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### Доповнення до статті “Козображення матричних алгебр Манна”

### Addendum to the paper “Corepresentations of Munn matrix algebras”.

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*Ця стаття є доповненням до статті, опублікованої у Віснику Київського національного університету імені Тараса Шевченка (Серія. фіз-мат. науки), 2022, 3, с. 42-44.*

*У оригінальному варіанті не приведено чітке означення козображення алгебри з одиницею, що привело до деяких формальних неточностей (щодо позначень, запису визначальних співвідношень, тощо). Усі результати оригінальної статті по суті правильні, але в останній частині доведення теореми 1 допущена помилка, яка не впливає на правильність ідеї доведення, проте вимагає розширених міркувань.*

*У цьому доповненні автор надає детальну інформацію про все, що стосується тверджень про твірні елементи і визначальні співвідношення алгебр Манна над алгебрами з одиницею.*

*Ключові слова: козображення, твірні елементи та визначальні співвідношення, вільна асоціативна алгебра, некомутативний поліном, алгебра Манна, напівгрупа Ріса, регулярна сендвіч-матриця.*

*This paper is an addendum to the paper, published in Bulletin of Taras Shevchenko National University of Kyiv (Series: Physics & Mathematics), 2022, No. 3, pp. 42-44.*

*In the original version does not given a clear definition of the corepresentation of an algebra with identity what led to some formal inaccuracies (regarding notations, writing of defining relations, etc.). All the results of the original paper are essentially correct, but an error was made in the last part of the proof of Theorem 1, which does not affect the correctness of the idea of proof, but requires extended considerations.*

*In this addendum the author provides detailed information on everything related to statements about the generating elements and defining relations of the Munn algebras over algebras with identity.*

*Key Words: corepresentation, generators and defining relations, free associative algebra, non-commutative polynomial, Mann algebra, Rees semigroup, regular sandwich matrix.*

## 1 Introduction

In this addendum to the paper [1] the author provide more details concerning Theorem 1 on corepresentations (in other words, presentations) of Munn matrix algebras over finite-dimensional  $K$ -algebras, where  $K$  is any field.

Munn matrix algebras arose as generalizations of semigroup algebras of Rees matrix semigroups, which in turn are closely related to 0-simple semigroups (see the book [2] and the original variant [1] of this paper).

For a finite-dimensional (associative)  $K$ -algebra  $\mathcal{A}$  with identity, natural numbers  $m, n$  and a

fixed  $n \times m$  matrix  $P = (p_{ji})$  over  $\mathcal{A}$ , the  $K$ -vector space  $\mathcal{M}(\mathcal{A}; m, n; P)$  of all  $m \times n$  matrices over  $\mathcal{A}$  can be made into an algebra with respect to the following operation  $\circ: B \circ C = BPC$  with the ordinary matrix multiplication on the right side of the equality. This algebra is called the *Munn  $m \times n$  matrix algebra over  $\mathcal{A}$  with sandwich matrix  $P$* .

By analogy with the Rees semigroups (taking into account the fact that the Munn algebras for equivalent sandwich matrices are isomorphic) we called in [1] a sandwich matrix *regular* if at least one its entry is invertible, and (by the same fact) assumed without loss of generality that  $p_{11}$  is equal

to the identity element of  $\mathcal{A}$ , and the remaining entries of the first row and first column are equal to zero. We denoted sandwich matrices of such form by  $P_0$  and further considered in [1], and will consider here, only such sandwich matrices.

We emphasize that, in contrast to the case of semigroups (when the regularity of sandwich matrices is directly related to the regularity of semigroups), the regularity of sandwich matrix for algebras is not associated with the regularity of algebras, and it is defined only as a formal generalization.

Thus, we study the Munn matrix algebras  $\mathcal{M} = \mathcal{M}(\mathcal{A}; m, n; P)$  over finite-dimensional  $K$ -algebras  $\mathcal{A}$ , and as in [1], we are interested in the problem of determining their corepresentations.

