (13), so $E^1_{our}(k,s) = E^1_{my}(k,s)$, so both matrices may be denoted as $E^1(k,s)$.

Based on the calculated matrices E, we construct matrices of advice regarding doubling $Db_{my}\ (k,s)$ та $Db_{our}\ (k,s)$ and accepting doubling $Ac(k,s)$. If in this situation it is advisable to double, then $Db_*\ (k,s) = 1$, if it is advisable not to double, then $Db_*\ (k,s) = 0$. In a similar way, $Ac(k,s) = 1$ or 0, depending on accepting double is advisable or not. Then

$$Db_{my}\ (k,s) = \chi\left(E^1(k,s) \geq E^0_{my}(k,s)\right), \quad (16)$$

$$Db_{our}\ (k,s) = \chi\left(E^1(k,s) \geq E^0_{our}(k,s)\right).$$

The sign of the non-strict inequality in these formulas is explained as follows. Let the 2nd player be hopelessly behind, for example, $k = 10, s = 100$ (in this case they say "there is no theory"). Then, obviously, all values in (16) are equal to each other and equal to 1. Obviously, in this position one should double and this is the end the game, the outcome of which is predetermined. If the inequality were strict, then the game would continue until the very end. When calculating the matrix $Ac(k,s)$ one should take into account that the optimal behavior of the 2nd player has already been taken into account when calculating $E^1(k,s)$, therefore

$$Ac(k,s) = \chi(E^1(k,s) < 1). \quad (17)$$

It turns out that in each row of the matrices $Db_{my}(k,s), Db_{our}(k,s)$ there are zeros at the beginning, and starting from some number - ones. Accordingly, in $Ac(k,s)$ there are first ones, then zeros. Let us construct the vectors $db_{my}(k)$ and $db_{our}(k)$, in which the k -th component is equal to the index of the first one in the row of the matrices $Db_{my}(k,s)$ and $Db_{our}(k,s)$ respectively. Let us also set the k -th component of the vector $ac(k)$ equal to the last one in the k -th row of the matrix $Ac(k,s)$. Then let

$$db^*_{my}(k) = db_{my}(k) - k,$$
$$db^*_{our}(k) = db_{our}(k) - k,$$
$$ac^*(k) = ac(k) - k$$

vectors characterizing the minimum advantage for doubling and the maximum possible lag for accepting it. It turns out that the components of these vectors form non-decreasing numerical sequences that can be compactly presented in the form of a table (see Tab. 3). It makes no sense to put the data for $k < 22$ into the table, since in this case the required number of throws depends strongly on the position

Table 3

Doubling Tips Table

Minimal advantage for doubling at our cube		Minimal advantage for doubling at my cube		Maximum lag for accepting a doubling	
k	$s-k$	k	$s-k$	k	$s-k$
22–27	0	23–27	1	23–26	3
28–32	1	28–33	2	27–31	4
33–38	2	34–38	3	32–37	5
39–45	3	39–44	4	38–44	6
46–53	4	45–51	5	45–51	7
54–62	5	52–60	6	52–59	8
63–72	6	61–69	7	60–68	9
73–82	7	70–79	8	69–78	10
83–93	8	80–89	9	79–88	11
94–105	9	90–100	10	89–98	12
106–117	10	101–111	11	99–110	13
118–129	11	112–124	12	111–122	14
130–143	12	125–136	13	123–134	15
144–157	13	137–150	14	135–148	16
158–170	14	151–163	15	149–161	17
		164–170	16	162–170	18

Let us consider the analysis of another possible scenario of the end of the game using dynamic programming. Let opponents A and B have m and n checkers respectively in position 1. With each move, the players discard 2 or 4 checkers with probabilities $5/6$ and $1/6$ respectively (if a double doesn't rolled or rolled respectively). Let us consider the following equivalent model.

Let $k = \left[\frac{m+1}{2}\right], s = \left[\frac{n+1}{2}\right]$ be conditional points, so,that one or two conditional points can be deducted (if double is rolled on dices or not respectively) for one roll. Since $m, n \leq 15$, then $k, s \leq 8$. Next, similarly to the previous case, the matrices E, Db, Ac are calculated, but now $j = 1$ or 2 with probabilities $5/6$ та $1/6$ respectively.

