Taras Shevchenko National University of Kyiv
Faculty of Computer Science and Cybernetics
Department of Complex Systems Modelling

GRADUATION THESIS
for a master's degree
in the specialty 113 „Applied Mathematics"

on the topic:

Adaptive control algorithms using sensitivity methods

of the 2 year student
Rohovchenko Tetiana Mykolaiivna

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Scientific advisor:
Associate Professor, Dr. Sci. in Physics
and Mathematics
Pichkur V. V.

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Head of Department
of Complex Systems Modelling

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Cherniy D. I.

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Introduction

The theory of adaptive systems studies problems in which a priori or current information is not enough to construct a controller. This theory develops methods that are aimed at improving the quality of system during its operation by changing (adaptation) the control algorithm. The problems which control systems developers face to study today increasingly require to use of adaptive methods. Indeed, in a long list of unpleasant properties that characterize the complexity of system (multidimensionality, multiple connection, nonlinearity, non-stationarity, stochasticity, etc.) uncertainty often comes first. Lack of information increases the complexity of the problem. The reasons of uncertainty are different. For example, the lack of „good" mathematical models of control object at the conceptual phase, missing or lack of information about the possible functioning conditions of the system, complexity or „high cost" of computing of systems factors, etc. Among these problems we can find such problems as control of aircraft, continuous technological processes, energy complexes, moving objects, navigation problems, etc. [5].

The methods of stability and stabilization theory are basic while studying the control systems. They are used in the design of systems with a specified quality of system operating, in the construction of automatic control systems, etc. The classical approach to solving the stabilization problem is the method of Lyapunov functions and also algebraic methods for the analysis of stable modes. At the same time, the requirements for the current control systems are related to the existence of phase constraints, to construct a controller under conditions of uncertainty. However, the characteristics of the noise that affects dynamics of the system may not be known in advance. These circumstances imply the creation of new and development of available mathematical tools and algorithms that take into account the specified features. Such approaches include methods of stabilization, practical stabilization for multivalued right-side systems [9,11,17]. Another technique involves the construction of robust stabilizing regulators [2,8,13]. Adaptive methods to the problems of parameter identification and adaptive system controlling were considered in the works [3,4,16]. Approaches to construct adaptive controllers based on the parametric representation of the Lyapunov function were highlighted in [7].

Study object: control systems.

Study subject: adaptive control algorithms.

Research aims: develop and study an adaptive method for solving the stabilization problem of control system with parametrically represented regulator; using the sensitivity function construct an iterative algorithm that implements adjustments to the stabilizing controller parameters at discrete points of time.

To achieve these goals the following **tasks** were imposed:

1) perform a review of research areas related to the adaptive control methods;

2) develop a method of adaptive stabilization which allows to adjust the parameters of parametric stabilizing controller at given discrete points of time; adjustments are provided in such a way as to minimize a *quality criterion* that describes the distance of the system trajectory to the origin;

3) consider a problem of modal control and adaptive stabilization in noisy environment;

4) considering as an example the problem of oscillation of two masses perform computational experiments to test the effectiveness of the suggested method; based on conducted researches and obtained practical results analyze and evaluate advantages and disadvantages of this method.

Research methods: methods of control theory, stability theory, sensitivity theory, optimization methods.

Research background and significance. The issues leading to the choice of research topic are as follows:

1) Necessity to create new and develop existing mathematical tools and algorithms that would take into account the features of control systems modelling and study in conditions of uncertainty.

2) Applying adaptive control methods to ensure the specified dynamic properties of the system in conditions of a priori uncertainty of the object parameters and characteristics of external perturbations.

Research scientific novelty is to develop a new adaptive control method using sensitivity function.

Research theoretical value. The theoretical value of the suggested method lies in its flexibility and versatility. Indeed, the method can be applied to the wide range of problems and modified according to a specific mathematical model in order to obtain better results.

Research practical value. The suggested method of adaptive stabilization using sensitivity function can be applied to the design of systems with a given quality of functioning, while constructing automatic control systems, etc. in conditions of a priori and current uncertainty.

1 Adaptive approach in control problems

1.1 Features of adaptive approach. Statement of the problem of adaptive control systems synthesis

There are many definitions of what should be understood by *training*, *self-learning* and *adaptation* [3]. Unfortunately, even regarding to automatic systems these definitions are quite controversial. By *training* we will mean the process of developing in a system a particular response to external signals through repeated effects on the system and external adjustment. Of course, the system is considered potentially „capable” of training. *Self-learning* differs from training in that it has no external adjustments. Additional information (that reaction is correct) is not reported to the system. By *adaptation* we will understand the process of changing the parameters and structure of the system (and possibly the control effects) based on current information in order to achieve certain (usually optimal) state of the system with initial uncertainty and changing operating conditions.

At the end of XX century evolution of the theory of automatic control systems and its practical application was characterized by intensive development of adaptive control methods. Such methods are used to construct control systems with significant uncertainty about the parameters of the control object and its functioning conditions (environmental characteristics) which are available at the synthesis stage or before the system is operating [1].

There are considered the following control problems in which the dynamic properties of the object can vary widely in an unknown way in advance. The initial (a priori) available information is not enough to construct control systems with optimal (or predetermined) quality parameters. In adaptive control systems the lack of a priori information is filled in process of its functioning based on current data of the object behavior. This data is processed in real time (at the pace of the controlled process) and used to improve the quality of the control system.

Application of adaptation principles allows [1]:

1) to provide system performance in conditions of significant change of dynamic properties of the object;

2) to optimize operation modes of the object while changing its parameters;

3) to minimize the technological requirements for constructing individual units and elements of the system;

4) to unify individual regulators or block of controllers by adapting them to work with different objects of the same type;

5) to shorten the time of design testing, increase the reliability of the system.

The adaptive control process can be considered as a process of interaction of three subsystems [1] such as *object*, *controller of main circuit (own regulator)* which is adjusted and *adaptation block („adapter")*. The last two are combined into *an adaptive controller* that has a two-tier hierarchical structure.

The regulator of main circuit directly forms the control effect $u(t)$ that influences the control object. The law (algorithm) of control in the main circuit depends on some set of controller parameters θ which are adjusted. The adjustment of these parameters is performed at the second level according to some law, i.e. the adaptation algorithm based on available current information and without direct using of parameter values which are a priori unknown. A priori information about parameter values is characterized by defining some set Ξ of theirs possible values. A specific set of object parameters (and environmental characteristics) forms a vector of unknown parameters $\xi \in \Xi$.

Some control objective is considered to be given. The adaptive controller must lead to the objective for all $\xi \in \Xi$. If this condition is satisfied, then the system is called *adaptive in the class* Ξ (or just *adaptive*). The control objective is usually given in form of some *quality functional* (*quality criterion*). Depending on the specific problem the control objective is considered to be achieved if the specified functional either takes an extreme value or its value is within the specified limits.

Apart from the control objective, the adaptation objective is also considered. It is also formalized by some functional and may either coincide with the control objective or differ from it playing the role of some subsidiary objective to accomplish the basic control problem. Such objective can be, for example, the object identification, i.e. to obtain estimates $\hat{\xi}$ for unknown parameters ξ.

Thus the most characteristic feature of adaptation is accumulation and immediate using current information to eliminate uncertainty due to insufficient a priori information in order to optimize the chosen quality criterion. In adaptive control systems information about the object and external affects is collected during system functioning, immediately processed and used to produce control influences. The development of the control algorithm corresponds to the stage of system synthesis, the most responsible stage. At the system synthesis stage the system structure is selected, the parameters of the control algorithm are calculated, the algorithm is checked and refined according to the results of the simulation (computational experiments).

Therefore, let the problem of adaptive control be set and formalized [5]. Consider a control object (control system) influenced by measured perturbations $r = r(t)$, unmeasured perturbations $\varphi = \varphi(t)$ and control effects $u = u(t)$. Output variables $y = y(t)$ are available for observation. The object behavior also depends on a number of unknown parameters. The set of such parameters we denote ξ. Let Ξ be a given set of possible values ξ which define the class of permissible objects and perturbations. Let the control objective be given in the form of quality functional (quality criterion) which determines desired behavior of the system.

We need to define (synthesize) an algorithm for calculating control effects that uses measurable values, does not depend on $\xi \in \Xi$ and leads to the chosen control objective for all $\xi \in \Xi$ [5].

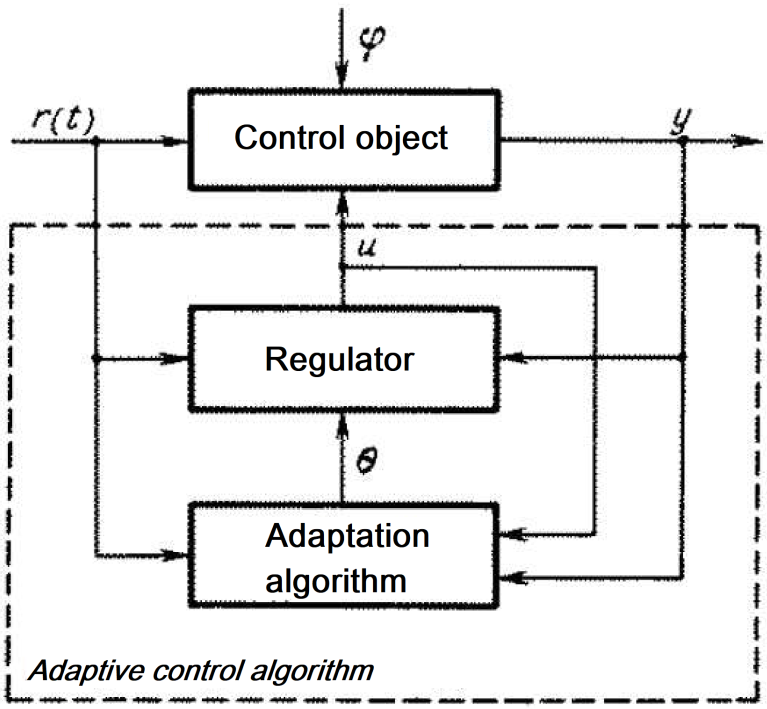

Figure 1: Adaptive control algorithm

The vector of unknown parameters ξ usually consists of the equations coefficients included in the mathematical description of the object as well as the coefficients that determine the change in external influences (state of the environment). We consider the vector ξ quasi-stationary, i.e. it is constant or slowly changing (slower than the dynamic processes in the object and external influences). The set (class) Ξ characterizes the available a priori information about the object. The less information about the object parameters is available to the system developer, the bigger set Ξ is.

The synthesis process of an adaptive controller can be divided into the following steps [1]:

Step 1. Choosing the „perfect" control law. There exists a control law that provides a fundamental opportunity to achieve the control objective. The vector of parameters ξ is considered to be known. The resulting control law cannot be directly implemented as it depends (in general case) on the unknown parameters of the object. In this sense it can be called the perfect control law. For example, such law may be based on the solution of the optimal control problem. But not only optimal (in general sense) control laws can be considered to be „perfect", since it is supposed that their synthesis assumes the existence of sufficiently accurate information about the parameters of the object and environment.

Step 2. Choosing the regulator parameters which are adjusted and adaptation objective. The „perfect" control law depends on unknown parameters which are replaced by parameters which are adjusted. As a result, there is a control algorithm that no longer contains unknown parameters, so it can be implemented by a controller. There are two approaches to the synthesis of adaptive regulators. In the direct approach the parameters of adjustment are directly the coefficients of the control law. The identification approach (indirect approach) uses estimation of the values required for the synthesis of the controller of unknown object parameters and characteristics of external influences. Then the procedure of combined synthesis is performed. The parameters estimates are used to calculate the coefficients of the control law. When adjustment parameters are chosen, the adaptation objective (some auxiliary objective condition) is set. In the direct approach the adaptation objective coincides with the control objective. In the identification approach the adaptation objective is usually to ensure that the unknown parameters estimates coincide or are close enough to their true values.

Step 3. Choosing of adaptation algorithm. Adaptation algorithms are usually recurrent procedures that belong to class of methods of sequential improvements.

Step 4. Study the efficiency of the adaptive system. The final step in the adaptive controller synthesis is to study the performance of the system taking into account the nature of perturbations, external influences, restrictions to the state of the object and other factors that were not taken into account at the beginning.

1.2 Speed gradient method (gradient descent)

The initial data for the synthesis of adaptation algorithm is the adjustment object equation and the control objective. Assume that the object is given by the equation [5]:

$$\dot{x} = F(x, \theta, t),$$

where $x \in R^n$ is the object state vector, $\theta \in R^m$ is the control vector (of adjustment parameters), the vector function F is defined for all $x \in R^n, \theta \in R^m, t \geq 0$, piecewise continuous with respect to the variable t and continuously differentiable with respect to x, θ.

Let the control objective be given in the form of local objective functional (quality criterion) $Q_t = Q(x(t), t))$. To construct control algorithm we calculate a scalar function $\dot{Q}_t = \omega(x, \theta, t)$ which describes the rate of change Q_t according the object equation $\dot{x} = F(x, \theta, t)$:

$$\omega(x, \theta, t) = \frac{\partial Q(x, t)}{\partial t} + (\nabla_x Q(x, t))^* F(x, \theta, t)$$

Find the gradient of $\omega(x, \theta, t)$ with respect to θ

$$\nabla_\theta \omega(x, \theta, t) = \left(\frac{\partial \omega}{\partial \theta}\right)^* = \left(\frac{\partial F}{\partial \theta}\right)^* \nabla_x Q(x, t)$$

The algorithm of $\theta(t)$ changing is given by differential equation

$$\frac{d\theta}{dt} = -\Gamma \nabla_\theta \omega(x, \theta, t),$$

where $\Gamma = \Gamma^T > 0$ is a symmetric, positive definite matrix.

The constructed algorithm is called *speed gradient method (gradient descent)*, since it changes $\theta(t)$ in proportion to the gradient of the rate of change of Q_t. The background of the method can be explained as follows: in order to achieve the control objective, it is desirable to change $\theta(t)$ in the direction of minimizing of Q_t. However, Q_t does not depend on $\theta(t)$ and finding such direction is quite difficult problem. Alternatively, we can try to minimize $\dot{Q}_t$ in order to satisfy the inequality $\dot{Q}_t < 0$. The function $\dot{Q}_t = \omega(x, \theta, t)$ clearly depends on $\theta(t)$. It allows us to construct this algorithm.

1.3 Adaptive control method for linear systems based on the second Lyapunov method

There are two basic approaches to system adaptation that are oriented toward observation or a reference model. In adaptive systems with a reference model the desired object dynamics is given by a model that is a sample or reference for the object. Therefore it it called a reference model. The model is a stationary dynamic system with known parameters, the input data is the same as for the control object.

