УДК 519.21 https://doi.org/10.17721/1812-5409.2022/4.1

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Стохастична модель хижак-жертва, що залежить від щільності популяції хижака

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A stochastic predator-prey model that depends on the population density of the predator

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Доведено існуваня і единість додатного, глобального розв'язку системи стохастичних диференціальних рівнянь, яка описує динаміку популяцій у стохастичній, залежній від щільності хижаків моделі хижак-жертва з функціональною відповіддю типу Холлінг II. Одержано достатні умови граничної стохастичної обмеженості і стохастичної перманентності даної моделі.

Ключові слова: стохастична модель хижак-жертва, залежність від щільності хижаків, глобальний розв'язок, гранична стохастична обмеженість, стохастична перманентність.

The system of stochastic differential equations describing a nonautonomous stochastic density-dependent predator-prey model with Holling-type II functional response disturbed by white noise, centered and non-centered Poisson noises is considered. So, in this model we take into account levels of predator density dependence and jumps, corresponding to the centered and non-centered Poisson measures. The existence and uniqueness theorem for the positive, global (no explosions in the finite time) solution of the considered system is proved. We obtain sufficient conditions of stochastic ultimate boundedness and stochastic permanence in the considered stochastic predator-prey model.

Key Words: stochastic predator-prey model, predator density dependence, existence and uniqueness of global positive solution, ultimate boundedness, stochastic permanence.

Communicated by Prof. Moklyachuk M.P.

1 Introduction

Predator-prey interaction is a basic mechanism for two species dynamics. The predator-prey model usually described by the system of differential equations

$$\begin{aligned} \frac{dx_1(t)}{dt} &= x_1(t)(a_1-b_1x_1(t))-f(x_1(t),x_2(t))x_2(t),\\ \frac{dx_2(t)}{dt} &= -x_2(t)a_2+\kappa f(x_1(t),x_2(t))x_2(t), \end{aligned}$$

where $x_1(t)$, $x_2(t)$ represent the population density of prey and predator respectively at time t, $a_1 > 0$ is the growth rate of prey, $b_1 > 0$ measures the strength of competition among individuals of species x_1, $a_2 > 0$ is the death rate of predator x_2, $\kappa > 0$ denotes the conversion coefficient, $f(x_1,x_2)$ is the functional response of the predator. The deterministic autonomous Rosenzweig-Mac'Arthur model ([1]) has a form

$$\begin{aligned} dx_1(t) &= x_1(t)\left(a_1-b_1x_1(t)-\tfrac{cx_2(t)}{1+mx_1(t)}\right)dt,\\ dx_2(t) &= x_2(t)\left(-a_2+\tfrac{\kappa c x_1(t)}{1+mx_1(t)}\right)dt, \end{aligned} \tag{1}$$

c is the maximum ingestion rate; $m > 0$ is the half-saturation.

In the paper [2] the stochastic version of predator-prey model (1) is considered in the following form

$$\begin{aligned} dx_1(t) &= x_1(t)\left(a_1-b_1x_1(t)-\tfrac{cx_2(t)}{1+mx_1(t)}\right)dt\\ &+\sigma_1x_1(t)dw_1(t),\\ dx_2(t) &= x_2(t)\left(-a_2+\tfrac{\kappa c x_1(t)}{1+mx_1(t)}\right)dt\\ &+\sigma_2x_2(t)dw_2(t), \end{aligned} \tag{2}$$

where $w_1(t)$ and $w_2(t)$ are mutually independent Wiener processes. In [2] the authors proved that there is a unique positive solution to the system (2), deduced the conditions that there is a stationary distribution of the system, which implies that the system is permanent and obtained the sufficient conditions for the system that is going to be extinct.

Population systems may suffer abrupt environmental perturbations, such as fires, hurricanes, earthquakes, etc. So it is natural to introduce Poisson noises into the population model for describing such discontinuous systems. There are considerable pieces of evidence in nature that predator species may be density-dependent. So we need to take into account levels of predator density dependence.

In this paper, we consider the non-autonomous density-dependent predator-prey model with Holling-type II functional response, disturbed by white noise and jumps generated by centered and non-centered Poisson measures. This model is driven by the system of stochastic differential equations

$$\begin{aligned} dx_i(t)=&x_i(t)\left[(-1)^{i-1}\left(a_i(t)-\tfrac{c_i(t)x_{3-i}(t)}{1+m(t)x_1(t)}\right)\right.\\ &\left.-b_i(t)x_i(t)\right]dt+\sigma_i(t)x_i(t)dw_i(t)\\ &+\int_{\mathbb{R}}\gamma_i(t,z)x_i(t^-)\tilde\nu_1(dt,dz)\\ &+\int_{\mathbb{R}}\delta_i(t,z)x_i(t^-)\nu_2(dt,dz),\\ x_i(0)=&x_{i0}>0,\ i=1,2. \end{aligned} \tag{3}$$