## 2 Corepresentations of the Munn matrix algebras

Throughout the paper  $K$  denotes arbitrary field, and all algebras (which are assumed to be associative) are considered over  $K$  and are finite-dimensional (except for 2.4). Since the Mann algebras over a unital algebra  $\mathcal{A}$  does not always have an identity (see [2, Lemma 5.18]), we consider both unital and non-unital algebras. If an algebra is unital, its identity is usually denoted by  $e$  (leaving the symbol 1 for fields and free algebras).

### 2.1 Generators and defining relations

In discussing this topic, we use the books [3, 4].

Let  $Alg(K)$ ,  $Alg^1(K)$  denotes, respectively, the class of all or only unit  $K$ -algebras.

**Definition 1.** Let  $X$  be a finite set. The free  $K$ -algebra  $F = K\langle X \rangle$  in the class  $Alg(K)$  (respectively,  $F = K^1\langle X \rangle$  in the class  $Alg^1(K)$ ) is defined by the universal mapping property: there exists a map  $i : X \rightarrow F$  such that every map  $\alpha$  from  $X$  into another  $K$ -algebra from the same class can be uniquely factored by  $i$  (i.e. there exists a unique homomorphism  $\alpha'$  between the  $K$ -algebras such that (in right-handed notation)  $\alpha = i\alpha'$ ).

These algebras exists and are uniquely, up to isomorphism, defined by the field  $K$  and the set  $X$ . There is simple normal forms for elements of the free algebras  $F = K\langle X \rangle$  and  $F = K^1\langle X \rangle$ . Namely, considering the set  $X$  as an alphabet,  $F = K\langle X \rangle$  is the algebra with  $K$ -basis the set

$X^+$  of all finite words  $x_1 \dots x_n$  of length  $n \geq 1$  in the case of  $Alg(K)$  and the set  $X^*$  of all finite words  $x_1 \dots x_n$  of length  $n \geq 0$  (i.e. including the empty word) in the case  $Alg^1(K)$ , and with juxtaposition as multiplication, i.e. multiplying  $x_{i_1} \dots x_{i_n}$  by  $x_{j_1} \dots x_{j_m}$  is  $x_{i_1} \dots x_{i_n} x_{j_1} \dots x_{j_m}$ . If one denotes a word  $x_{i_1} \dots x_{i_n}$  by  $x_I$  with  $I$  to be the sequence  $(i_1, \dots, i_n)$ , its elements are uniquely expressible in the form  $f = \sum a_I x_I$ , where  $a_I \in K$ , almost all is equal to 0. The empty word (in the case of  $Alg^1(K)$ ) represents the identity of the free algebra  $F$ , which is denoted by 1. So  $K^1\langle X \rangle = K\langle X \rangle \cup 1$ .

The set  $X$  is called a *free generating set* of the algebra  $F$ .

Free algebra  $F = K^1\langle X \rangle$  is often called the algebra of non-commutative polynomials in variables  $X$ , In the case  $F = K\langle X \rangle$  one needs to take the polynomials with zero free term.

**Definition 2.** Let  $\mathcal{B}$  be a non-unital (respectively, unital) finite-dimensional algebra. It is said to be defined by the presentation  $\langle X|R \rangle$  with  $R$  being a subset of the free algebra  $K\langle X \rangle$  (respectively,  $K^1\langle X \rangle$ ), if there exists an isomorphism  $\varphi : K\langle X \rangle/I \rightarrow \mathcal{B}$  (respectively,  $K^1\langle X \rangle/I \rightarrow \mathcal{B}$ ) of non-unital (respectively, unital) algebras, where  $I$  is the ideal generated by  $R$ .