Consider motion equation of the object and reference model [6]:

$$\frac{dy^{(p)}}{dt} = A^{(p)}y^{(p)} + B^{(p)}v, \tag{1.3.1}$$

$$\frac{dy^{(m)}}{dt} = A^{(m)}y^{(m)} + B^{(m)}u, \tag{1.3.2}$$

where $u(t)$ is a test signal, $A^{(m)}$ is an asymptotically stable matrix, $A^{(p)}$, $A^{(m)}$,$B^{(p)}$, $B^{(m)}$ are known $n \times n$- and $n \times r$-dimensional matrices respectively, $y^{(p)}(t)$ and $y^{(m)}(t)$ are n-dimensional state vectors of the object and reference model respectively. The r-dimensional control vector is chosen in the form

$$v = Q(t)\left[F(t)y^{(p)}(t) + u(t)\right],$$

where $Q(t)$ is $r \times r$-dimensional unknown direct control matrix, $F(t)$ is $r \times n$-dimensional unknown feedback matrix. The problem is to find such matrices $Q(t)$ and $F(t)$ that the following condition is satisfied:

$$\lim_{x\to\infty}\left\|y^{(p)}(t) - y^{(m)}(t)\right\| = 0$$

Then the system (1.3.1) looks like

$$\frac{dy^{(p)}}{dt} = \left[A^{(p)} + B^{(p)}QF\right]y^{(p)} + \left[B^{(p)}Q\right]u$$

Assume there exists such Q and F that

$$B^{(p)}Q^* = B^{(m)}, \; A^{(p)} + B^{(p)}Q^*F^* = A^{(m)} \tag{1.3.3}$$

These conditions are called *consistency conditions.* Write the system of ordinary differential equations for the error vector $e = y^{(m)} - y^{(p)}$

$$\frac{de}{dt} = A^{(m)}e + \left[A^{(m)} - A^{(p)} - B^{(p)}QF\right] y^{(p)} + + \left[B^{(m)} - B^{(p)}Q\right] u \qquad (1.3.4)$$

Denote

$$\Phi(t) = A^{(m)} - A^{(p)} - B^{(p)}QF,\ \Psi(t) = B^{(m)} - B^{(p)}Q$$

and consider the Lyapunov function

$$V = \langle Pe, e\rangle + \frac{1}{2}tr(\Phi^*(t)\Phi(t) + \Psi^*(t)\Psi(t))$$

The matrix P is defined from the Lyapunov matrix differential equation

$$A^{(m)^*}P + PA^{(m)} = -Q$$

Then

$$\frac{dV}{dt} = -\langle Qe, e\rangle + \left\langle P\Phi(t)y^{(p)}, e\right\rangle + \langle P\Psi(t)u, e\rangle +$$

$$+tr\left[\frac{d\Phi^*}{dt}\Phi + \frac{d\Psi^*}{dt}\Psi\right] = -\langle Qe, e\rangle + tr\left[y^{(p)}e^*P\Phi(t) + ue^*P\Psi(t) + \frac{d\Phi^*}{dt}\Phi + \frac{d\Psi^*}{dt}\Psi\right].$$

To satisfy the condition

$$\frac{dV}{dt}_{(4)} = -\langle Qe, e\rangle$$

we put

$$\frac{d\Phi^*}{dt} = -y^{(p)}e^*P,\ \frac{d\Psi^*}{dt} = -ue^*P$$

In (1.3.1), (1.3.2) consider the case

$$A^{(p)} = A^{(m)} = A,\ B^{(p)} = B^{(m)} = B$$

Thus we have the systems of differential equations

$$\frac{dy^{(p)}}{dt} = Ay^{(p)} + Bv, \frac{dy^{(m)}}{dt} = Ay^{(m)} + Bu, \qquad (1.3.5)$$

where $v = Q(t)\left[F(t)y^{(p)} + u\right]$. The the system (1.3.5) can be rewritten as:

$$\frac{dy^{(p)}}{dt} = \left[A + BQ(t)F(t)\right]y^{(p)} + \left[BQ(t)\right]u(t).$$

We assume that A is asymptotically stable matrix. Write the differential equation for the error vector:

$$\frac{de}{dt} = Ae - BQFy^{(p)} - BQu. \tag{1.3.6}$$

Denote $\Phi(t) = Q(t)F(t),\ \Psi(t) = Q(t)$. Construct the Lyapunov function

$$V = \langle Pe, e\rangle + tr\left[\Phi^*(t)\Phi(t) + \Psi^*(t)\Psi(t)\right].$$

Compute the total derivative of V with respect to t on the system trajectories of ordinary differential equations (1.3.6):

$$\frac{dV}{dt} = -\langle Pe, e\rangle - \left\langle P\Phi(t)y^{(p)}, e\right\rangle - \langle P\Psi(t)u, e\rangle +$$

$$+tr\left[\frac{d\Phi^*}{dt}\Phi + \frac{d\Psi^*}{dt}\Psi\right].$$

Put

$$\frac{d\Phi^*}{dt} = y^{(p)}e^*PB,\ \frac{d\Psi^*}{dt} = ue^*PB. \tag{1.3.7}$$

From (1.3.7) we obtain:

$$\frac{dQ}{dt} = B^*Peu^*,\ \frac{d}{dt}\left(Q(t)F(t)\right) = B^*Pey^{(p)*}.$$

Thus

$$\frac{dQ}{dt}F + Q(t)\frac{dF}{dt} = B^*Pey^{(p)*},$$

$$\frac{dF}{dt} = Q^{-1}(t)\left[B^*Pey^{(p)*} - \frac{dQ}{dt}F\right] =$$

$$= Q^{-1}(t)\left[B^*Pey^{(p)*} - B^*Peu^*F\right] = Q^{-1}(t)B^*Pe\left[y^{(p)*} - u^*F\right].$$

Finally we get

$$\frac{dQ}{dt} = B^*Peu^*, \frac{dF}{dt} = Q^{-1}(t)B^*Pe\left[y^{(p)*} - u^*F\right]. \tag{1.3.8}$$

We assume that in (1.3.8) matrix $Q(t)$ is nondegenerate. We obtained in explicit form the matrix differential equations for $Q(t)$ and $F(t)$ in order to adjust the object trajectories (1.3.1) to the reference model trajectories (1.3.2).

1.4 Adaptive identification method

Assume the object motion equation is given in the form [6]:

$$\frac{dy^{(p)}}{dt} = A^{(p)}y^{(p)} + B^{(p)}u, \tag{1.4.1}$$

where $y^{(p)}(t)$ is n-dimensional state vector, u is r-dimensional control vector, $A^{(p)}$ and $B^{(p)}$ are $n \times n$- and $n \times r$-dimensional unknown matrices respectively which are slowly changing with respect to time. Consider the reference model

$$\frac{dy^{(m)}}{dt} = A^{(m)}y^{(m)} + B^{(m)}u. \tag{1.4.2}$$

The variables $y^{(m)}$, $A^{(m)}$, u, $B^{(m)}$ have the same dimensions as the corresponding variables in the system of differential equations (1.4.1). The matrices $A^{(m)}$, $B^{(m)}$ are chosen in such a way that the reference model has the desired dynamic characteristics and the matrix $A^{(m)}$ is asymptotically stable. The problem is to determine by the test control signal $u(t)$ and the corresponding vector of observations $y^{(p)}$ the unknown matrices $A^{(p)}$ and $B^{(p)}$ of the object. To solve this problem we use the second Lyapunov method. Define the error vector

$$e = y^{(m)} - y^{(p)},$$

which satisfies the differential equation

$$\begin{aligned}\frac{de}{dt} &= A^{(m)}e + \left(A^{(m)} - A^{(p)}\right) y^{(p)} + \left(B^{(m)} - B^{(p)}\right) u = \\ &= A^{(m)}e + A(t)y^{(p)} + B(t)u.\end{aligned} \tag{1.4.3}$$

Suppose that all elements of $A^{(p)}$ and $B^{(p)}$ can be separately matched to the corresponding elements of $A^{(m)}$ and $B^{(m)}$. Define

$$A(t) = A^{(m)} - A^{(p)} = \left(a_{ij}(t)\right)_{i,j=1}^{n},$$

$$B(t) = B^{(m)} - B^{(p)} = \left(b_{ij}(t)\right)_{i=1,j=1}^{n,r}.$$

Take the Lyapunov function in the form of

$$V = \langle Pe, e\rangle + \sum_{i,j=1}^{n} \frac{1}{\alpha_{ij}} a_{ij}^2 + \sum_{i=1}^{n}\sum_{j=1}^{r} \frac{1}{\beta_{ij}} b_{ij}^2$$

To obtain asymptotic stability in the space of errors vectors we choose the unknown parameters of the system of ordinary differential equations (1.4.3) in such a way that the total time derivative of the function $V(e,t)$ according to the system (1.4.3) is negative definite, i.e.

$$\left(\frac{dV}{dt}\right)_{(1.4.3)} = -\langle Qe, e\rangle, \tag{1.4.4}$$

where $A^{(m)*}P + PA^{(m)} = -Q$, Q is $n\times n$-dimensional positive definite, symmetric matrix. Thus we obtain

$$\frac{dV}{dt} = \left\langle P\frac{de}{dt}, e\right\rangle + \left\langle Pe, \frac{de}{dt}\right\rangle + \sum_{i,j=1}^{n} \frac{2}{\alpha_{ij}} a_{ij} \frac{da_{ij}}{dt} +$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{r} \frac{2}{\beta_{ij}} b_{ij} \frac{db_{ij}}{dt} = -\langle Qe, e\rangle +$$

$$+y^{(p)*}A^*(t)Pe + u^*B^*(t)Pe + e^*PA(t)y^{(p)}(t) +$$

$$+e^*PB(t)u(t) + \sum_{i,j=1}^{n} \frac{2}{\alpha_{ij}} a_{ij} \frac{da_{ij}}{dt} + \sum_{i=1}^{n}\sum_{j=1}^{r} \frac{2}{\beta_{ij}} b_{ij} \frac{db_{ij}}{dt}.$$

In order to satisfy (1.4.4) we assume

$$e^*PA(t)y^{(p)}(t) + e^*PB(t)u(t) +$$

$$+\sum_{i,j=1}^{n} \frac{a_{ij}}{\alpha_{ij}} \frac{da_{ij}}{dt} + \sum_{i=1}^{n}\sum_{j=1}^{r} \frac{b_{ij}}{\beta_{ij}} \frac{db_{ij}}{dt} = 0.$$

Here is taken into account that

$$y^{(p)*}A^*(t)Pe = e^*PA(t)y^{(p)},\ u^*B^*(t)Pe = e^*PB(t)u.$$

We rewrite the obtained equations as follows

$$\sum_{k=1}^{n}\sum_{i,j=1}^{n} e_k p_{ki} a_{ij}(t) y_j^{(p)} + \sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{r} e_k p_{ki} b_{ij}(t) u_j(t) +$$

$$+\sum_{i,j=1}^{n} \frac{a_{ij}}{\alpha_{ij}} \frac{da_{ij}}{dt} + \sum_{i=1}^{n}\sum_{j=1}^{r} \frac{b_{ij}}{\beta_{ij}} \frac{db_{ij}}{dt} = 0,$$

$$\sum_{i,j=1}^{n} a_{ij}\left[\frac{1}{\alpha_{ij}}\frac{da_{ij}}{dt}+\sum_{k=1}^{n} e_k p_{ki} y_j^{(p)}\right]+$$
$$+\sum_{i=1}^{n}\sum_{j=1}^{r} b_{ij}\left[\frac{1}{\beta_{ij}}\frac{db_{ij}}{dt}+\sum_{k=1}^{n} e_k p_{ki} u_j(t)\right]=0.$$

Choose the elements of the matrices $A(t)$ and $B(t)$ according to the differential equations

$$\frac{da_{ij}}{dt}=-\alpha_{ij}\left(\sum_{k=1}^{n} e_k p_{ki}\right) y_j^{(p)},\ i,j=1\dots n, \tag{1.4.5}$$

$$\frac{db_{ij}}{dt}=-\beta_{ij}\left(\sum_{k=1}^{n} e_k p_{ki}\right) u_j,\ i=1\dots n, j=1\dots r. \tag{1.4.6}$$

If in the adaptive adjustment algorithms (1.4.5), (1.4.6) there is significant attenuation, then the second algorithm which can be obtained using the Lyapunov function is applied to adjust the object parameters based on the observation signals of the model parameters.

$$\begin{aligned} V=\langle Pe,e\rangle+\sum_{i,j=1}^{n}\frac{1}{\alpha_{ij}}\left[a_{ij}(t)+\alpha_{ij}\gamma_{ij}\left(\sum_{k=1}^{n} e_k p_{ki}\right) y_j^{(p)}\right]^2+\\ +\sum_{i=1}^{n}\sum_{j=1}^{r}\frac{1}{\beta_{ij}}\left[b_{ij}(t)+\beta_{ij}\delta_{ij}\left(\sum_{k=1}^{n} e_k p_{ki}\right) u_j\right]^2. \end{aligned} \tag{1.4.7}$$

The object parameters are chosen such that

$$\begin{aligned} \frac{dV}{dt}=-\langle Qe,e\rangle-2\sum_{i,j=1}^{n}\alpha_{ij}\gamma_{ij}\left(\sum_{k=1}^{n} e_k p_{ki} y_j^{(p)}\right)^2-\\ -\sum_{i=1}^{n}\sum_{j=1}^{r}\beta_{ij}\delta_{ij}\left(\sum_{k=1}^{n} e_k p_{ki} u_j(t)\right)^2=-\langle Qe,e\rangle-\\ -2\sum_{i,j=1}^{n}\alpha_{ij}\gamma_{ij}\mu_{ij}^{A^2}-\sum_{i=1}^{n}\sum_{j=1}^{r}\beta_{ij}\delta_{ij}\left(\mu_{ij}^{B}\right)^2. \end{aligned} \tag{1.4.8}$$

Here $\mu_{ij}^A=\sum_{k=1}^{n} e_k p_{ki} y_j^{(p)}$, $\mu_{ij}^B=\sum_{k=1}^{n} e_k p_{ki} u_j$. Based on (1.4.7), (1.4.8) we obtain adaptive adjustment algorithm of the elements of matrices $A(t)$ and $B(t)$:

$$\frac{dV}{dt}=-\langle Qe,e\rangle+2\left\langle PA(t)y^{(p)},e\right\rangle+2\langle PB(t)u(t),e\rangle+$$

$$+\sum_{i,j=1}^{n}\frac{2}{\alpha_{ij}}\left[\frac{da_{ij}}{dt}+\alpha_{ij}\gamma_{ij}\frac{d\mu_{ij}^{(A)}}{dt}\right]\left[a_{ij}(t)+\alpha_{ij}\gamma_{ij}mu_{ij}^{(A)}(t)\right]+$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{r}\frac{2}{\beta_{ij}}\left[\frac{db_{ij}}{dt}+\beta_{ij}\delta_{ij}\frac{d\mu_{ij}^{(B)}}{dt}\right]\left[b_{ij}(t)+\beta_{ij}\delta_{ij}mu_{ij}^{(B)}(t)\right]=$$

$$=-\langle Qe,e\rangle-2\sum_{i,j=1}^{n}\alpha_{ij}\gamma_{ij}\mu_{ij}^{(A)^2}-\sum_{i=1}^{n}\sum_{j=1}^{r}\beta_{ij}\delta_{ij}\left(\mu_{ij}^{(B)}\right)^2.$$