The results of the calculations can be formulated as easy-to-remember advice: in the "my cube" situation, double, if $k < s$ або $k = s \leq 4$, in the "our cube" situation, if $k < s$ or $k = s \leq 8$. However, the advice to double in situations (7,7) and (8,8) gives a small benefit on average, but increases the risk, since

$$E^1_{our}(8,8) - E^0_{our}(8,8) = 1.6 \cdot 10^{-3}, E^1_{our}(7,7) - E^0_{our}(7,7) = 7.2 \cdot 10^{-3}.$$

So whether to double or not in this situation is a matter of risk appetite. It should also be noted that for these results to apply, it is not necessary for all checkers to be on 1. For example, the position "1 checker on 4, 3 on 3, 6 on 2 and 5 on 1" is not significantly worse than 15 on 1", so all the above considerations can be applied to it.

Discussion and conclusions

This paper provides a comprehensive analysis of the final stage of backgammon game ("races"), a phase characterized by minimal interaction between players' checkers and the outcome depending first of all on the speed of the checkers' movement. During the study:

1. A mathematical model of the game process is proposed to assess the effectiveness of decisions, based on the pip-count indicator, which allows quantitatively assessing the player's advantage.
2. Optimal strategies for using the double cube are created based on probabilistic calculations and dynamic programming principles. It is shown that the critical points of doubling and accepting doubling depend on the probabilities of winning, modeled as Wiener processes.
3. Practical algorithms for evaluating game positions, in particular for one-step and two-step games, have been found. The results confirm the importance of accurate probability calculations in decision-making.
4. Recommendations for doubling down and accepting doubling down in various game situations have been developed, allowing players to maximize their advantage in the game.

The obtained results demonstrated the possibility of increased accuracy of modeling of "races" phases by reducing the state space through the localization of critical moments and the use of the central limit theorem. The practical consequences of the work are the simplification of strategic decision-making in competitive conditions and the creation of a basis for improving training methods for players.

The results of the study confirmed the high importance of using probabilistic and mathematical models for effective evaluation of positions in the "races" phase. The proposed pip-count calculation method and its adaptation to backgammon allow for accurate prediction of the winning chances. Such calculations are especially relevant for sophisticated positions.

Future research could be aimed at expanding the model to adapt to individual playing styles of opponents and take into account psychological aspects of decision-making. Another promising direction is the integration of the developed methods into artificial intelligence to automate learning and analysis of game data.

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АЛГОРИТМІЧНІ ПІДХОДИ ДО МОДЕЛЮВАННЯ ТА АНАЛІЗУ ФІНАЛЬНИХ ПОЗИЦІЙ У ГРІ В НАРДИ

Розглянуто кінцеву фазу гри в нарди, відому як "races" або "гра за швидкістю", що характеризується мінімізацією взаємодії шашок супротивників та зведенням ігрового процесу переважно до оцінки швидкості переміщення власних шашок у "дім" і їхнього подальшого виведення з дошки. Представлений аналіз описує специфіку цієї стадії в різних варіантах нард, зокрема й у класичних коротких нардах (backgammon), із застосуванням математичних моделей і теорії ймовірностей. Запропоновано методи оптимізації стратегій у кінцевій фазі гри, що базуються на комплексному аналізі можливих ігрових траєкторій та розрахунку ймовірностей випадіння значень гральних кубиків. Окрему увагу приділено алгоритмічним підходам до ухвалення рішень, що дають змогу обчислювати найефективніші ігрові дії для досягнення оптимального результату.

Результати дослідження є цінними як для вдосконалення ігрових стратегій гравців-практиків і новачків, так і для розвитку теоретичних підходів до аналізу ігор на основі моделювання й оцінки ефективності прийняття рішень в умовах обмеженої взаємодії між учасниками.

Ключові слова: ***нарди (backgammon), фаза "races", короткі нарди, теорія ймовірностей, математичне моделювання, оптимальні стратегії, алгоритмічний аналіз, дабл-куб (doubling cube), pip-count, ігрова оптимізація.***

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