where $x_i(t^-)$, $i = 1,2$ is a left limit of $x_i(t), i = 1,2$, $b_2(t) > 0$ is the predator density dependence rate, $c_2(t) = \kappa c_1(t)$, $w_i(t), i = 1,2$ are independent standard one-dimensional Wiener processes, $\nu_i(t,A), i = 1,2$ are independent Poisson measures, which are independent on $w_i(t), i = 1,2$, $\tilde\nu_1(t,A) = \nu_1(t,A) - t\Pi_1(A)$, $E[\nu_i(t,A)] = t\Pi_i(A), i = 1,2$, $\Pi_i(A), i = 1,2$ are finite measures on the Borel sets A in $\mathbb{R}$.

To the best of our knowledge, there have been no papers devoted to the dynamic properties of the stochastic predator-prey model (3).

In this paper, we prove that the system (3) has a unique, positive, global (no explosion in a finite time) solution for any positive initial value, and that this solution is stochastically ultimately bounded. The sufficient conditions for stochastic permanence is deduced. It is worth noting that the impact of centered and non-centered Poisson noises to the stochastic non-autonomous predator-prey models is studied in the papers [3] – [5].

In the following we will use the notations

$$\begin{aligned} &X(t)=(x_1(t),x_2(t)), X_0=(x_{10},x_{20}),\\ &|X(t)|=\sqrt{x_1^2(t)+x_2^2(t)},\\ &\mathbb{R}^2_+=\{X\in\mathbb{R}^2:\ x_1>0,x_2>0\},\\ &\alpha_i(t)=a_i(t)+\int_{\mathbb{R}}\delta_i(t,z)\Pi_2(dz),\\ &\beta_i(t)=\tfrac{\sigma_i^2(t)}{2}+\int_{\mathbb{R}}[\gamma_i(t,z)-\ln(1+\gamma_i(t,z))]\Pi_1(dz)\\ &-\int_{\mathbb{R}}\ln(1+\delta_i(t,z))]\Pi_2(dz), i=1,2. \end{aligned}$$

$$\bar f(t)=\frac{1}{t}\int_0^t f(s)\,ds,$$
$$f_*=\liminf_{t\to\infty}f(t),\ \ f^*=\limsup_{t\to\infty}f(t).$$

For the bounded, continuous functions $f_i(t), t \in [0,+\infty), i = 1,2$, let us denote

$$f_{i\sup}=\sup_{t\geqslant0}f_i(t), f_{i\inf}=\inf_{t\geqslant0}f_i(t), i=1,2.$$

2 Existence and uniqueness of global solution

Let $(\Omega,\mathcal{F},\mathbf{P})$ be a probability space, $w_i(t), i = 1,2, t \geqslant 0$ are independent standard one-dimensional Wiener processes on $(\Omega,\mathcal{F},\mathbf{P})$, and $\nu_i(t,A), i = 1,2$ are independent Poisson measures defined on $(\Omega,\mathcal{F},\mathbf{P})$ independent on $w_i(t), i = 1,2$. Here $\mathsf{E}[\nu_i(t,A)] = t\Pi_i(A), i = 1,2$, $\tilde\nu_i(t,A) = \nu_i(t,A) - t\Pi_i(A), i = 1,2$, $\Pi_i(\cdot), i = 1,2$ are finite measures on the Borel sets in $\mathbb{R}$. On the probability space $(\Omega,\mathcal{F},\mathbf{P})$ we consider an increasing, right continuous family of complete sub-σ-algebras $\{\mathcal{F}_t\}_{t\geqslant0}$, where $\mathcal{F}_t = \sigma\{w_i(s),\nu_i(s,A), s \leqslant t, i = 1,2\}$.

We need the following assumption.

Assumption 1. It is assumed, that $a_i(t), b_i(t), c_i(t), \sigma_i(t), \gamma_i(t,z), \delta_i(t,z), i = 1,2$, $m(t)$ are bounded, continuous on t functions, $a_i(t) > 0, i = 1,2$, $b_{i\inf} > 0$, $c_{i\inf} > 0, i = 1,2$, $m_{\inf} > 0$, and $\ln(1+\gamma_i(t,z)), \ln(1+\delta_i(t,z)), i = 1,2$ are bounded, $\Pi_i(\mathbb{R}) < \infty, i = 1,2$.

In what follows we will assume that Assumption 1 holds.