From the equation above we have

$$\sum_{k,i,j=1}^{n}e_kp_{ki}a_{ij}y_j^{(p)}+\sum_{i,j=1}^{n}\frac{1}{\alpha_{ij}}\left[\frac{da_{ij}}{dt}+\alpha_{ij}\gamma_{ij}\frac{d\mu_{ij}^{(A)}(t)}{dt}\right]a_{ij}=0,$$

$$\sum_{k,i=1}^{n}\sum_{j=1}^{r}e_kp_{ki}b_{ij}u_j+\sum_{i=1}^{n}\sum_{j=1}^{r}\frac{1}{\beta_{ij}}\left[\frac{db_{ij}}{dt}+\beta_{ij}\delta_{ij}\frac{d\mu_{ij}^{(B)}(t)}{dt}\right]b_{ij}=0,$$

$$\sum_{i,j=1}^{n}\frac{1}{\alpha_{ij}}\left[\frac{da_{ij}}{dt}+\alpha_{ij}\gamma_{ij}\frac{d\mu_{ij}^{(A)}(t)}{dt}\right]\alpha_{ij}\gamma_{ij}\mu_{ij}^{(A)}(t)=$$

$$=-\sum_{i,j=1}^{n}\alpha_{ij}\gamma_{ij}\mu_{ij}^{(A)}(t),$$

$$\sum_{i=1}^{n}\sum_{j=1}^{r}\frac{1}{\beta_{ij}}\left[\frac{db_{ij}}{dt}+\beta_{ij}\delta_{ij}\frac{d\mu_{ij}^{(B)}(t)}{dt}\right]\beta_{ij}\delta_{ij}\mu_{ij}^{(B)}(t)=$$

$$=-\sum_{i=1}^{n}\sum_{j=1}^{r}\beta_{ij}\delta_{ij}\mu_{ij}^{(B)}(t).$$

Based on previous system adaptive adjustment algorithms are defined:

$$\frac{da_{ij}}{dt}=-\alpha ij\left(\mu_{ij}^{(A)}+\gamma_{ij}\frac{d\mu_{ij}^{(A)}}{dt}\right),$$

$$\frac{db_{ij}}{dt}=-\beta ij\left(\mu_{ij}^{(B)}+\delta_{ij}\frac{d\mu_{ij}^{(B)}}{dt}\right),$$

$$\mu_{ij}^{(A)}=\left(\sum_{k=1}^{n}e_kp_{ki}\right)y_j^{(p)},\ \mu_{ij}^{(B)}=\left(\sum_{k=1}^{n}e_kp_{ki}\right)u_j.$$

Here α_{ij}, β_{ij}, γ_{ij}, δ_{ij} are parameters of algorithm convergence adjustment.

2 Adaptive stabilization of control systems

2.1 Adaptive stabilization based on the second Lyapunov method

The system

$$\dot{x} = f(x) + F(x)\theta + g(x)u \tag{2.1.1}$$

called *absolutely adaptive stabilizing* [7], if there exists a function $\alpha(x, \hat{\theta})$ defined on $(\mathbb{R}^n \backslash \{0\}) \times \mathbb{R}^p$ such that $\alpha(0, \hat{\theta}) \equiv 0$, a smooth function $\tau(x, \hat{\theta})$ and also $p \times p$-dimensional symmetric, positive definite matrix Γ such that

$$u = \alpha(x, \hat{\theta}) \tag{2.1.2}$$

$$\dot{\hat{\theta}} = \Gamma\tau(x, \hat{\theta}). \tag{2.1.3}$$

The control in (2.1.2), (2.1.3) guarantees that the solution $(x, \hat{\theta}(t))$ is absolutely bounded and $x(t) \to 0$ while $t \to \infty$ $\forall\theta \in \mathbb{R}^p$. The approach is to replace the adaptive stabilization problem of the original system (2.1.1) with the problem of non-adaptive stabilization of the modified system.

Definition 1. A smooth function $V_a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ which is positive definite and unbounded in x $\forall\theta$ is called Lyapunov function for the adaptive control system (2.1.1), if there exist a positive definite, symmetric matrix $\Gamma \in \mathbb{R}^{p\times p}$ such that $\forall\theta \in \mathbb{R}^p, V_a(x, \theta)$ is Lyapunov function of control system

$$\dot{x} = f(x) + F(x)\left(\theta + \Gamma\left(\frac{\partial V_a}{\partial\theta}\right)^*\right) + g(x)u, \tag{2.1.4}$$

and V_a satisfies

$$\inf_{u\in\mathbb{R}}\left\{\left(\frac{\partial V_a}{\partial x}\right)^*\left[f(x) + F(x)\left(\theta + \Gamma\left(\frac{\partial V_a}{\partial\theta}\right)^*\right) + g(x)u\right]\right\} < 0. \tag{2.1.5}$$

Theorem 1. *The following statements are equivalent:*

1. There exists a triple (α, V_a, Γ) such that $\alpha(x, \theta)$ absolutely stabilizes (2.1.4) in $x = 0$ $\forall\theta \in \mathbb{R}^p$ with $V_a(x, \theta)$ which is the Lyapunov function of this system.

2. There exists Lyapunov function $V_a(x, \theta)$ for the adaptive control system (2.1.1). In addition, if such Lyapunov function exists, then (2.1.1) is the abso-

lutely adaptive stabilizing system.

Remark. The question remains. If there exists Lyapunov function for the adaptive control system (2.1.1), is this system absolutely asymptotically stabilizing $\forall\theta$ and vice versa? In some literature there is found Lyapunov function construction in the case of not only an absolutely stabilizing but also an absolutely asymptotically stabilizing system.

2.2 Properties of sensitivity function

The basic method of research in sensitivity theory is using the so-called *sensitivity function* [15]. Let $\alpha_1, \dots, \alpha_m$ be sets of parameters which together form the set α. Therewith state variables $Y_1, \dots, Y_n$ and quality criterions $J_1, \dots, J_s$ are one-to-one functions of parameters $\alpha_1, \dots, \alpha_m$, i.e.

$$Y_i(t, \alpha) = Y_i(t, \alpha_1, \dots, \alpha_m)\ \forall i,$$

$$J_j(\alpha) = J_j(\alpha_1, \dots, \alpha_m)\ \forall j,$$

Definition 2. Partial derivatives

$$\frac{\partial Y_i(t, \alpha)}{\partial \alpha_k}, \frac{\partial J_j(\alpha)}{\partial \alpha_k} \tag{2.2.1}$$

are called *sensitivity functions of the first order* of Y_i, J_j with respect to corresponding parameters [15].

Definition 3. Partial derivatives of k-th order of Y_i, J_j with respect to $\alpha_1, \dots, \alpha_m$

$$\frac{\partial^k Y_i(t, \alpha)}{\partial \alpha_1^{k_1} \dots \partial \alpha_m^{k_m}}, \frac{\partial^k J_j(\alpha)}{\partial \alpha_1^{k_1} \dots \partial \alpha_m^{k_m}}, \ k_1 + \dots + k_m = k \tag{2.2.2}$$

are called *sensitivity functions of the k-th order* with respect to corresponding combinations of parameters [15].

It is obvious that the sensitivity functions of the state variables $Y_i(t, \alpha)$ depend on t and parameters $\alpha_1, \dots, \alpha_m$ and the sensitivity functions of the quality criterions depend only on the parameters $\alpha_1, \dots, \alpha_m$. It is supposed that derivatives (2.2.1), (2.2.2) exist. Definition (2) also assumes that the sensitivity functions are independent of the order of differentiation with respect to relevant parameters. For this it is sufficient to assume that in some neighborhood of the point

$\alpha(\alpha_1, \ldots, \alpha_m)$ of the space of parameters there exist all possible derivatives of $(k-1)$-th order and also mixed partial derivatives of k-th order. Moreover all these derivatives are continuous.

The sensitivity functions of different orders are solutions of special equations that can be directly obtained from a known parametric system model. Such equations are called *sensitivity equations.* The set of the initial mathematical model and some auxiliary relations that determine the sensitivity functions is commonly called the *system sensitivity model.*

Consider the dynamical system

$$\frac{dx(t)}{dt} = f(x(t), p, t), \; t \in [t_0, T], \tag{2.2.3}$$

with the Cauchy condition

$$x(t_0) = x_0(p). \tag{2.2.4}$$

Here $x = (x_1, x_2, \ldots, x_n)^*$ is the state vector, $f(x, p, t) = (f_1(x, p, t), ..., f_n(x, p, t))^*$ is continuously differentiable function with respect to x and p, $p = (p_1, p_2, \ldots, p_m)$ is parameter, $x_0(p)$ is continuously differentiable function from $\mathbb{R}^m$ to $\mathbb{R}^n$.

Definition 4. The *sensitivity function* of the system (2.2.3) at the point $p = \hat{p}$ is a function which according to (2.2.1), (2.2.2) is given by the relation $U(t) = \frac{\partial x(t,\hat{p})}{\partial p}$.

Here $x(t) = x(t, \hat{p})$ is the solution of the system (2.2.3) while $p = \hat{p}$. From (2.2.3), (2.2.4) we obtain an equivalent integral equation

$$x\,(t, p) = x_0(p) + \int_{t_0}^{t} f\,(x(s), p, s)\, ds.$$

Differentiate the last equality with respect to the variable p and substitute $p = \hat{p}$. We get an integral relation for finding the sensitivity function

$$U\,(t) = \frac{\partial x_0(p)}{\partial p} + \int_{t_0}^{t} \left(\frac{\partial f\,(x(s), \hat{p}, s)}{\partial x} U\,(s) + \frac{\partial f\,(x(s), \hat{p}, s)}{\partial p} \right) ds.$$

Differentiate the last equality. Thus sensitivity matrix satisfies the matrix differential equation (sensitivity equation)

$$\begin{aligned} \frac{dU\,(t)}{dt} &= \frac{\partial f\,(x(t), \hat{p}, t)}{\partial x} U\,(t) + \frac{\partial f\,(x(s), \hat{p}, t)}{\partial p}. \\ U\,(t_0) &= \frac{\partial x_0(p)}{\partial p}. \end{aligned} \tag{2.2.5}$$

2.3 Adaptive stabilization method using sensitivity function

Let the control system be given in the form [9, 10, 14]

$$\frac{dx}{dt} = f(x, u(x, p, t), t),\ t \geq t_0. \tag{2.3.1}$$

Here $x = (x_1, x_2, ..., x_n)^*$ is the state vector (of phase coordinates), $u = (u_1, u_2, ..., u_m)^*$ is the control vector (of control affects), $u(0, p) = 0$, $p = (p_1, p_2, ..., p_r)^*$ is the vector of parameters, $f(x, u, t)$ is vector function of right hand sides of system (2.3.1) which is continuously differentiable with respect to the variables x, u and continuous with respect to time t on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^1$, $f(0, 0, t) = 0$.

Suppose the control $u = u(x, p, t)$ solves the stabilization problem of system (2.3.1) for all $p \in P$, where $P \subset \mathbb{R}^r$ is an open set. It should be noted that there exists a number of factors that significantly affect the behavior of system, choosing of parameter $p \in P$ and which may be unknown before the stabilization algorithm start. For example, the initial conditions (or initial data) may be unknown; there are noises affecting the right side of system and the nature of these noises cannot be determined in advance.

Thus we suggest to make adaptive adjustments of the parameter p depending on the current system state. We will implement change in the value of parameter p at discrete points of time $t_1 < t_2 < ... < t_i < ....$ Suppose that the parameter p takes constant values $p = p^{(i)}$, $i = 0, 1, ...$ on intervals (t_i, t_{i+1}). At the time points t corresponding to different intervals the values of parameter p can be different.

By $x(t, p)$ we denote the solution of system (2.3.1), $x(t_0) = x_0$ while $u = u(x, p, t)$ is the control function. According to the theorem on *continuous differentiation of solutions of the system of differential equations by parameters* the function $x(t, p)$ is continuously differentiable with respect to the variable p under the condition the control function $u = u(x, p, t)$ is continuously differentiable with respect to the variable p. At the time points $t = t_i$ the realization $x(t, p)$ is known as the solution of (2.3.1) which corresponds to the parameter value $p = p^{(i-1)}$. Based on this realization we look for such parameter value $p^{(i)}$ that minimizes the quality criterion in the form below

$$I_i\left(p\right) = \left\|x\left(t_i, p\right)\right\|^2 \to \min_{p \in P}. \tag{2.3.2}$$

Apply the approaches that are characteristic for the second-order optimization methods and also use the properties of the sensitivity matrix. We perturbate the parameter p at the point $p^{(i-1)}$ by h and write the solution of system (2.3.1) taking into account the linear approximation

$$x\left(t_i, p^{(i-1)} + h\right) = x\left(t_i, p^{(i-1)}\right) + W(t_i, p^{(i-1)})h + r_i(h). \tag{2.3.3}$$

Here $W(t) = W(t, p^{(i-1)})$ is the sensitivity matrix of system (2.3.1) which corresponds to the solution $x(t,p)$ while $p = p^{(i-1)}$, h belongs to $\mathbb{R}^r$, $r_i(h)$ is infinitesimal value with respect to h while $h \to 0$. The sensitivity matrix satisfies the matrix differential equation

$$\frac{dW(t)}{dt} = F(t,p)W(t) + g(t,p),\ W(t_i) = 0,\ t \in [t_i, t_{i+1}], \tag{2.3.4}$$

where $F(t,p) = \frac{\partial f(t,p)}{\partial x} + \frac{\partial f(t,p)}{\partial u}\frac{\partial u(t,p)}{\partial p}$, $g(t,p) = \frac{\partial f(t,p)}{\partial u}\frac{\partial u(t,p)}{\partial p}$, $p = p^{(i-1)}$. Here $f(t,p) = f(x(t,p), u(x(t,p),p,t)$, $u(t,p) = u(x(t,p),p,t)$.