Let  $X = \{x_1, \dots, x_s\}$ . One often considers instead of the elements  $x_i \in X$  and polynomials  $f(x_1, \dots, x_s) \in R$ , respectively, the elements  $b_i = \varphi(x_i + I)$  of  $\mathcal{B}$  and equalities  $f(b_1, \dots, b_s) = 0$ . Then the set  $B = \{b_1, \dots, b_s\}$  is a system of generators of the algebra  $\mathcal{B}$ , and the equalities  $f(b_1, \dots, b_s) = 0$  with  $f \in R$  are called the defining relations of  $\mathcal{B}$  with respect to the system of generators  $B$ . In this case, one means by  $R$  the set of all equalities  $f(b_1, \dots, b_s) = 0$ , calls  $\langle B|R \rangle_K$  the presentation of  $\mathcal{B}$  and writes  $\mathcal{B} = \langle B|R \rangle_K$  or simply  $\mathcal{B} = \langle B|R \rangle$  if  $K$  is fixed (when the multiplication in  $\mathcal{B}$  is denoted by some symbol, say,  $\Delta$ , then it is placed as an upper index, i.e.  $\langle B|R \rangle_K^\Delta$  and, respectively,  $\langle B|R \rangle^\Delta$ ). The author will adhere to this point of view.

From definition 2 it follows that the identity element of a unital algebra never enters a system of generators.

Let's compare two simple examples.

**Example 1.** Let  $\mathcal{B}$  be the two-dimensional non-unital  $K$ -algebra with basis  $b_1, b_2$  and the

following multiplication table:

	$b_1$	$b_2$
$b_1$	$b_2$	$0$
$b_2$	$0$	$0$

Then  $\mathcal{B} = \langle b_1 | b_1^3 = 0 \rangle$  since the homomorphism  $\varphi : K\langle x \rangle \rightarrow \mathcal{B}$  with  $\varphi(x) = b_1$  is an epimorphism with the kernel to be generated (as an ideal) by the polynomial  $f(x) = x^3$ .

**Example 2.** Let  $\mathcal{B}$  be the two-dimensional unital  $K$ -algebra with basis  $e, b_1$  and the following multiplication table:

	$e$	$b_1$
$e$	$e$	$b_1$
$b_1$	$b_1$	$e$

Then  $\mathcal{B} = \langle b_1 | b_1^2 = e \rangle$  since the homomorphism  $\varphi : K^1\langle x \rangle \rightarrow \mathcal{B}$  with  $\varphi(1) = e, \varphi(x) = b_1$  is an epimorphism with the kernel to be generated (as an ideal) by the polynomial  $f(x) = x^2 - 1$ .

## 2.2 Formulation of the main theorem

By the above, a corepresentation of an algebra  $\mathcal{A}$  is by definition, a pair  $\langle A | R \rangle$ , where  $A$  is a set of generators and  $R$  a set of defining relations for the given set of generators. For convenience, the relations (non-commutative polynomials in elements of  $A$ ) will be written not only in the form  $f = 0$ , but also in the form  $f_1 = f_2$  (if  $f = f_1 - f_2$ ).

The following theorem defines a corepresentation of any Munn algebras  $\mathcal{M}(\mathcal{A}; m, n; P)$  with a regular sandwich matrix.

**Theorem 1.** *Let  $\mathcal{A}$  be a finite-dimensional algebra with identity  $e$  over a field  $K$  and  $\langle A | R \rangle$  be its corepresentation with*

$$A = \{a_1, \dots, a_s\},$$

$$R = \{f_1 = g_1, \dots, f_r = g_r\},$$

where  $f_i = f_i(a_1, \dots, a_s), g_i = (a_1 \dots, a_s)$  are non-commutative polynomials over  $K$ .