Theorem 2.1. *There exists a unique global solution $X(t)$ to the system (3) for any initial value $X(0) = X_0 \in \mathbb{R}^2_+$, and* $\mathbf{P}\{X(t)\in\mathbb{R}^2_+\} = 1$.

Proof. Let us consider the system of stochastic differential equations

$$\begin{aligned} d\xi_i(t) = &\Big[(-1)^{i-1}\left(a_i(t) - \tfrac{c_i(t)\exp\{\xi_{3-i}(t)\}}{1+m(t)\exp\{\xi_1(t)\}}\right) \\ &-b_i(t)\exp\{\xi_i(t)\} - \beta_i(t)\Big]\,dt + \sigma_i(t)dw_i(t) \\ &+ \int_{\mathbb{R}} \ln(1+\gamma_i(t,z))\tilde{\nu}_1(dt,dz) \\ &+ \int_{\mathbb{R}} \ln(1+\delta_i(t,z))\tilde{\nu}_2(dt,dz), \\ \xi_i(0) = &\ln x_{i0},\ i=1,2. \end{aligned} \tag{4}$$

The coefficients of the system (4) are local Lipschitz continuous. So, for any initial value $(\xi_1(0), \xi_2(0))$ there exists a unique local solution $\Xi(t) = (\xi_1(t), \xi_2(t))$ on $[0, \tau_e)$, where $\sup_{t<\tau_e} |\Xi(t)| = +\infty$ (cf. Theorem 6, p.246, [6]). Therefore, from the Itô's formula, we derive that the process $X(t) = (\exp\{\xi_1(t)\}, \exp\{\xi_2(t)\})$ is a unique, positive local solution to the system (3). To show this solution is global, we need to show that $\tau_e = +\infty$ a.s. Let $n_0 \in \mathbb{N}$ be sufficiently large for $x_{i0} \in [1/n_0, n_0], i = 1,2$. For any $n \geqslant n_0$ we define the stopping time

$$\tau_n = \inf\left\{t \in [0,\tau_e) : X(t) \notin \left(\frac{1}{n}, n\right) \times \left(\frac{1}{n}, n\right)\right\}.$$

It is easy to see that τ_n is increasing as $n \to +\infty$. Denote $\tau_\infty = \lim_{n\to\infty} \tau_n$, whence $\tau_\infty \leqslant \tau_e$ a.s. If we prove that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. and $X(t) \in \mathbb{R}^2_+$ a.s. for all $t \in [0, +\infty)$. So we need to show that $\tau_\infty = \infty$ a.s. If it is not true, there are constants $T > 0$ and $\varepsilon \in (0,1)$, such that $\mathbf{P}\{\tau_\infty < T\} > \varepsilon$. Hence, there is $n_1 \geqslant n_0$ such that

$$\mathbf{P}\{\tau_n < T\} > \varepsilon, \quad \forall n \geqslant n_1. \tag{5}$$

For the non-negative function

$$V(X) = \sum_{i=1}^{2} k_i(x_i - 1 - \ln x_i),$$

$X = (x_1, x_2)$, $x_i > 0$, $k_i > 0$, $i = 1,2$ by the Itô's formula we obtain

$$\begin{aligned} dV(X(t)) = &\sum_{i=1}^{2} k_i\Big\{(-1)^{i-1}(x_i(t)-1)\Big(a_i(t) \\ &-\tfrac{c_i(t)x_{3-i}(t)}{1+m(t)x_1(t)}\Big) - b_i(t)x_i(t)(x_i(t)-1) + \beta_i(t) \\ &+ \int_{\mathbb{R}} \delta_i(t,z)x_i(t)\Pi_2(dz)\Big\}\,dt \\ &+ \sum_{i=1}^{2} k_i\Big\{(x_i(t)-1)\sigma_i(t)dw_i(t) \\ &+ \int_{\mathbb{R}} [\gamma_i(t,z)x_i(t^-) - \ln(1+\gamma_i(t,z))]\tilde{\nu}_1(dt,dz) \\ &+ \int_{\mathbb{R}} [\delta_i(t,z)x_i(t^-) - \ln(1+\delta_i(t,z))]\tilde{\nu}_2(dt,dz)\Big\} \end{aligned} \tag{6}$$

Let us consider the function

$$\begin{aligned} f(t,x_1,x_2) = &-k_1b_1(t)x_1^2 + k_1\Big(\alpha_1(t) + b_1(t)\Big)x_1 \\ &+ k_1(\beta_1(t) - a_1(t)) - k_2b_2(t)x_2^2 + k_2\big(-a_2(t) \\ &+ b_2(t) + \textstyle\int_{\mathbb{R}} \delta_2(t,z)\Pi_2(dz)\big)x_2 + k_2(\beta_2(t) + a_2(t)) \\ &+ \tfrac{k_2(x_2-1)c_2(t)x_1 - k_1(x_1-1)c_1(t)x_2}{1+m(t)x_1},\ x_i > 0, i = 1,2. \end{aligned}$$