Denote $x^{(i)} = x(t_i, p^{(i-1)})$. Omitting in (2.3.3) the function $r_i(h)$ we substitute linear approximation to $x\left(t_i, p^{(i-1)} + h\right)$ into the functional (2.3.2). We obtain the quadratic functional of the form

$$J(h) = \left\langle x^{(i)} + W(t_i)h, x^{(i)} + W(t_i)h \right\rangle =$$

$$= \left\langle x^{(i)}, x^{(i)} \right\rangle + \langle W^*(t_i)W(t_i)h, h\rangle + 2\left\langle h, W^*(t_i)x^{(i)} \right\rangle.$$

Look for the parameter $h = h^{(i)}$ using the minimizing condition of $J(h)$. From $\frac{\partial J(h^{(i)})}{\partial h} = 0$ we obtain the system of linear algebraic equations for finding $h^{(i)}$

$$W^*(t_i)W(t_i)h = -W^*(t_i)x^{(i)}.$$

Suppose that the matrix $W^*(t_i)W(t_i)$ is nondegenerate. Then

$$h^{(i)} = -(W^*(t_i)W(t_i))^{-1}W^*(t_i)x^{(i)}. \tag{2.3.5}$$

Taking into account (2.3.1), (2.3.4), (2.3.5), the method can be rewritten as

$$\begin{gathered}\frac{dx}{dt} = f(x, u(x, p^{(i)}, t), t), \\ x(t_0) = x_0 \text{ if } i = 0,\ x(t_i) = x\left(t_i, p^{(i-1)}\right) \text{ if } i = 1, 2, ..., \\ \frac{dW(t)}{dt} = F(t, p^{(i)})W(t) + g(t, p^{(i)}),\ W(t_i) = 0,\ t \in [t_i, t_{i+1}], \\ p^{(i+1)} = p^{(i)} - s_i(W^*(t_{i+1})W(t_{i+1}))^{-1}W^*(t_{i+1})x^{(i+1)},\ p^{(0)} = p_0,\end{gathered} \tag{2.3.6}$$

where $s_i \in (0, 1]$ is chosen such that $p^{(i+1)} \in P$, $i = 0, 1, ...$

If the matrix $W^*(t_{i+1})W(t_{i+1})$ is degenerate of close enough to degenerate, then regularization of the method can be applied (2.3.6). For this purpose we introduce the regularization parameter $\varepsilon > 0$. For this parameter we have

$$\begin{gathered}\frac{dx}{dt} = f(x, u(x, p^{(i)}, t), t), \\ x(t_0) = x_0 \text{ if } i = 0,\ x(t_i) = x\left(t_i, p^{(i-1)}\right) \text{ if } i = 1, 2, ..., \\ \frac{dW(t)}{dt} = F(t, p^{(i)})W(t) + g(t, p^{(i)}),\ W(t_i) = 0,\ t \in [t_i, t_{i+1}], \\ p^{(i+1)} = p^{(i)} - s_i(W^*(t_{i+1})W(t_{i+1}) + \varepsilon I)^{-1}W^*(t_{i+1})x^{(i+1)},\ p^{(0)} = p_0,\end{gathered} \tag{2.3.7}$$

where $s_i \in (0, 1]$ is chosen such that $p^{(i+1)} \in P$, $i = 0, 1, ...$, I is $m \times m$-dimensional identity matrix. It should be noted that

$$\lim_{\varepsilon \to +0} (W^*(t_{i+1})W(t_{i+1}) + \varepsilon I)^{-1}W^*(t_{i+1}) = W^+(t_{i+1}),$$

where $W^+(t_{i+1})$ is a pseudo-inverse matrix to the matrix $W(t_{i+1})$. Thus in (2.3.7) the last line can be modified as

$$p^{(i+1)} = p^{(i)} - s_i W^+(t_{i+1})x^{(i+1)},\ p^{(0)} = p_0. \tag{2.3.8}$$

Let the system (2.3.1) be linear and look like

$$\frac{dx}{dt} = Ax + Bu,\ t \geq t_0,$$

where A, B are $n \times n$- and $n \times m$-dimensional matrices with constant coefficients.

Assume that the control which solves the stabilization problem has a form of

$$u = C(p)x,$$

where $C(p)$ is $m \times n$-dimensional matrix, whose components depend smoothly on p. Then the methods (2.3.6), (2.3.7) will be written similarly taking into account that $F(t,p) = A + BC(p)$, $g(t,p) = \frac{\partial (A+BC(p))x}{\partial p}$.

2.4 Computational experiments

To perform computational experiments consider the adaptive adjustments of controller parameters in the stabilization problem of the system of oscillation of two masses [9, 10, 14].

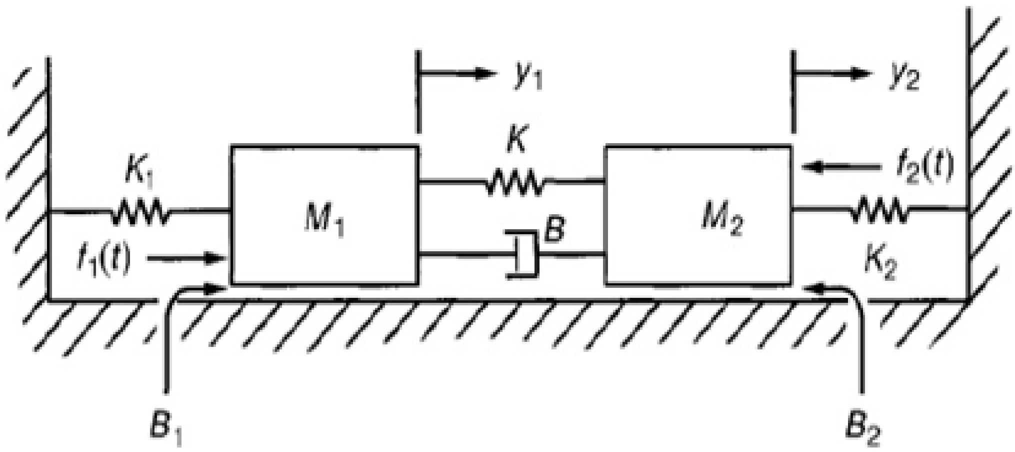

Figure 2: Mathematical model of two-masses oscillation with friction

The mathematical model of oscillation of two masses M_1, M_2 which interact through the friction forces B, B_1, B_2 and are connected by springs with appropriate rigidities K, K_1, K_2 has the form

$$\begin{cases} M_1 \frac{d^2 y_1(t)}{dt^2} + (B + B_1) \frac{dy_1(t)}{dt} + (K + K_1) y_1(t) - B \frac{dy_2(t)}{dt} - K y_2(t) = u_1(t), \\ M_2 \frac{d^2 y_2(t)}{dt^2} + (B + B_2) \frac{dy_2(t)}{dt} + (K + K_2) y_2(t) - B \frac{dy_1(t)}{dt} - K y_1(t) = -u_2(t). \end{cases}$$

Here y_1, y_2 are deviations of masses M_1, M_2 from the equilibrium, $u_1(t)$, $u_2(t)$ are external forces acting on the masses M_1, M_2 respectively. Suppose that M_1, M_2, B, B_1, B_2, K, K_1, K_2 are given positive constants. Such system can be presented in the form

$$\begin{cases} \frac{dy_1(t)}{dt} = y_3\left(t\right), \ \frac{dy_2(t)}{dt} = y_4(t), \\ M_1\frac{dy_3(t)}{dt} + \left(K+K_1\right)y_1\left(t\right) - Ky_2\left(t\right) + \left(B+B_1\right)y_3\left(t\right) - By_4\left(t\right) = u_1\left(t\right), \\ M_2\frac{dy_4(t)}{dt} - Ky_1\left(t\right) + \left(K+K_2\right)y_2\left(t\right) - By_3\left(t\right) + \left(B+B_2\right)y_4\left(t\right) = -u_2\left(t\right). \end{cases} \tag{2.4.1}$$

We consider the problem of stabilizing the equilibrium point of system (2.4.1) in which the external forces $u_1\left(t\right)$, $u_2\left(t\right)$ play the role of control functions. The problem is to find the controller in the form of feedback control.

$$\begin{aligned} u_1\left(y_1, y_2, y_3, y_4\right) &= A_1y_1 + C_1y_2 + D_1y_3 + E_1y_4, \\ u_2\left(y_1, y_2, y_3, y_4\right) &= A_2y_1 + C_2y_2 + D_2y_3 + E_2y_4. \end{aligned} \tag{2.4.2}$$

Here A_1, C_1, D_1, E_1, A_2, C_2, D_2, E_2 are coefficients determined so that the trivial solution of system (2.4.1) is asymptotically stable. The following statement holds.

Theorem 2 (On the controller parametric representation)**.** *Suppose that the coefficients A_1, C_1, D_1, E_1 satisfy the following conditions*

$$\begin{gathered} A_1 \neq K + K_1 - \frac{K+C_1}{B+E_1}\left(B + B_1 - D_1 - \frac{M_1(K+C_1)}{B+E_1}\right), \ \textit{if } E_1 \neq -B, \\ C_1 \neq -K, \ \textit{if } E_1 = -B. \end{gathered} \tag{2.4.3}$$

Suppose also that A_2, C_2, D_2, E_2 are defined as the solution of system of linear *algebraic equations* $L\overline{x} = \overline{g}$. *Here* $\overline{x} = (A_2, C_2, D_2, E_2)^*$,

$$\overline{g} = \left(k_1 + g_1, \frac{k_2^2 k_3 + k_3^2 + k_4 k_1^2}{k_1 k_2 k_3} + g_2, \frac{k_3^2 + k_4 k_1^2}{k_2 k_3} + g_3, \frac{k_4}{k_3} + g_4 \right)^*,$$

$$g_1 = -\frac{BM_1 + BM_2 + B_1 M_2 + B_2 M_1 - D_1 M_2}{M_1 M_2},$$

$$g_2 = \frac{A_1 M_2 - BB_1 - BB_2 + BD_1 + BE_1 - B_1 B_2 + B_2 D_1 - KM_1}{M_1 M_2} - \\ - \frac{-KM_2 - K_1 M_2 - K_2 M_1}{M_1 M_2},$$

$$g_3 = \frac{A_1 B + A_1 B_2 + BC_1 - BK_1 - BK_2 - B_1 K - B_1 K_2 - KB_2 - K_1 B_2}{M_1 M_2} + \\ + \frac{D_1 K + K_2 D_1 + E_1 K}{M_1 M_2}, g_4 = \frac{A_1 K + A_1 K_2 + KC_1 - KK_1 - KK_2 - K_1 K_2}{M_1 M_2},$$

$$L = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & a & b & d \\ b & d & c & e \\ c & e & 0 & 0 \end{pmatrix}, \ a = \frac{1}{M_2}, \ b = \frac{B + E_1}{M_1 M_2}, \ c = \frac{K + C_1}{M_1 M_2},$$

$$d = \frac{B + B_1 - D_1}{M_1 M_2}, \ e = \frac{K + K_1 - A_1}{M_1 M_2},$$

k_1, k_2, k_3, k_4 *are arbitrary positive constants. Then the controller (2.4.2) is the solution of stabilization problem of system (2.4.1).*

Proof. Substitute the control function (2.4.2) into (2.4.1). Obtain the system of ordinary differential equations with its characteristic polynomial $P(\lambda) = \lambda^4 +$

$p_1\lambda^3 + p_2\lambda^2 + p_3\lambda + p_4$. Moreover

$$p_1 = \frac{E_2}{M_2} + \frac{BM_1 + BM_2 + B_1M_2 + B_2M_1 - D_1M_2}{M_1M_2},$$
$$p_2 = \frac{C_2}{M_2} + \frac{(B+E_1)\,D_2}{M_1M_2} + \frac{(B+B_1-D_1)\,E_2}{M_1M_2} - \frac{1}{M_1M_2}[A_1M_2 - BB_1 -$$
$$- BB_2 + BD_1 + BE_1 - B_1B_2 + B_2D_1 - KM_1 - KM_2 - K_1M_2 - K_2M_1],$$
$$p_3 = \frac{(B+E_1)\,A_2}{M_1M_2} + \frac{(B+B_1-D_1)\,C_2}{M_1M_2} + \frac{(C_1+K)\,D_2}{M_1M_2} +$$
$$+ \frac{(K+K_1-A_1)\,E_2}{M_1M_2} - \frac{1}{M_1M_2}[A_1B + A_1B_2 + BC_1 - BK_1 - BK_2 -$$
$$- B_1K - B_1K_2 - KB_2 - K_1B_2 + D_1K + K_2D_1 + E_1K],$$
$$p_4 = \frac{(C_1+K)\,A_2}{M_1M_2} + \frac{(K+K_1-A_1)\,C_2}{M_1M_2} -$$
$$- \frac{1}{M_1M_2}[A_1K + A_1K_2 + KC_1 - KK_1 - KK_2 - K_1K_2]. \tag{2.4.4}$$

According to Routh–Hurwitz stability criterion the zero equilibrium point of the system is asymptotically stable if and only if

$$\Delta_1 = p_1 > 0,\ \Delta_2 = p_1p_2 - p_3 > 0,\ \Delta_3 = p_3\Delta_2 - p_4p_1^2 > 0,\ \Delta_4 = p_4\Delta_3 > 0.$$

Put $\Delta_1 = k_1$, $\Delta_2 = k_2$, $\Delta_3 = k_3$, $\Delta_4 = k_4$, where k_1, k_2, k_3, k_4 are arbitrary positive constants. Then $p_1 = k_1$, $p_2 = \frac{k_2^2k_3 + k_3^2 + k_4k_1^2}{k_1k_2k_3}$, $p_3 = \frac{k_3^2 + k_4k_1^2}{k_2k_3}$, $p_4 = \frac{k_4}{k_3}$. The system (2.4.4) can be rewritten as $L\overline{x} = \overline{g}$. Its solution is a vector $(A_2, C_2, D_2, E_2)^*$. The conditions for the nondegeneracy of matrix L are as follows: $e \neq \frac{bcd - ac^2}{b^2}$ if $b \neq 0$; $c \neq 0$ if $b = 0$. It is equivalent to (2.4.3) and gives the conditions to find $(A_1, C_1, D_1, E_1)^*$. The theorem is proved. □

Remark. If in the theorem formulation $k_1 = k_2 = k_3 = k_4 = p$, then

$$\overline{g} = \left(p + g_1, 2 + \frac{1}{p} + g_2, 1 + p + g_3, 1 + g_4\right)^*$$

Computational experiments. Assume the control function (2.4.2) is such that A_1, C_1, D_1, E_1, $A_2 = A_2(p)$, $C_2 = C_2(p)$, $D_2 = D_2(p)$, $E_2 = E_2(p)$ satisfy the conditions of the theorem *on the controller parametric representation* while

$k_1 = k_2 = k_3 = k_4 = p$. In this case substitution (2.4.2) into (2.4.1) gives

$$\begin{cases} \frac{dy_1(t)}{dt} = y_3(t), \ \frac{dy_2(t)}{dt} = y_4(t), \\ \frac{dy_3(t)}{dt} = \frac{(-K-K_1+A_1)}{M_1} y_1(t) + \frac{(K+C_1)}{M_1} y_2(t) + \frac{(-B-B_1+D_1)}{M_1} y_3(t) + \frac{(B+E_1)}{M_1} y_4(t), \\ \frac{dy_4(t)}{dt} = \frac{(K-A_2(p))}{M_2} y_1(t) + \frac{(-K-K_2-C_2(p))}{M_2} y_2(t) + \\ \quad + \frac{(B-D_2(p))}{M_2} y_3(t) + \frac{(-B-B_2-E_2(p))}{M_2} y_4(t). \end{cases}$$

Here

$$A_2(p) = e_{11}(p+g_1) + e_{12}\left(2+p^{-1}+g_2\right) + e_{13}(1+p+g_3) + e_{14}(1+g_4),$$
$$C_2(p) = e_{21}(p+g_1) + e_{22}\left(2+p^{-1}+g_2\right) + e_{23}(1+p+g_3) + e_{24}(1+g_4),$$
$$D_2(p) = e_{31}(p+g_1) + e_{32}\left(2+p^{-1}+g_2\right) + e_{33}(1+p+g_3) + e_{34}(1+g_4),$$
$$E_2(p) = e_{41}(p+g_1) + e_{42}\left(2+p^{-1}+g_2\right) + e_{43}(1+p+g_3) + e_{44}(1+g_4),$$

e_{ij} are components of matrix L^{-1}, $i,j = 1,2,3,4$. The equation for sensitivity function $W(t) = (w_1(t), w_2(t), w_3(t), w_4(t))^*$ looks like

$$\begin{cases} \frac{dw_1(t)}{dt} = w_3(t), \ \frac{dw_2(t)}{dt} = w_4(t), \\ \frac{dw_3(t)}{dt} = \frac{(-K-K_1+A_1)}{M_1} w_1(t) + \frac{(K+C_1)}{M_1} w_2(t) + \frac{(-B-B_1+D_1)}{M_1} w_3(t) + \frac{(B+E_1)}{M_1} w_4(t), \\ \frac{dw_4(t)}{dt} = \frac{(K-A_2(p))}{M_2} w_1(t) + \frac{(-K-K_2-C_2(p))}{M_2} w_2(t) + \frac{(B-D_2(p))}{M_2} w_3(t) + \\ \quad + \frac{(-B-B_2-E_2(p))}{M_2} w_4(t) - \frac{A_2'(p)}{M_2} y_1(t) - \frac{C_2'(p)}{M_2} y_2(t) - \frac{D_2'(p)}{M_2} y_3(t) - \frac{E_2'(p)}{M_2} y_4(t), \end{cases}$$

$t \in [t_i, t_{i+1}]$, $w_1(t_i) = w_2(t_i) = w_3(t_i) = w_4(t_i) = 0$. Here $A_2'(p) = e_{11} + e_{13} - e_{12}p^{-2}$, $C_2'(p) = e_{21} + e_{23} - e_{22}p^{-2}$, $D_2'(p) = e_{31} + e_{33} - e_{32}p^{-2}$, $E_2'(p) = e_{41} + e_{43} - e_{42}p^{-2}$.