For  $m, n$ , introduce the sets

$$\Gamma = \{\gamma_2, \dots, \gamma_m\}, \quad \Lambda = \{\lambda_2, \dots, \lambda_n\}.$$

We have the corepresentation  $\langle M | \bar{R} \rangle^\circ$  of a Munn algebra  $\mathcal{M} = \mathcal{M}(\mathcal{A}; m, n; P_0)$  with a regular  $P_0 = (p_{ji})$ , where  $M = A \cup \{e\} \cup \Gamma \cup \Lambda$  and  $\bar{R}$  consists of the following relations:

- (1) the relations from  $R$ ;
- (2)  $e \circ e = e, e \circ a_i = a_i, a_i \circ e = a_i$  ( $i = 1, \dots, s$ );
- (3)  $e \circ \gamma_i = 0, \gamma_i \circ e = \gamma_i$  ( $i = 2, \dots, m$ );
- (4)  $e \circ \lambda_j = \lambda_j, \lambda_j \circ e = 0$  ( $j = 2, \dots, n$ );
- (5)  $\gamma_i \circ \gamma_{i'} = 0$  ( $i, i' = 2, \dots, m$ );
- (6)  $\lambda_j \circ \lambda_{j'} = 0$  ( $j, j' = 2, \dots, n$ );
- (7)  $\lambda_j \circ \gamma_i = p_{ji}$  ( $i = 2, \dots, m; j = 2, \dots, n$ ).

## 2.3 Proof of the theorem

First, indicate which element of  $\mathcal{M}$  corresponds to each generator.

For  $x \in \mathcal{A}$  and  $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$ , denote by  $(x)_{ij}$  the matrix from  $\mathcal{M}$  with its  $(i, j)$ th entry being  $x$ , and its remaining entries being  $0$ ; the expression  $(0)_{ij}$  means the  $m \times n$  zero matrix  $0$ . Then, by the above definition of  $\circ$ ,

$$(b)_{ij} \circ (c)_{i'j'} = (bp_{j'i'}c)_{ij'}$$

and in particular (taking into account the equality  $p_{11} = 1$ )

$$(bc)_{ij} = (b)_{i1} \circ (c)_{1j}. \quad (*)$$

As the generators of  $\mathcal{M}$  indicated in the condition of the theorem, we take the following matrices:

$$\begin{aligned} e &= (e)_{11}, \quad a_k = (a_k)_{11}, \\ \gamma_i &= (e)_{i1}, \quad \lambda_j = (e)_{1j} \end{aligned} \quad (**)$$

for  $k \in \{1, \dots, s\}, i \in \{2, \dots, m\}, j \in \{2, \dots, n\}$ .

Prove that these elements form a system of generators of  $\mathcal{M}$ .

From (\*) it follows that the homomorphism, as non-unital algebras,  $i_0 : \mathcal{A} \rightarrow \mathcal{M}$ , given by the equality  $i_0(x) = (x)_{11}$  for any  $x \in \mathcal{A}$  is an embedding, and we identify the corresponding elements of both algebras. Hence the set of generators  $A$  together with  $e$  is a subset of the set of generators  $M$ .

With the help of matrix multiplication (as elements of the Munn matrix algebra), it is easy to verify that conditions (3)-(7) of the theorem are satisfied, and the equalities  $(a)_{i1} = \gamma_i \circ (a)_{11}, (a)_{1j} = (a)_{11} \circ \lambda_j, (a)_{ij} = (a)_{i1} \circ (e)_{1j}$ , which follow from (\*) and (\*\*), show that the indicated matrices  $(e)_{11}, (a)_{11}, \gamma_i, \lambda_j$ , where  $a \in A, i = 2, \dots, m; j = 2, \dots, n$ , form a system of generators of the algebra  $\mathcal{M}$ .

Now we show that the set of relations  $\bar{R}$  defines the algebra  $\mathcal{M}$  (this is the only place where a mistake was made in the paper [1]). In other words, there is no relation between generators from the set  $M$  which would not follow from the relations

of  $\bar{R}$ . Assume the contrary and let  $F$  be a non-commutative polynomial in variables from  $M$  such that  $F = 0$  and it is not a consequence of the relations from the set  $\bar{R}$ .