We have estimate

$$\begin{aligned} f(t,x_1,x_2) \leqslant &\sum_{i=1}^{2} k_i\big(-b_{i\,\inf}x_i^2 + (\alpha_{i\,\sup} \\ &+ b_{i\,\sup})x_i\big) + k_1c_{1\,\sup}x_2 + k_1(\beta_{1\,\sup} - a_{1\,\inf}) \\ &+ k_2(\beta_{2\,\sup} + a_{2\,\sup}) + \frac{c_1(t)x_1x_2(k_2\kappa - k_1)}{1+m(t)x_1} \end{aligned}$$

So for $k_2 = k_1/\kappa$ there is a constant $L(k_1,k_2) > 0$, such that $f(t,x_1,x_2) \leqslant L(k_1,k_2)$. From (6) we obtain by integrating

$$\begin{aligned} V(X(T\wedge\tau_n)) \leqslant &V(X_0) + L(k_1,k_2)(T\wedge\tau_n) \\ &+ \sum_{i=1}^{2} k_i\Big\{\int_0^{T\wedge\tau_n} (x_i(t)-1)\sigma_i(t)dw_i(t) \\ &+ \int_0^{T\wedge\tau_n}\int_{\mathbb{R}} [\gamma_i(t,z)x_i(t^-) - \ln(1+\gamma_i(t,z))]\tilde{\nu}_1(dt,dz) \\ &+ \int_0^{T\wedge\tau_n}\int_{\mathbb{R}} [\delta_i(t,z)x_i(t^-) - \ln(1+\delta_i(t,z))]\tilde{\nu}_2(dt,dz)\Big\} \end{aligned} \tag{7}$$

Taking expectations we derive from (7)

$$\mathsf{E}\left[V(X(T\wedge\tau_n))\right] \leqslant V(X_0) + L(k_1,k_2)T. \tag{8}$$

Set $\Omega_n = \{\tau_n \leqslant T\}$ for $n \geqslant n_1$. Then by (5), $\mathbf{P}(\Omega_n) = \mathbf{P}\{\tau_n \leqslant T\} > \varepsilon$, $\forall n \geqslant n_1$. Note that for every $\omega \in \Omega_n$ at least one of $x_1(\tau_n,\omega)$ and $x_2(\tau_n,\omega)$ equals either n or $1/n$. So

$$\begin{aligned} &V(X(\tau_n)) \geqslant \min\{k_1,k_2\} \\ &\times \min\left\{n - 1 - \ln n, \frac{1}{n} - 1 + \ln n\right\}. \end{aligned}$$

From (8) it follows

$$\begin{aligned} &V(X_0) + L(k_1,k_2)T \geqslant \mathsf{E}[\mathbf{1}_{\Omega_n}V(X(\tau_n))] \\ &\geqslant \varepsilon\min\{k_1,k_2\}\min\left\{n-1-\ln n, \frac{1}{n}-1+\ln n\right\}, \end{aligned}$$

where $\mathbf{1}_{\Omega_n}$ is the indicator function of Ω_n. Letting $n \to \infty$ leads to the contradiction $\infty > V(X_0) + L(k_1,k_2)T = \infty$. This completes the proof of the theorem.

3 Long-time behavior

Lemma 3.1. *Let $p > 0$. Then for any initial value $x_{i0} > 0, i = 1, 2$ we have*

$$\limsup_{t\to\infty} E\left[x_i^p(t)\right] \leqslant K_i(p),\ \ i = 1, 2,$$

where $K_i(p) > 0, i = 1, 2$ are some constants depending on p.

Proof. Let τ_n be the stopping time defined in Theorem 2.1. Applying the Itô's formula to the process $V(t, x_i(t)) = e^t x_i^p(t),\ i = 1, 2, p > 0$, we obtain for $i = 1, 2$