We apply in this case the method (2.3.8) and present the results of computational experiments.

Experiment 1. Input data: $M_1 = 2$, $M_2 = 5$, $B = 5$, $B_1 = B_2 = 0.001$, $K = K_1 = K_2 = 3$. Initial conditions: $y_1(0) = y_2(0) = y_3(0) = y_4(0) = 10$. The condition of algorithm termination

$$\|y(T)\| \le \varepsilon,$$

where $\varepsilon = 0.1$, T is the time of algorithm termination. We choose $s_i = 1$, if $p^{(i+1)} > 0$, otherwise $s_i = 0.5s_i$ while $p^{(i+1)} \le 0$, $i = 0,1,\ldots$. Denote $\tau = t_{i+1} - t_i$, $i = 0,1,\ldots$, i.e. the step of method is constant.

At $p = 0.3$ provided that the controller is chosen according to the statement 2 without applying (2.3.8) the algorithm termination time is $T = 243.974$. The plot is given in the Figure (3). We choose $p_0 = 0.3$ as initial approximation and apply the method (2.3.8). We obtain $T = 110$ while $\tau = 5$. The plot is given in the Figure (4).

Experiment 2. Input data: $M_1 = 125$, $M_2 = 249$, $B = 12$, $B_1 = 9$, $B_2 = 10$, $K = 47$, $K_1 = 35$, $K_2 = 46$. Initial conditions: $y_1(0) = 100$, $y_2(0) = -100$, $y_3(0) = 100$, $y_4(0) = 100$. The condition of algorithm termination

$$\|y(T)\| \leq \varepsilon,$$

where $\varepsilon = 0.1$, T is the time of algorithm termination. We choose $s_i = 1$, if $p^{(i+1)} > 0$, otherwise $s_i = 0.5s_i$ while $p^{(i+1)} \leq 0$, $i = 0, 1, \ldots$. Denote $\tau = t_{i+1} - t_i$, $i = 0, 1, \ldots$, i.e. the step of method is constant.

At $p = 10$ provided that the controller is chosen according to the statement 2 without applying (2.3.8) the algorithm termination time is $T = 1920.145$. The plot is given in the Figure (5). We choose $p_0 = 10$ as initial approximation and apply the method (2.3.8). We obtain $T = 120$ while $\tau = 10$. The plot is given in the Figure (6).

Features of the experiment. To perform the computational experiment the developed algorithm was implemented using the programming language Python in web-based interactive computational environment Jupyter Notebook.

Analysis of the method advantages and disadvantages. The stabilizing regulator can be moving into the desired neighborhood of initial conditions for very long time. The idea of developing the adaptive stabilization algorithm using the sensitivity function is predetermined to improve the time of entry into a given neighborhood of initial conditions. The main advantage of the method is its adaptability which allows to take into account the current system state, adapting to changing environmental conditions, and to work in the conditions of a priori or current uncertainty. The stabilizing regulator may not depend on the parameter, but the stabilization time can be quite long. Thus it is reasonable to consider the dependence of controller on the parameter that we will adjust at discrete points of time.

The main disadvantage of the suggested method is that it is local, because due to the linearization of the system solution the method works only in the neighborhood of linearization. It cannot work in any arbitrary neighborhood.

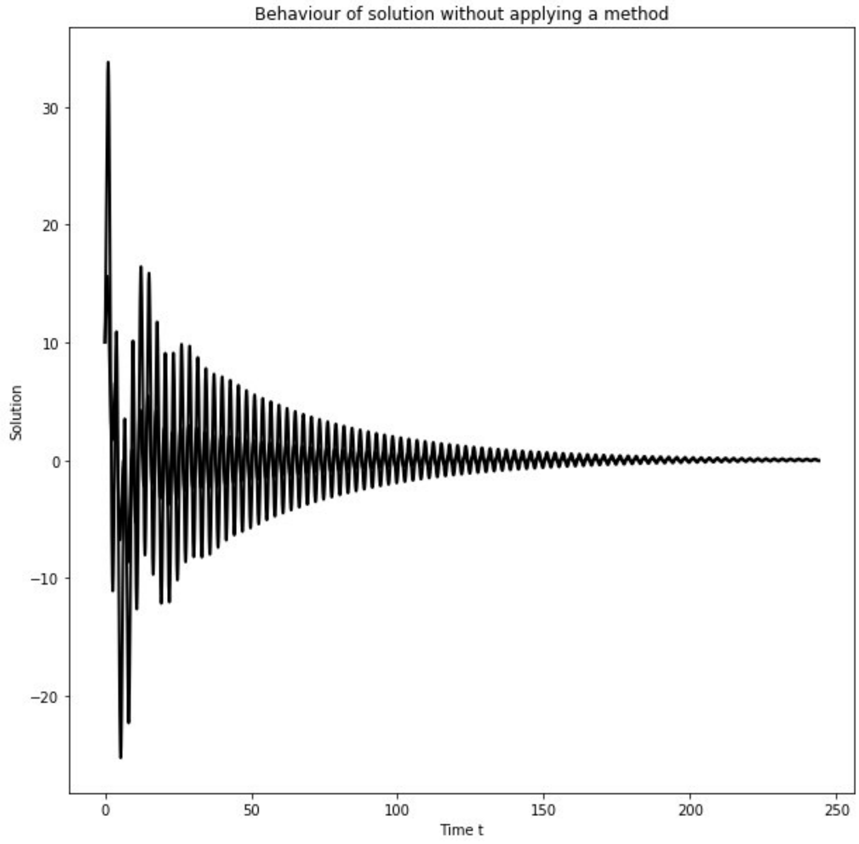

Figure 3: Behavior of solution without applying a method – Experiment 1

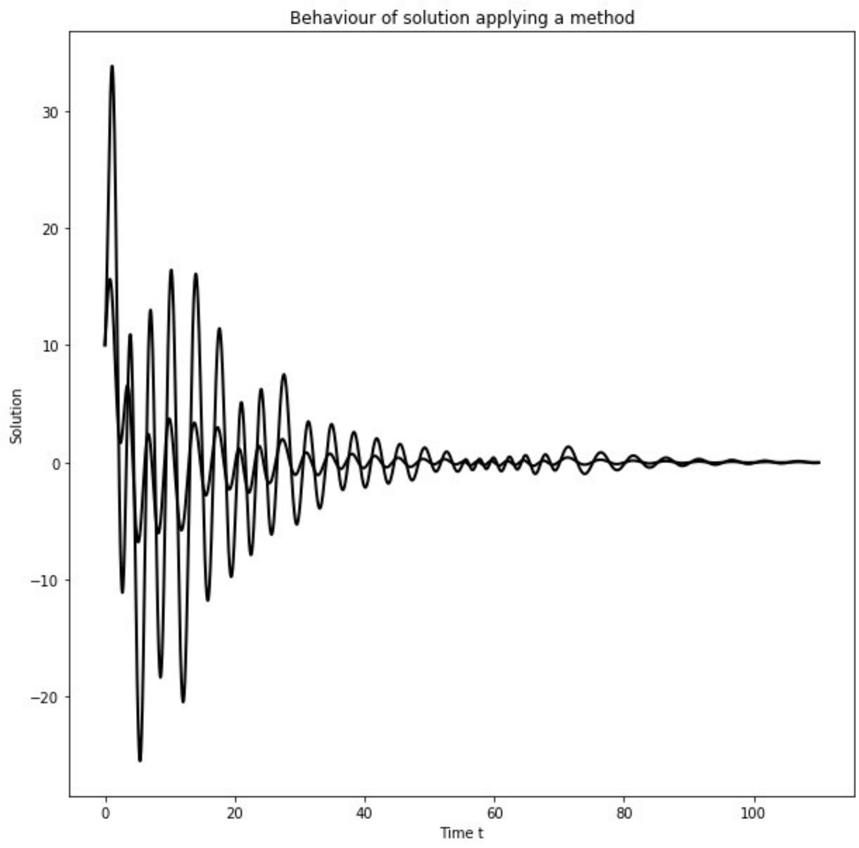

Figure 4: Behavior of solution using the method – Experiment 1

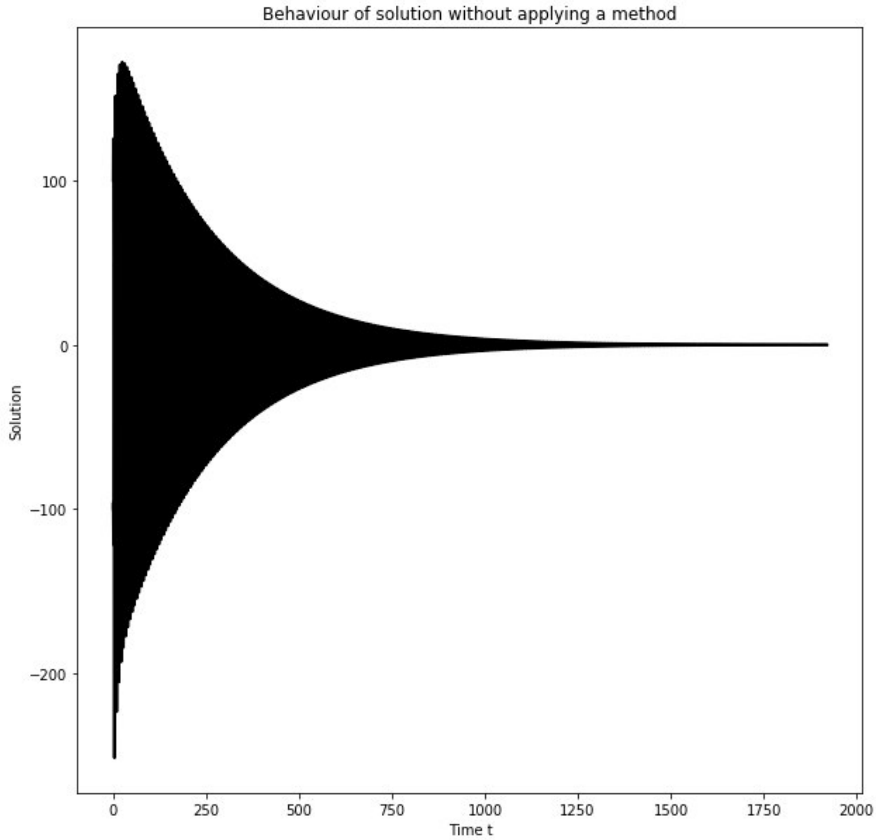

Figure 5: Behavior of solution without applying a method – Experiment 2

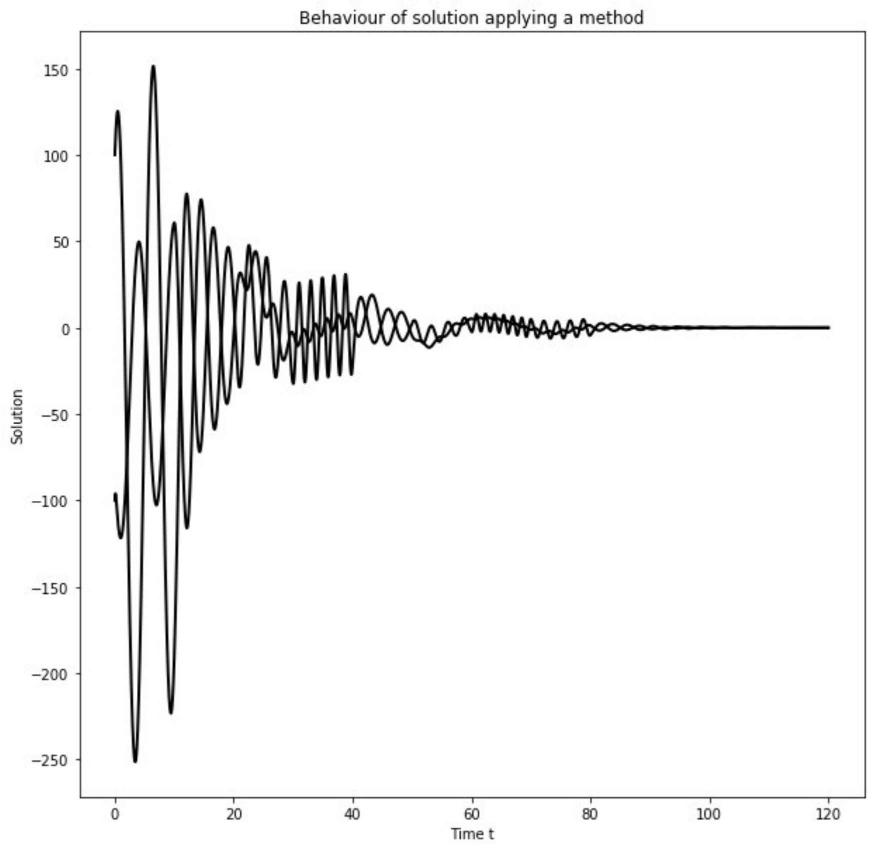

Figure 6: Behavior of solution using the method – Experiment 2

3 Adaptive stabilization in a problem of modal control

3.1 A problem of modal control in a linear system

If the dynamic system is unstable, we can try to achieve the stability of a closed system by introducing feedback control [12]. In many cases, using feedback control we can not only stabilize the system, but achieve any given spectrum of the matrix of closed-loop control system (that is the so-called *modal control problem*). In other words, any desired characteristic polynomial of a closed-loop control system can be obtained. We give the main results for the closed-loop control systems with scalar control

$$\dot{x} = Ax + bu, \tag{3.1.1}$$

where $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}$.