We first find out what form a word over  $M$  can have. Denote by  $\bar{A}$  and  $\bar{M}$  the sets of all finite words from  $A$  and  $M$  (with respect to multiplication  $\circ$ ). In addition to conditions (1)-(7), we will use the equalities

$$(3') a_k \circ \gamma_i = 0 \quad (k = 2, \dots, m);$$

$$(4') \lambda_j \circ a_s = 0 \quad (s = 2, \dots, n);$$

they follow from condition (2) and, respectively, conditions (3), (4), namely  $a_k \circ \gamma_i = (a_k \circ e) \circ \gamma_i = a_k \circ (e \circ \gamma_i)$  and  $\lambda_j \circ a_s = \lambda_j \circ (e \circ a_s) = (\lambda_j \circ e) \circ a_s$ .

Let a non-empty word  $w = w_1 \cdots w_q \neq e$ . Then

(a) it can be assumed by (2)-(4) that all  $w_i$  are not equal  $e$ ;

(b)  $w_t = a_k$  implies by (3') that  $w_{t+1} = a_s$  or  $w_{t+1} = \lambda_j$ ;

(c)  $w_t = a_s$  implies by (4') that  $w_{t-1} = a_k$  or  $w_{t-1} = \gamma_i$ ;

(d)  $w_t = \lambda_j$  implies by (4') and (6) that  $w_{t+1} = \gamma_i$ , and therefore by (7)  $w_t w_{t+1} \in \bar{A}$ ;

(e)  $w_t = \gamma_i$  implies by (3') and (5) that  $w_{t-1} = \lambda_j$ , and therefore by (7)  $w_{t-1} w_t \in \bar{A}$ ;

From (a)-(e) it follows that any non-empty word  $w \in \bar{M} \setminus (\bar{A} \cup \{e\})$ , is equal to  $\bar{a}_1 \circ \lambda_j$  or  $\gamma_i \circ \bar{a}_2$  or  $\gamma_i \circ \bar{a}_3 \circ \lambda_j$  with  $\bar{a}_1, \bar{a}_2, \bar{a}_3 \in \bar{A}$ .

Thus, the polynomial  $F$  introduced above has

the following form:  $F = f_0 +$

$$\sum_{2 \leq i \leq m} \gamma_i \circ f_{1i} + \sum_{2 \leq j \leq n} f_{2j} \circ \lambda_j + \sum_{\substack{2 \leq i \leq m \\ 2 \leq j \leq n}} \gamma_i \circ f_{3ij} \circ \lambda_j = 0,$$

where  $f_0, f_{1i}, f_{2j}, f_{3ij}$  are polynomials in variables  $a \in A$ .

The non-zero parts of the matrices that correspond (according to the definitions of the generators from  $M$ ) the terms of the polynomial  $F$  do not intersect, since by the definition of the embedding  $i_0$  and (\*\*)

$$f_0 = (f_0)_{11}, \quad \gamma_i \circ f_{1i} = (f_{1i})_{i1},$$

$$f_{2j} \circ \lambda_j = (f_{2j})_{1j}, \quad \gamma_i \circ f_{3ij} \circ \lambda_j = (f_{3ij})_{ij}.$$

Therefore the equality  $F = 0$  is equivalent to the following system of the equalities in variables from  $A$ :

$$f_0 = 0, \quad f_{1i} = 0, \quad f_{2j} = 0, \quad f_{3ij} = 0,$$

where  $i = 1, \dots, m, \quad j = 1, \dots, n$ .

We have come to a contradiction because these relations are consequences of the relations from  $R$ . Theorem 1 is proved.

## 2.4 Remark

Since we did not use the finite dimensionality of the algebra  $\mathcal{A}$  in the proof of the theorem (and the definitions of a free algebra and a corepresentation are naturally generalized to the infinite-dimensional case), Theorem 1 is also true for an infinite-dimensional algebra  $\mathcal{A}$ .

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