$$\begin{aligned} &V(t\wedge\tau_n, x_i(t\wedge\tau_n)) = x_{i0}^p \\ &+\int_0^{t\wedge\tau_n} e^s x_i^p(s)\left\{1+p\left[(-1)^{i-1}\left(a_i(s)\right.\right.\right. \\ &\left.\left.-\tfrac{c_i(s)x_{3-i}(s)}{1+m(s)x_1(s)}\right) - b_i(s)x_i(s) + \tfrac{p(p-1)\sigma_i^2(s)}{2}\right] \\ &+\int_{\mathbb{R}}\left[(1+\gamma_i(s,z))^p - 1 - p\gamma_i(s,z)\right]\Pi_1(dz) \\ &\left.+\int_{\mathbb{R}}\left[(1+\delta_i(s,z))^p - 1\right]\Pi_2(dz)\right\} ds \\ &+\int_0^{t\wedge\tau_n} p e^s x_i^p(s)\sigma_i(s)dw_i(s) \\ &+\int_0^{t\wedge\tau_n}\int_{\mathbb{R}} e^s x_i^p(s^-)\left[(1+\gamma_i(s,z))^p - 1\right]\tilde{\nu}_1(ds,dz) \\ &+\int_0^{t\wedge\tau_n}\int_{\mathbb{R}} e^s x_i^p(s^-)\left[(1+\delta_i(s,z))^p - 1\right]\tilde{\nu}_2(ds,dz). \end{aligned} \tag{9}$$

Under Assumption 1 there are a constants $K_i(p) > 0, i = 1, 2$, such that

$$\begin{aligned} &e^s x_i^p(s)\left\{1+p\left[(-1)^{i-1}\left(a_i(s) - \tfrac{c_i(s)x_{3-i}(s)}{1+m(s)x_1(s)}\right)\right.\right. \\ &\left.-b_i(s)x_i(s)\right] + \tfrac{p(p-1)\sigma_i^2(s)}{2} + \int_{\mathbb{R}}\left[(1+\gamma_i(s,z))^p\right. \\ &\left.-1 - p\gamma_i(s,z)\right]\Pi_1(dz) + \int_{\mathbb{R}}\left[(1+\delta_i(s,z))^p\right. \\ &\left.\left.-1\right]\Pi_2(dz)\right\} \leqslant e^s K_i(p) \end{aligned} \tag{10}$$

From (9) and (10), taking the expectation, we obtain

$$\mathsf{E}\left[V(t\wedge\tau_n, x_i(t\wedge\tau_n))\right] \leqslant x_{i0}^p + K_i(p)e^t, i = 1, 2.$$

If $n \to \infty$, then we get

$$e^t\mathsf{E}\left[x_i^p(t)\right] \leqslant x_{i0}^p + K_i(p)e^t,\ \ i = 1, 2.$$

Hence

$$\limsup_{t\to\infty} \mathsf{E}\left[x_i^p(t)\right] \leqslant K_i(p),\ \ i = 1, 2.$$

Lemma 3.2. *If $p_{2\inf} > 0$, where $p_2(t) = -a_2(t) - \beta_2(t)$, then for any initial value $x_{20} > 0$, the predator population density $x_2(t)$ has the property that*

$$\limsup_{t\to\infty} E\left[\left(\frac{1}{x_2(t)}\right)^\theta\right] \leqslant K(\theta),\ \ 0 < \theta < 1, \tag{11}$$

Proof. For the process $U(t) = 1/x_2(t)$ by the Itô's formula we have

$$\begin{aligned} &U(t) = U(0) + \int_0^t U(s)\left[a_2(s) - \tfrac{c_2(s)x_1(s)}{1+m(s)x_1(s)}\right. \\ &\left.+b_2(s)x_2(s) + \sigma_2^2(s) + \int_{\mathbb{R}}\tfrac{\gamma_2^2(s,z)}{1+\gamma_2(s,z)}\Pi_1(dz)\right] ds \\ &-\int_0^t U(s)\sigma_2(s)dw_2(s) - \int_0^t\int_{\mathbb{R}} U(s^-)\tfrac{\gamma_2(s,z)}{1+\gamma_2(s,z)}\tilde{\nu}_1(ds,dz) \\ &-\int_0^t\int_{\mathbb{R}} U(s^-)\tfrac{\delta_2(s,z)}{1+\delta_2(s,z)}\nu_2(ds,dz). \end{aligned}$$