We will look for control in the form of a static output feedback control

$$u = k^T x, \quad k \in \mathbb{R}^n; \tag{3.1.2}$$

then the matrix of closed-loop system will take the form

$$A_c = A + bk^T. \tag{3.1.3}$$

Let Λ be an arbitrary set of n numbers $\lambda_i \in \mathbb{C}$, $i = 1, ..., n$, which satisfies the following condition: if $\lambda_i \in \Lambda$, then $\lambda_i^* \in \Lambda$. Then the polynomial

$$p(s) = \prod_{i=1}^{n} (s - \lambda_i) = s^n + p_{n-1}s^{n-1} + \cdots + p_1 s + p_0$$

has real coefficients. The converse also holds. The set of roots of any polynomial with real coefficients satisfies the indicated condition [12].

Theorem 3. *If the pair (A, b) is controllable, then there exists a vector $k \in \mathbb{R}^n$ such that the eigenvalues of matrix A_c (3.1.3) coincide with a pre-given set Λ.*

Proof. As it is known, any control system with scalar control can be transformed into canonical form by nondegenerate transformation of variables.

Precisely, using linear transformation

$$\widetilde{x} = Tx$$

with nondegenerate matrix

$$T^{-1} = \begin{pmatrix} A^{n-1}b \dots Ab & b \end{pmatrix} \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ a_{n-1} & 1 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ a_2 & & \ddots & \ddots & 0 \\ a_1 & a_2 & \dots & a_{n-1} & 1 \end{pmatrix}$$

the system (3.1.1) is rewritten as

$$\dot{\widetilde{x}} = \widetilde{A}\widetilde{x} + \widetilde{b}u,$$

where

$$\widetilde{A} = TAT^{-1} = \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & & 1 \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{pmatrix}, \quad \widetilde{b} = Tb = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{3.1.4}$$

If we look for control in the form

$$u = \widetilde{k}^T \widetilde{x}, \quad \widetilde{k} = \begin{pmatrix} \widetilde{k}_0 \\ \widetilde{k}_1 \\ \vdots \\ \widetilde{k}_{n_1} \end{pmatrix} \in \mathbb{R}^n,$$

the matrix of closed-loop control system will look like

$$\widetilde{A}_c = \widetilde{A} + \widetilde{b}\widetilde{k}^T = \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & & 1 \\ -a_0 + \widetilde{k}_0 & -a_1 + \widetilde{k}_1 & \dots & -a_{n-1} + \widetilde{k}_{n-1} \end{pmatrix}$$

and its characteristic polynomial is

$$s^n + (a_{n-1} - \widetilde{k}_{n-1})s^{n-1} + \cdots + (a_0 - \widetilde{k}_0).$$

Choosing

$$\widetilde{k}_i = a_i - p_i, \quad i = 1, ..., n,$$

we get that its coefficients coincide with the coefficients of polynomial $p(s)$, i.e. the eigenvalues of matrix $\widetilde{A}_c$ coincide with the set Λ.

Finally, taking $k = T^T \widetilde{k}$ we obtain

$$\widetilde{A}_c = \widetilde{A} + \widetilde{b}\widetilde{k}^T = TAT^{-1} + Tbk^TT^{-1} = T(A + bk^T)T^{-1} = TA_cT^{-1}.$$

Since $det(T^{-1}) = \frac{1}{det(T)}$, the characteristic polynomials of $\widetilde{A}_c$ and TA_cT^{-1} are the same. Thus the eigenvalues of $\widetilde{A}_c$ and A_c coincide. The theorem is proved. □

Corollary 3.1. *Under the assumptions of theorem (3) a closed-loop control system can be made stable, if the set* Λ *consists of all such* λ_i *that* $Re\lambda_i < 0$, $i = 1, ..., n$.

The required regulator k is given in analytical form by the following formula (Ackermann's formula):

$$k = -e^T U^{-1} p(A), \tag{3.1.5}$$

where $U = \begin{bmatrix} B & AB & A^2B \dots A^{n-1}B \end{bmatrix}$ is the controllability matrix of system (3.1.1), $p(A) = (A - \lambda_1 I)(A - \lambda_2 I) \dots (A - \lambda_n I)$ is the matrix characteristic polynomial, I is identity matrix, and

$$e^T = (0 \quad 0 \dots 1)^T \in \mathbb{R}^n.$$

The practical applicability of this formula is limited by scalar systems of low order.

3.2 Computational experiments

To perform computational experiments we consider the same problem of two masses oscillations as in subsection 2.4, but now we assume that the control function $u(t)$ is scalar, i.e. $u_1(t) = u_2(t) = u(t)$. We obtain the system

$$\begin{cases} \frac{dy}{dt} = Ay + bu, \quad t > t_0 \\ y(t_0) = y_0 \end{cases} \tag{3.2.1}$$

where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{K+K_1}{M_1} & \frac{K}{M_1} & -\frac{B+B_1}{M_1} & \frac{B}{M1} \\ \frac{K}{M_2} & -\frac{K+K_2}{M_2} & \frac{B}{M_2} & -\frac{B+B_2}{M_2} \end{pmatrix}, b = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{M_1} \\ -\frac{1}{M_2} \end{pmatrix}$$

Computational experiments will take place in three stages:

Stage 1. Solving the problem of modal control

Consider the problem of modal control where we need to find feedback control

$$u(t) = k^T y = k_1 y_1 + k_2 y_2 + k_3 y_3 + k_4 y_4 \tag{3.2.2}$$

such that characteristic polynomial of the system (3.2.4) has pre-given roots $\lambda_i \in \mathbb{R}^1$, $i = 1, ..., 4$. Choosing $\lambda_i \in \mathbb{R}^1$, $i = 1, ..., 4$ such that $Re\lambda_i < 0$, $i = 1, ..., 4$ we obtain the stabilizing controller of the form (3.2.2).

During the first stage we solve the modal control problem – compute the vector k by Ackermann's formula (3.1.5) and observe the behavior of solution of such system

$$\frac{dy}{dt} = (A + bk^T)y, \quad t \geq t_0 \tag{3.2.3}$$

Stage 2. Behavior in noisy environment

Assume there exist some noises $N(t)$, where $N(t)$ is 4×4-dimensional matrix with elements $n_{ij}(t)$, $i, j = 1, ..., 4$ such that at each time moment t $n_{ij}(t)$ is the realization of random variable uniformly distributed on $[-R, R]$, R is pre-given positive constant. Assume also that at some moment $t_* \geq t_0$ these noises start to affect the system (3.2.3):

$$\begin{cases} \frac{dy}{dt} = (A + bk^T)y, \quad t \in [t_0, t_*) \\ \frac{dy}{dt} = (\widetilde{A} + bk^T)y, \quad t \geq t_* \end{cases} \tag{3.2.4}$$

where $\widetilde{A} = A + N(t)$.

We observe the behavior of solution of system (3.2.4). We expect that in noisy environment the quality of stabilization is getting worse and in some cases the control system can even become unstable.

Stage 3. Applying the method of adaptive stabilization

At time moment t_* to the vector $k^T = (k_1, k_2, k_3, k_4)^T$ which defines the control function u we add a parameter $p^T = (p_1, p_2, p_3, p_4)^T$ and obtain a parametric controller

$$u(t,p) = k_*^T(p)y = (k_1 + p_1)y_1 + (k_2 + p_2)y_2 + (k_3 + p_3)y_3 + (k_4 + p_4)y_4.$$

We apply the method of adaptive stabilization using sensitivity function (2.3.8) and observe the behavior of solution of such system

$$\begin{cases} \frac{dy}{dt} = (A + bk^T)y, \quad t \in [t_0, t_*) \\ \frac{dy}{dt} = (\widetilde{A} + bk_*^T(p))y, \quad t \geq t_* \end{cases} \tag{3.2.5}$$

Results of computational experiments

Experiment 1. Input data: $M_1 = 70$, $M_2 = 100$, $B = B_1 = B_2 = 1$, $K = K_1 = K_2 = 30$, $t_0 = 0$, $t_* = t_0 = 0$, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = -0.1$, $R = 0.01$. Initial conditions: $y_1(0) = -20$, $y_2(0) = 30$, $y_3(0) = 20$, $y_4(0) = 10$. First, we solve the problem of modal control. We obtain such results:

$$k^T = (116.085, 26.108, -86.824, -88.891)^T,$$

$$y_1(T) = 0.071, \quad y_2(T) = -0.07, \quad y_3(T) = -0.006, \quad y_4(T) = 0.006,$$

$$T = 202.134 \quad \text{is stabilization time.}$$

The plot is given in the Figure (7). Then we perform noises simulations. As we can see from the Figure (8) stabilization process becomes slower:

$$y_1(T) = -0.608, \quad y_2(T) = 0.611, \quad y_3(T) = -0.558, \quad y_4(T) = 0.566,$$

$$T = 1000.009 \quad \text{is algorithm termination time.}$$

Finally, we introduce parameter $p^T = (p_1, p_2, p_3, p_4)^T$ by adding it to the vector k^T. Taking $p_0^T = (0, 0, 0, 0)^T$ as initial approximation we apply a method of adaptive stabilization using sensitivity function (2.3.8) and correct the parameter value starting from the moment $t_* = 0$ after each time gap $\tau = 20$. We obtain

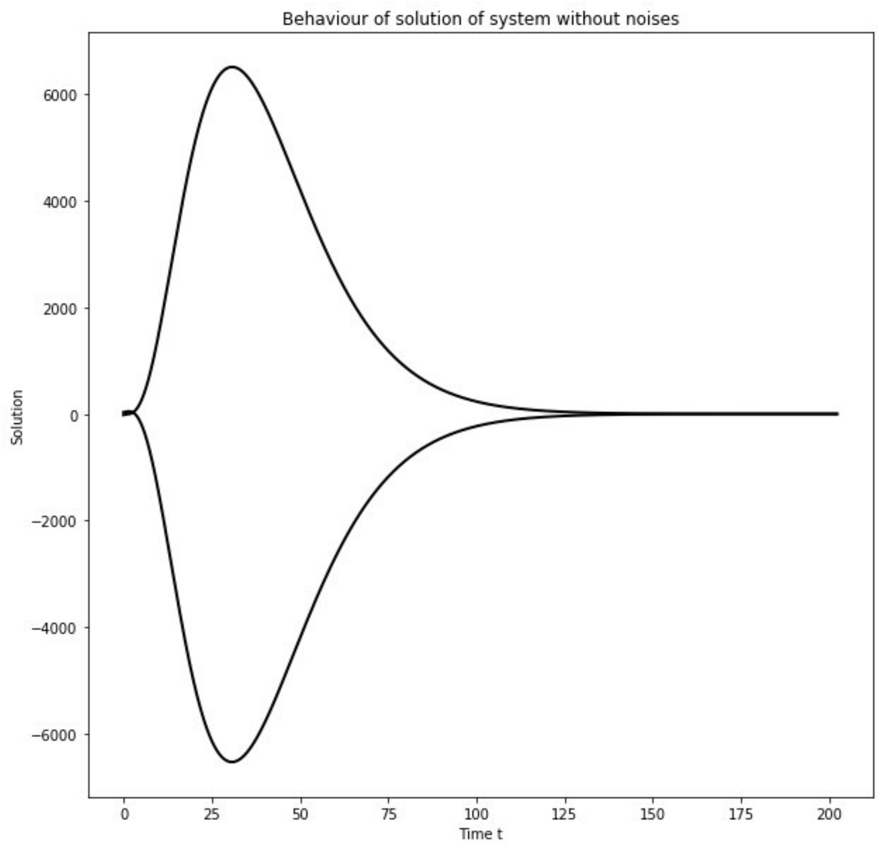

Figure 7: Behavior of solution without noises – Experiment 1

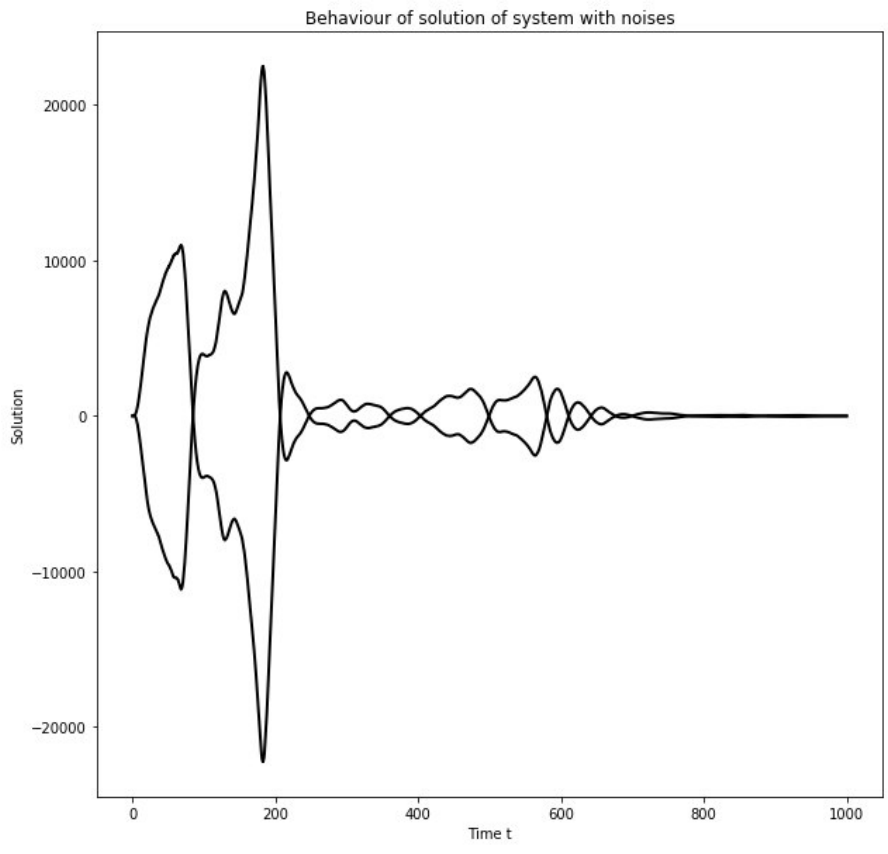

Figure 8: Behavior of solution in noisy environment – Experiment 1

such results:

$$y_1(T) = -0.007, \quad y_2(T) = 0.007, \quad y_3(T) = 0.002, \quad y_4(T) = -0.002,$$

$$T = 160.009 \quad \text{is stabilization time.}$$

After applying the method we get even less stabilization time than it was before noises simulations. The plot is given in the Figure (9).