Then by the Itô's formula we obtain for $0 < \theta < 1$

$$\begin{aligned} &(1+U(t))^\theta = (1+U(0))^\theta + \int_0^t \theta(1+U(s))^{\theta-2} \\ &\times\left\{(1+U(s))U(s)\left[a_2(s) - \tfrac{c_2(s)x_1(s)}{1+m(s)x_1(s)}\right.\right. \\ &\left.+b_2(s)x_2(s)\sigma_2^2(s) + \int_{\mathbb{R}}\tfrac{\gamma_2^2(s,z)}{1+\gamma_2(s,z)}\Pi_1(dz)\right] \\ &+\tfrac{\theta-1}{2}U^2(s)\sigma_2^2(s) + \tfrac{1}{\theta}\int_{\mathbb{R}}\left[(1+U(s))^2\right. \\ &\times\left(\left(\tfrac{1+U(s)+\gamma_2(s,z)}{(1+\gamma_2(s,z))(1+U(s))}\right)^\theta - 1\right) \\ &\left.+\theta(1+U(s))\tfrac{U(s)\gamma_2(s,z)}{1+\gamma_2(s,z)}\right]\Pi_1(dz) \\ &+\tfrac{1}{\theta}\int_{\mathbb{R}}(1+U(s))^2\left[\left(\tfrac{1+U(s)+\delta_2(s,z)}{(1+\delta_2(s,z))(1+U(s))}\right)^\theta\right. \\ &\left.\left.-1\right]\Pi_2(dz)\right\} ds - \int_0^t\theta(1+U(s))^{\theta-1}U(s)\sigma_2(s)dw_2(s) \\ &+\int_0^t\int_{\mathbb{R}}\left[\left(1+\tfrac{U(s^-)}{1+\gamma_2(s,z)}\right)^\theta - (1+U(s^-))^\theta\right]\tilde{\nu}_1(ds,dz) \\ &+\int_0^t\int_{\mathbb{R}}\left[\left(1+\tfrac{U(s^-)}{1+\delta_2(s,z)}\right)^\theta - (1+U(s^-))^\theta\right]\tilde{\nu}_2(ds,dz) \\ &= (1+U(0))^\theta + \int_0^t\theta(1+U(s))^{\theta-2}J(s)ds \\ &-I_{1,stoch}(t) + I_{2,stoch}(t) + I_{3,stoch}(t), \end{aligned} \tag{12}$$

where $I_{j,stoch}(t), j = \overline{1,3}$ are corresponding stochastic integrals in (12). Under Assumption 1 there exist constants $|K_1(\theta)| < \infty$, $|K_2(\theta)| < \infty$

such, that for the process $J(t)$ we have the estimate

$$J(t) \leqslant (1+U(t))U(t)\Big[a_2(t)+b_2(t)U^{-1}(t) +\sigma_2^2(t)+\int_{\mathbb{R}}\gamma_2(s,z)\Pi_1(dz)\Big]+\tfrac{\theta-1}{2}U^2(s)\sigma_2^2(s) +\tfrac{1}{\theta}\int_{\mathbb{R}}(1+U(s))^2\Big[\Big(\tfrac{1}{1+\gamma_2(s,z)}+\tfrac{1}{1+U(s)}\Big)^{\theta}-1\Big]\Pi_1(dz) +\tfrac{1}{\theta}\int_{\mathbb{R}}(1+U(s))^2\Big[\Big(\tfrac{1}{1+\delta_2(s,z)}+\tfrac{1}{1+U(s)}\Big)^{\theta}-1\Big]\Pi_2(dz) \leqslant U^2(t)\Big[a_2(t)+\tfrac{\sigma_2^2(t)}{2}+\int_{\mathbb{R}}\gamma_2(t,z)\Pi_1(dz) +\tfrac{\theta}{2}\sigma_2^2(t)+\tfrac{1}{\theta}\int_{\mathbb{R}}[(1+\gamma_2(t,z))^{-\theta}-1]\Pi_1(dz) +\tfrac{1}{\theta}\int_{\mathbb{R}}[(1+\delta_2(t,z))^{-\theta}-1]\Pi_2(dz)\Big]+K_1(\theta)U(t) +K_2(\theta)=-K_0(t,\theta)U^2(t)+K_1(\theta)U(t)+K_2(\theta).$$

Here we used the inequality $(x+y)^{\theta} \leqslant x^{\theta}+\theta x^{\theta-1}y$, $0<\theta<1$, $x,y>0$, in the first integral term for $x=(1+\gamma_2(s,z))^{-1}$, $y=(1+U(s))^{-1}$, and then in the second integral term for $x=(1+\delta_2(s,z))^{-1}$, $y=(1+U(s))^{-1}$. From

$$\lim_{\theta\to 0+}\Big[\tfrac{\theta}{2}\sigma_i^2(t)+\tfrac{1}{\theta}\int_{\mathbb{R}}[(1+\gamma_i(t,z))^{-\theta}-1]\Pi_1(dz) +\tfrac{1}{\theta}\int_{\mathbb{R}}[(1+\delta_i(t,z))^{-\theta}-1]\Pi_2(dz) +\int_{\mathbb{R}}\ln(1+\gamma_i(t,z))\Pi_1(dz)+\int_{\mathbb{R}}\ln(1+\delta_i(t,z))\Pi_2(dz)\Big] =\lim_{\theta\to 0+}\Delta(\theta,t)=0,$$

and the condition $p_{2\inf}>0$ we can choose a sufficiently small $0<\theta<1$ so that

$$K_0(\theta)=\inf_{t\geqslant 0}K_0(t,\theta)=\inf_{t\geqslant 0}[p_2(t)-\Delta(\theta,t)]>0$$

is satisfied. So from (12) and the estimate for $J(t)$ we derive

$$d\left[(1+U(t))^{\theta}\right]\leqslant\theta(1+U(t))^{\theta-2}[-K_0(\theta)U^2(t) +K_1(\theta)U(t)+K_2(\theta)]dt -\theta(1+U(t))^{\theta-1}U(t)\sigma_2(t)dw_2(t) +\int_{\mathbb{R}}\Big[\Big(1+\tfrac{U(t^-)}{1+\gamma_2(t,z)}\Big)^{\theta}-(1+U(t^-))^{\theta}\Big]\tilde{\nu}_1(dt,dz) +\int_{\mathbb{R}}\Big[\Big(1+\tfrac{U(t^-)}{1+\delta_2(t,z)}\Big)^{\theta}-(1+U(t^-))^{\theta}\Big]\tilde{\nu}_2(dt,dz). \tag{13}$$

By the Itô's formula and (13) we have

$$d\left[e^{\lambda t}(1+U(t))^{\theta}\right]=\lambda e^{\lambda t}(1+U(t))^{\theta}dt +e^{\lambda t}d\left[(1+U(t))^{\theta}\right]\leqslant e^{\lambda t}\theta(1+U(t))^{\theta-2} \times\left[-\left(K_0(\theta)-\tfrac{\lambda}{\theta}\right)U^2(t)+\left(K_1(\theta)+\tfrac{2\lambda}{\theta}\right)U(t) +K_2(\theta)+\tfrac{\lambda}{\theta}\right]dt-\theta e^{\lambda t}(1+U(t))^{\theta-1} \times U(t)\sigma_2(t)dw_2(t) +e^{\lambda t}\int_{\mathbb{R}}\Big[\Big(1+\tfrac{U(t^-)}{1+\gamma_2(t,z)}\Big)^{\theta}-(1+U(t^-))^{\theta}\Big]\tilde{\nu}_1(dt,dz) +e^{\lambda t}\int_{\mathbb{R}}\Big[\Big(1+\tfrac{U(t^-)}{1+\delta_2(t,z)}\Big)^{\theta}-(1+U(t^-))^{\theta}\Big]\tilde{\nu}_2(dt,dz). \tag{14}$$

Let us choose $\lambda=\lambda(\theta)>0$ such, that $K_0(\theta)-\lambda/\theta>0$. Then there is a constant $K>0$, such that

$$(1+U(t))^{\theta-2}\left[-\left(K_0(\theta)-\frac{\lambda}{\theta}\right)U^2(t) +\left(K_1(\theta)+\frac{2\lambda}{\theta}\right)U(t)+K_2(\theta)+\frac{\lambda}{\theta}\right]\leqslant K. \tag{15}$$

Let τ_n be the stopping time defined in the Theorem 2.1. Then by integrating (14), using (15) and taking the expectation we obtain

$$\mathsf{E}\left[e^{\lambda(t\wedge\tau_n)}(1+U(t\wedge\tau_n))^{\theta}\right] \leqslant\left(1+\frac{1}{x_{20}}\right)^{\theta}+\frac{\theta}{\lambda}K\left(e^{\lambda t}-1\right).$$

Letting $n\to\infty$ leads to the estimate

$$e^{t}\mathsf{E}\left[(1+U(t))^{\theta}\right] \leqslant\left(1+\frac{1}{x_{20}}\right)^{\theta}+\frac{\theta}{\lambda}K\left(e^{\lambda t}-1\right). \tag{16}$$

From (16) we obtain

$$\limsup_{t\to\infty}\mathsf{E}\left[\left(\frac{1}{x_2(t)}\right)^{\theta}\right]=\limsup_{t\to\infty}\mathsf{E}\left[U^{\theta}(t)\right] \leqslant\limsup_{t\to\infty}\mathsf{E}\left[(1+U(t))^{\theta}\right]\leqslant\frac{\theta}{\lambda(\theta)}K,$$

this implies (11).