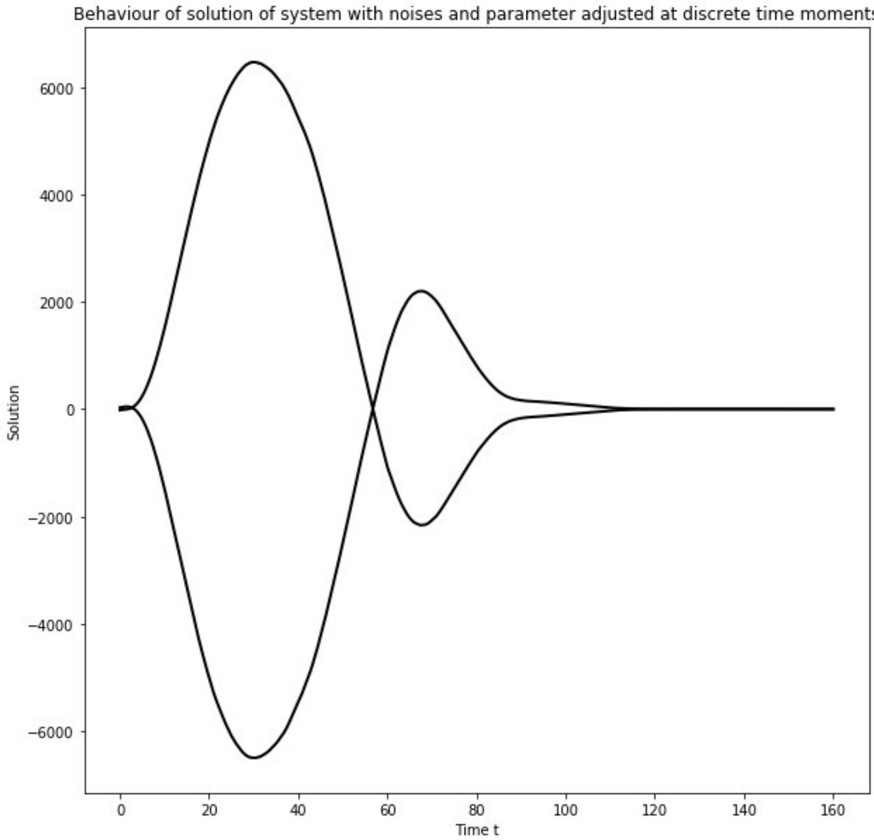

Figure 9: Behavior of solution with method – Experiment 1

Experiment 2. Input data: $M_1 = 120$, $M_2 = 390$, $B = B_1 = B_2 = 5$, $K = K_1 = K_2 = 20$, $t_0 = 0$, $t_* = t_0 = 0$, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = -0.1$, $R = 0.1$. Initial conditions: $y_1(0) = -50$, $y_2(0) = 50$, $y_3(0) = 50$, $y_4(0) = 50$. First, we solve the problem of modal control. We obtain such results:

$$k^T = (38.765, -21.001, -52.977, 58.675)^T,$$

$$y_1(T) = 0.073, \quad y_2(T) = -0.067, \quad y_3(T) = -0.006, \quad y_4(T) = 0.005,$$

$$T = 179.229 \quad \text{is stabilization time.}$$

The plot is given in the Figure (10). Then we perform noises simulations. As we can see from the Figure (11) stabilization process becomes slower:

$$y_1(T) = -0.079, \quad y_2(T) = 0.041, \quad y_3(T) = 0.031, \quad y_4(T) = -0.033,$$

$$T = 302.889 \quad \text{is stabilization time.}$$

Finally, we introduce parameter $p^T = (p_1, p_2, p_3, p_4)^T$ by adding it to the vector

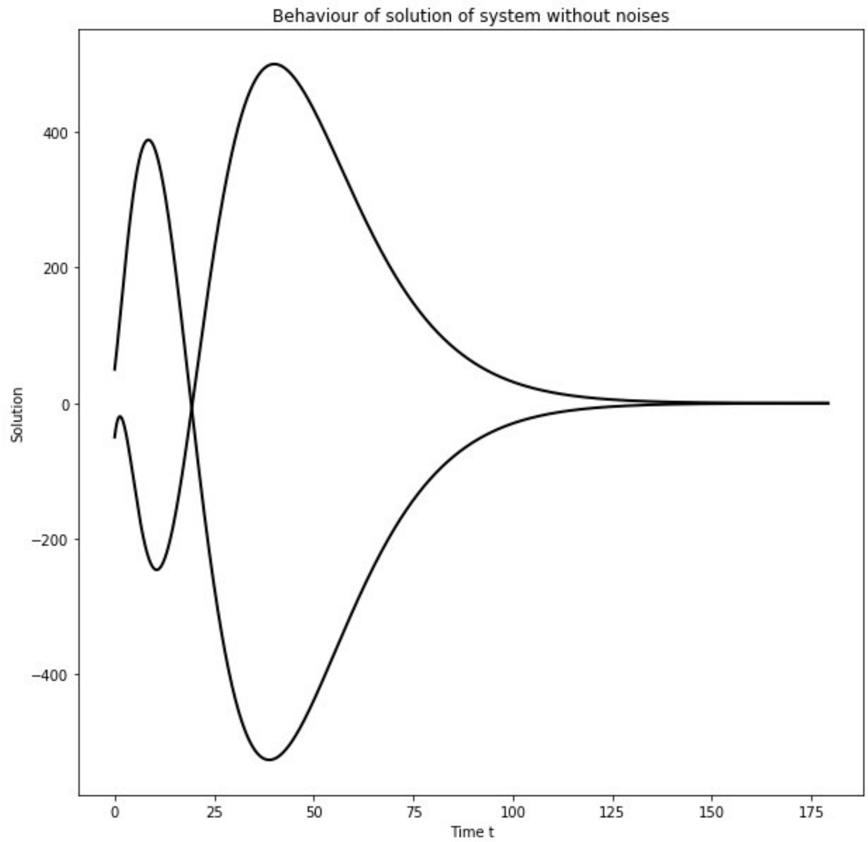

Figure 10: Behavior of solution without noises – Experiment 2

k^T. Taking $p_0^T = (0, 0, 0, 0)^T$ as initial approximation we apply a method of adaptive stabilization using sensitivity function (2.3.8) and correct the parameter value starting from the moment $t_* = 0$ after each time gap $\tau = 10$. We obtain such results:

$$y_1(T) = -0.014, \quad y_2(T) = -0.036, \quad y_3(T) = 0.011, \quad y_4(T) = -0.004,$$

$$T = 270.009 \quad \text{is stabilization time.}$$

After applying the method we get less stabilization time than in case of noises simulations. The plot is given in the Figure (12).

Experiment 3. Input data: $M_1 = 220$, $M_2 = 320$, $B = B_1 = B_2 = 5$, $K = K_1 = K_2 = 20$, $t_0 = 0$, $t_* = 0.1$, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = -0.1$, $R = 0.1$.

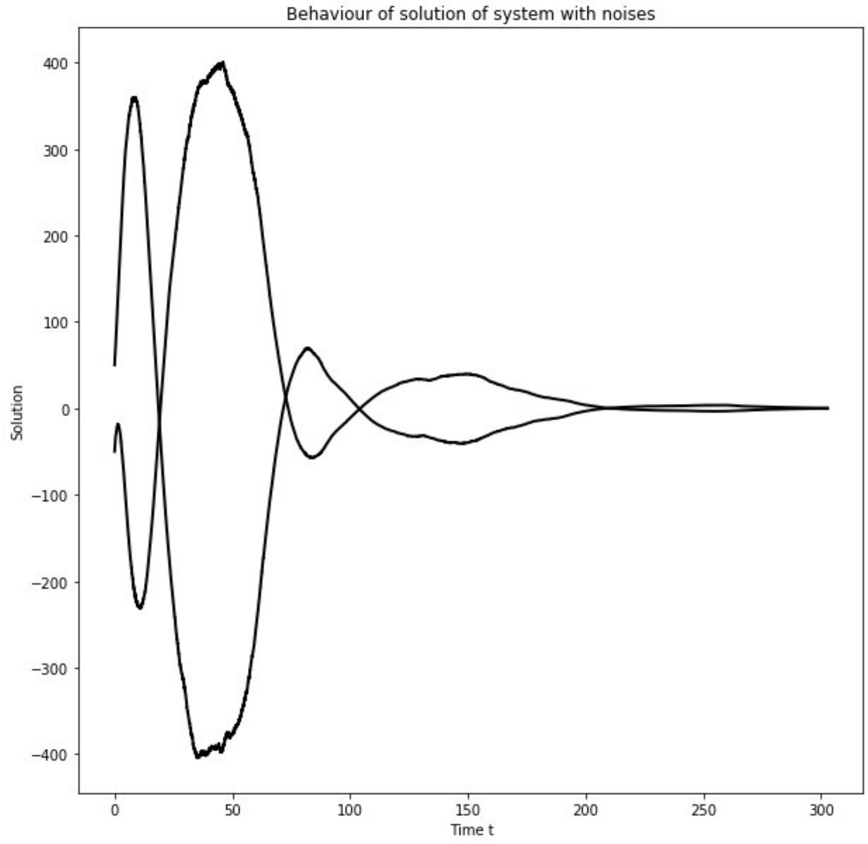

Figure 11: Behavior of solution in noisy environment – Experiment 2

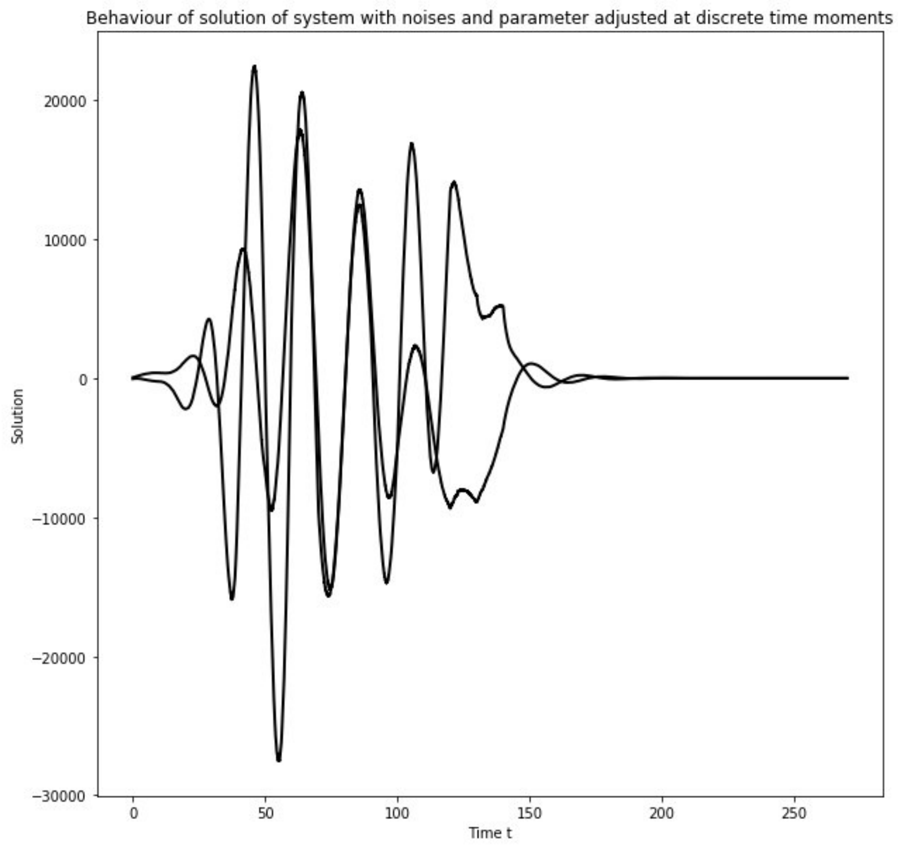

Figure 12: Behavior of solution with method – Experiment 2

Initial conditions: $y_1(0) = -100$, $y_2(0) = 100$, $y_3(0) = 20$, $y_4(0) = 30$. First, we solve the problem of modal control. We obtain such results:

$$k^T = (43.234, -16.414, -229.817, -230.825)^T,$$

$$y_1(T) = 0.071, \quad y_2(T) = -0.069, \quad y_3(T) = -0.006, \quad y_4(T) = 0.006,$$

$$T = 181.889 \quad \text{is stabilization time.}$$

The plot is given in the Figure (13). Then we perform noises simulations. As we can see from the Figure (14) stabilization process becomes slower:

$$y_1(T) = 152.770, \quad y_2(T) = -153.705, \quad y_3(T) = 7.499, \quad y_4(T) = -9.667,$$

$$T = 1000.009 \quad \text{is algorithm termination time.}$$

Finally, we introduce parameter $p^T = (p_1, p_2, p_3, p_4)^T$ by adding it to the vector k^T. Taking $p_0^T = (0, 0, 0, 0)^T$ as initial approximation we apply a method of adaptive stabilization using sensitivity function (2.3.8) and correct the parameter value starting from the moment $t_* = 0.1$ after each time gap $\tau = 20$. We obtain such results:

$$y_1(T) = -0.056, \quad y_2(T) = 0.013, \quad y_3(T) = 0.043, \quad y_4(T) = -0.067,$$

$$T = 280.109 \quad \text{is stabilization time.}$$

After applying the method we get less stabilization time than in case of noises simulations. The plot is given in the Figure (15). *Experiment 4.* Input data: $M_1 = 20$, $M_2 = 10$, $B = B_1 = B_2 = K = K_1 = K_2 = 0.1$, $t_0 = 0$, $t_* = 0.05$, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = -0.1$, $R = 0.1$. Initial conditions: $y_1(0) = -10$, $y_2(0) = 10$, $y_3(0) = 20$, $y_4(0) = 10$. First, we solve the problem of modal control. We obtain such results:

$$k^T = (0.722, 0.622, -7.6, -0.1)^T,$$

$$y_1(T) = 0.057, \quad y_2(T) = -0.082, \quad y_3(T) = -0.004, \quad y_4(T) = 0.006,$$