Definition 3.1. ([8]) The solution $X(t)$ to the system (3) will be said stochastically ultimately bounded, if for any $\varepsilon\in(0,1)$, there is a positive constant $\chi=\chi(\varepsilon)>0$, such that for any initial value $X_0\in\mathbb{R}_+^2$, the solution to the system (3) has the property that

$$\limsup_{t\to\infty}\mathbf{P}\left\{|X(t)|>\chi\right\}<\varepsilon.$$

Theorem 3.1. *The solution $X(t)$ to the system (3) is stochastically ultimately bounded for any initial value $X_0\in\mathbb{R}_+^2$.*

Proof. From Lemma 3.1 we have an estimate

$$\limsup_{t\to\infty} E[x_i(t)] \leqslant K_i,\ i = 1, 2. \tag{17}$$

For $X = (x_1, x_2) \in \mathbb{R}^2_+$ we have $|X| \leqslant x_1 + x_2$, therefore, from (17) we obtain

$$\limsup_{t\to\infty} E[|X(t)|] \leqslant L = K_1 + K_2.$$

Let $\chi > L/\varepsilon$, $\forall\varepsilon \in (0, 1)$. Then applying the Chebyshev inequality yields

$$\begin{aligned}&\limsup_{t\to\infty} \mathbf{P}\{|X(t)| > \chi\}\\ &\leqslant \frac{1}{\chi}\limsup_{t\to\infty} E[|X(t)|] \leqslant \frac{L}{\chi} < \varepsilon.\end{aligned}$$

The theorem is proved.

Definition 3.2. The population density $x(t)$ will be said stochastically permanent if for any $\varepsilon > 0$, there are positive constants $H = H(\varepsilon)$, $h = h(\varepsilon)$ such that

$$\begin{aligned}&\liminf_{t\to\infty} \mathbf{P}\{x(t) \leqslant H\} \geqslant 1 - \varepsilon,\\ &\liminf_{t\to\infty} \mathbf{P}\{x(t) \geqslant h\} \geqslant 1 - \varepsilon,\end{aligned}$$

for any inial value $x(0) = x_0 > 0$.

Theorem 3.2. *If $p_{2\,\inf} > 0$, where $p_2(t) = -a_2(t) - \beta_2(t)$, then for any initial value $x_{20} > 0$, the predator population density $x_2(t)$ is stochastically permanent.*

Proof. From Lemma 3.1 we have an estimate

$$\limsup_{t\to\infty} E[x_2(t)] \leqslant K.$$

Thus for any given $\varepsilon > 0$, let $H = K/\varepsilon$, by virtue of Chebyshev's inequality, we can derive that

$$\limsup_{t\to\infty} \mathbf{P}\{x_2(t) \geqslant H\} \leqslant \frac{1}{H}\limsup_{t\to\infty} E[x_2(t)] \leqslant \varepsilon.$$

Consequently

$$\liminf_{t\to\infty} \mathbf{P}\{x_2(t) \leqslant H\} \geqslant 1 - \varepsilon.$$

From Lemma 3.2 we have the estimate

$$\limsup_{t\to\infty} \mathsf{E}\left[\left(\frac{1}{x_2(t)}\right)^{\theta}\right] \leqslant K(\theta),\ 0 < \theta < 1.$$

For any given $\varepsilon > 0$, let $h = (\varepsilon/K(\theta))^{1/\theta}$, then by Chebyshev's inequality, we have

$$\begin{aligned}&\limsup_{t\to\infty} \mathbf{P}\{x_2(t) < h\}\\ &\leqslant \limsup_{t\to\infty} \mathbf{P}\left\{\left(\tfrac{1}{x_2(t)}\right)^{\theta} > h^{-\theta}\right\}\\ &\leqslant h^{\theta}\limsup_{t\to\infty} \mathsf{E}\left[\left(\tfrac{1}{x_2(t)}\right)^{\theta}\right] \leqslant \varepsilon.\end{aligned}$$

Consequently

$$\liminf_{t\to\infty} \mathbf{P}\{x_2(t) \geqslant h\} \geqslant 1 - \varepsilon.$$

The theorem is proved.

If the predator is absent, i.e. $x_2(t) \equiv 0$ a.s., then the equation for the prey population is the non-autonomous stochastic logistic differential equation. The sufficient conditions for the stochastic permanence of the solution to the stochastic logistic differential equation are obtained in [3]:

Theorem 3.3. *If the predator is absent, i.e. $x_2(t) \equiv 0$ a.s., and $p_{1\,\inf} > 0$, where $p_1(t) = a_1(t) - \beta_1(t)$, then for any initial value $x_{10} > 0$, the prey population density $x_1(t)$ is stochastically permanent.*

4 Conclusions

In this paper, we consider the non-autonomous stochastic density-dependent predator-prey model with Holling-type II functional response driven by the system of stochastic differential equations with white noise, centered and non-centered Poisson noises. So, we take into account levels of predator density dependence, "small" jumps, corresponding to the centered Poisson measure, and the "large" jumps, corresponding to the non-centered Poisson measure. For the considered system, sufficient criteria for the existence and uniqueness of a global positive solution is obtained. We derive sufficient conditions of stochastic ultimate boundedness and stochastic permanence in the considered stochastic predator-prey model.

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Received: 29.12.2022