$$T = 172.619 \quad \text{is stabilization time.}$$

The plot is given in the Figure (16). Then we perform noises simulations. As we

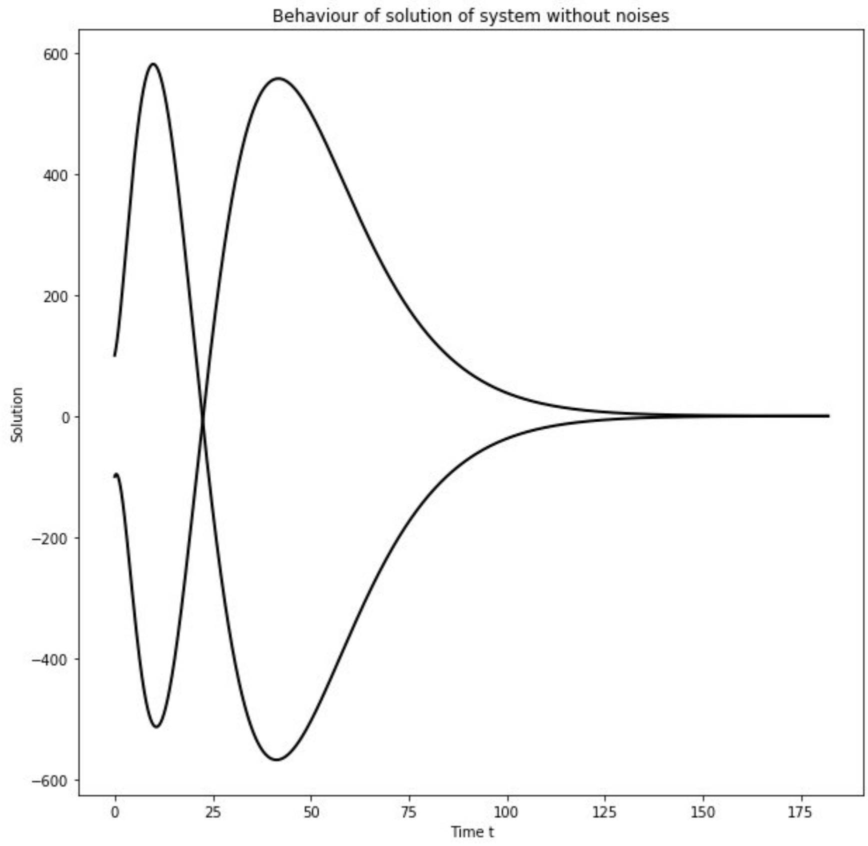

Figure 13: Behavior of solution without noises – Experiment 3

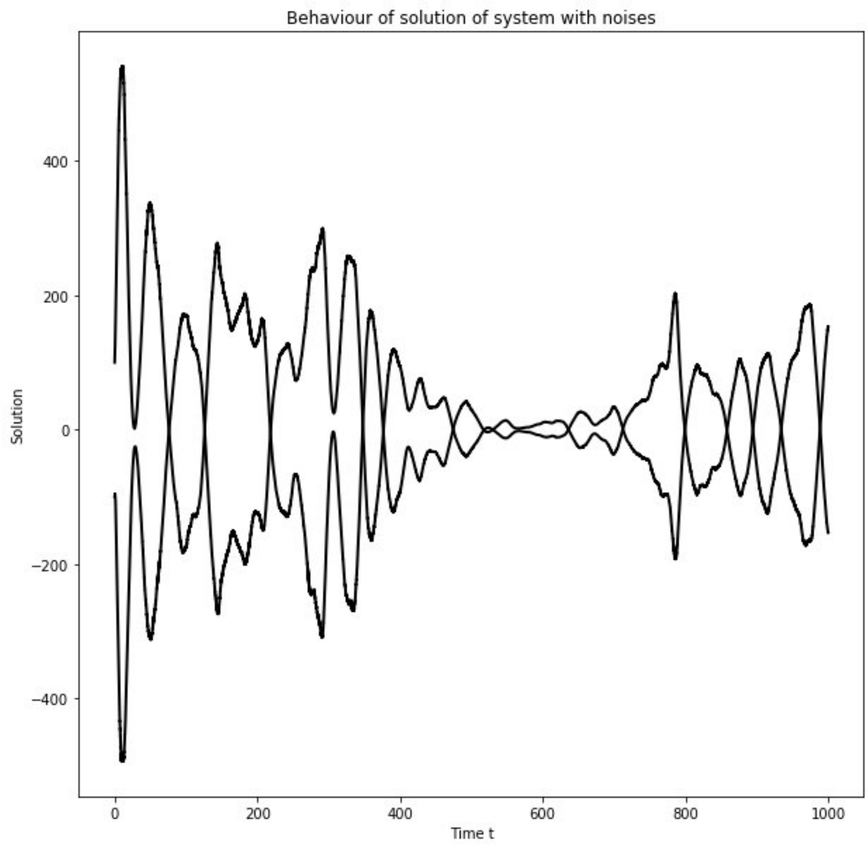

Figure 14: Behavior of solution in noisy environment – Experiment 3

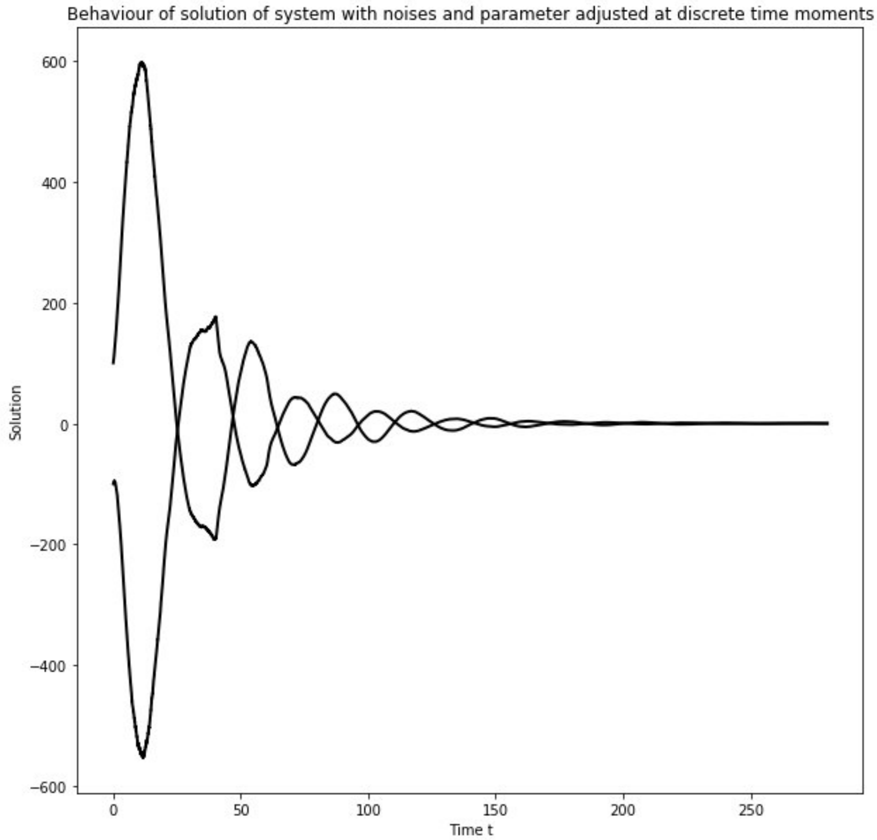

Figure 15: Behavior of solution with method – Experiment 3

can see from the Figure (17) stabilization process becomes slower:

$$y_1(T) = 0.052, \quad y_2(T) = -0.079, \quad y_3(T) = -0.012, \quad y_4(T) = 0.027,$$

$$T = 663.079 \quad \text{is algorithm termination time.}$$

Finally, we introduce parameter $p^T = (p_1, p_2, p_3, p_4)^T$ by adding it to the vector k^T. Taking $p_0^T = (0,0,0,0)^T$ as initial approximation we apply a method of adaptive stabilization using sensitivity function (2.3.8) and correct the parameter value starting from the moment $t_* = 0.05$ after each time gap $\tau = 150$. We obtain such results:

$$y_1(T) = -0.069, \quad y_2(T) = 0.041, \quad y_3(T) = -0.002, \quad y_4(T) = 0.001,$$

$$T = 300.059 \quad \text{is stabilization time.}$$

After applying the method we get less stabilization time than in case of noises simulations. The plot is given in the Figure (15).

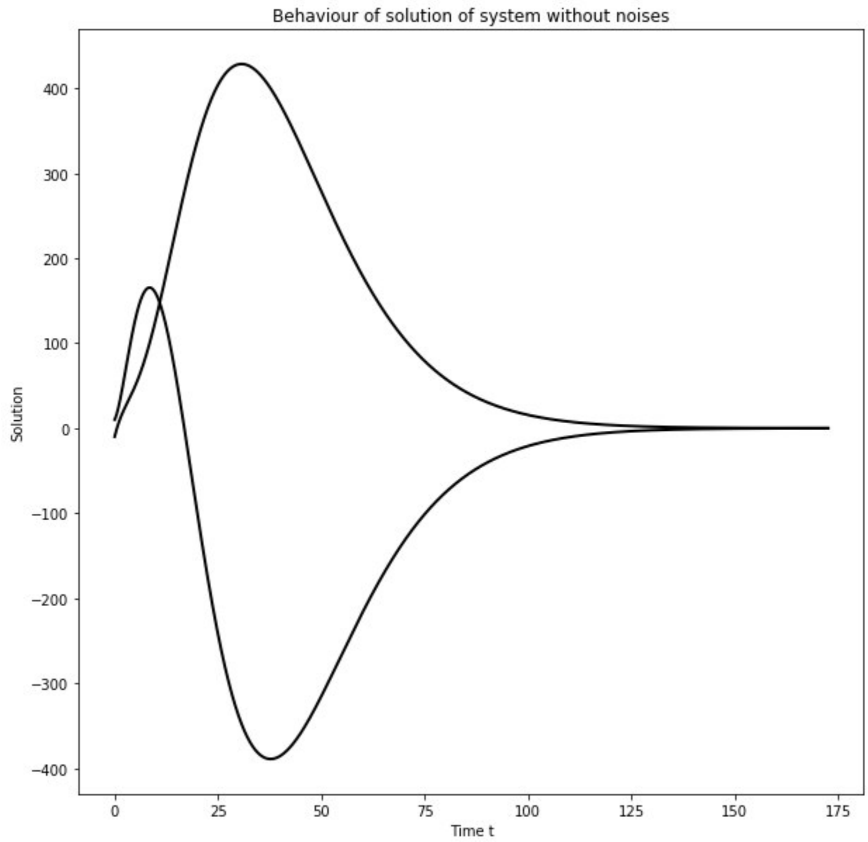

Figure 16: Behavior of solution without noises – Experiment 4

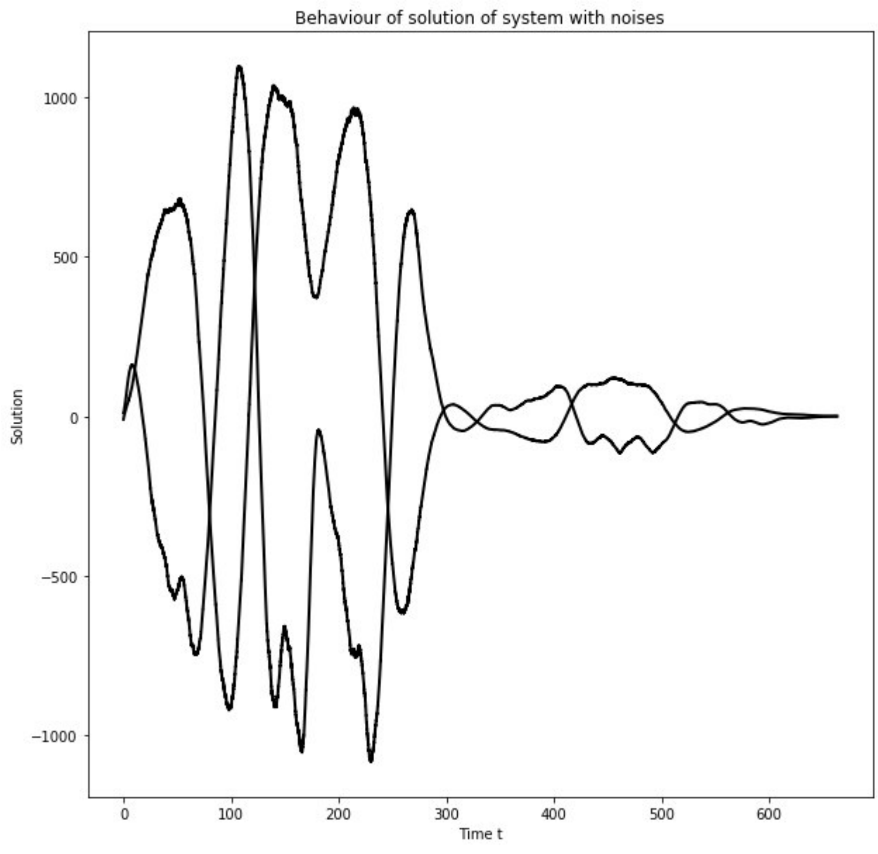

Figure 17: Behavior of solution in noisy environment – Experiment 4

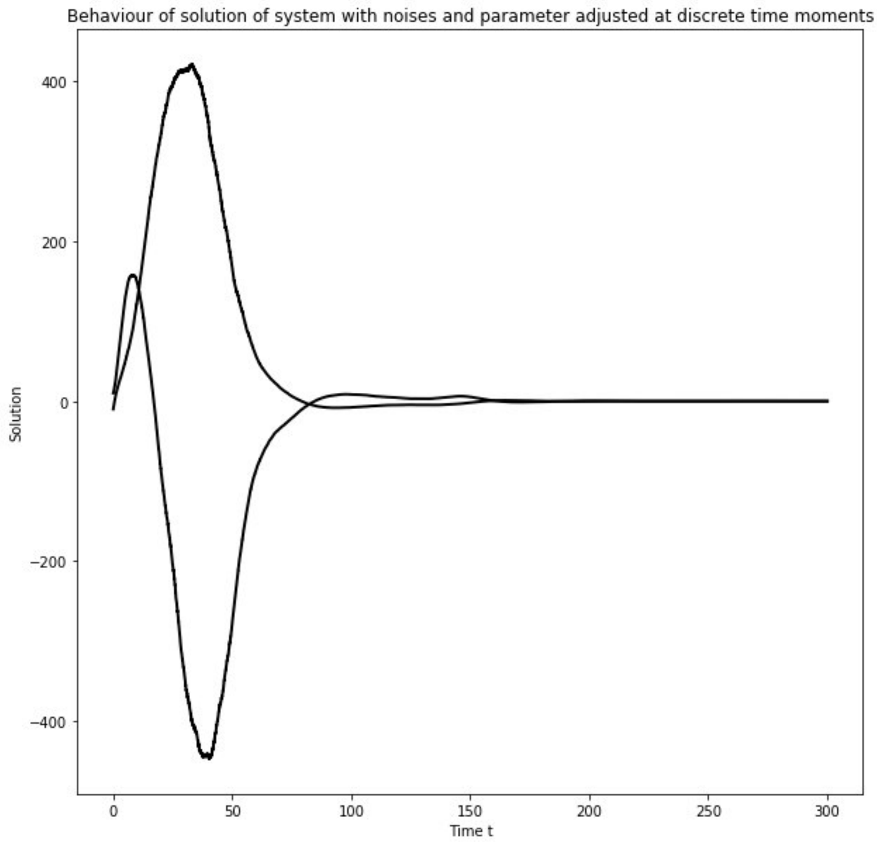

Figure 18: Behavior of solution with method – Experiment 4

Based on conducted computational experiments we can conclude that applying the method of adaptive stabilization using sensitivity function (2.3.8) we can obtain different results and this depends on the choice of method parameters. Varying the method parameters we can achieve better quality of system stabilization in noisy environment.

Conclusion

During preparation of the graduation thesis for a master's degree there was performed:

1) an overview of the research areas related to adaptive control methods (the first section highlights the features of the adaptive approaches in control problems, gradient descent method, adaptive control method for linear systems based on the second Lyapunov method and adaptive identification method; the beginning of second section describes adaptive stabilization of control systems based on the second Lyapunov method); an overview of the sensitivity function properties and the formulation of modal control problem;

2) development of the adaptive stabilization method using sensitivity function; this method allows to adjust the parameters of stabilizing controller at discrete moments of time minimizing the quality criterion which describes the distance of system trajectory to the origin;

3) consideration of the problem of modal control and adaptive stabilization in noisy environment;

3) conduction of computational experiments to check the efficiency of suggested method; the experiments were based on the problem of stabilization of two masses oscillation system;

4) analysis of the developed approach advantages and disadvantages based on conducted research and practical results.

The research result is the suggested method of adaptive stabilization using sensitivity function, the practical value is its application in the conditions of a priori and current uncertainty in the design of systems with a given quality of functioning, in the construction of automatic control systems, etc. The method can be modified and improved in order to obtain better results.

The results of research work were highlighted in the scientific article and also reported at the two scientific conferences.

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