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Andrii ANIKUSHYN, PhD (Phys. & Math.), Assoc. Prof.
ORCID ID: 0000-0003-1109-6290
e-mail: andrii.anikushyn@knu.ua
Taras Shevchenko National University of Kyiv, Kyiv, Ukraine

Viktor LYASHKO, PhD Student
ORCID ID: 0000-0003-2647-1188
e-mail: lyashko91@gmail.com
V. M. Glushkov Institute of Cybernetics, of National Academy of Science of Ukraine, Kyiv, Ukraine

Oleksandr SAMOSONOK, PhD (Phys. & Math.), Researcher
ORCID ID: 0009-0007-6239-4753
e-mail: samosyonok@gmail.com
V. M. Glushkov Institute of Cybernetics, of National Academy of Science of Ukraine, Kyiv, Ukraine

OPTIMAL CONTROL OF THE SYSTEMS GOVERNED BY LINEAR HYPERBOLIC INTEGRO-DIFFERENTIAL EQUATIONS

The work is dedicated to the study of hyperbolic integro-differential equations with partial derivatives. Integro-differential equations of this type have long been a standard object of study in applied mathematics and often arise in the investigation of processes in viscoelastic media (amorphous polymers, semicrystalline polymers, biopolymers, metals at very high temperatures, bituminous materials, etc.).

The main goal is to prove the existence and uniqueness of generalized solutions to the corresponding initial-boundary value problem, as well as to investigate the existence of optimal control for systems described by these models. The main results regarding the well-posedness of the initial-boundary value problem and the existence of optimal control are obtained using the method of a priori inequalities in negative norms. In particular, inequalities in negative norms for the integro-differential operator in certain special spaces are obtained. By utilizing the results of S. Lyashko and building upon the proven a priori inequalities, various definitions of generalized solutions are formulated in the work, and a result regarding their equivalence is provided. A theorem concerning the well-posedness, i.e., the existence, uniqueness, and continuous dependence of the generalized solution on the right-hand side of the equation, is presented.

The work begins with a review of relevant results with similar formulations, referencing works with physical justification of the model. References to the application of the a priori inequalities methodology for differential and integro-differential equations are provided. The problem statement, constraints on equation parameters, and functional spaces necessary for the investigation are described. Section 2 provides proven a priori inequalities, which are a central part of the work. Section 3 contains definitions of generalized solutions and theorems describing their properties, including the well-posedness of the initial-boundary value problem. Section 4 is dedicated to the investigation of the existence of optimal control. Control is exerted through the right-hand side of the equation using a certain special operator. The work examines some examples of such control operators (illustrating various control mechanisms) and the corresponding function spaces. By utilizing proven inequalities and relying on general theorems from the theory of a priori inequalities in negative norms, conditions for the existence of optimal control are established. In particular, restrictions are imposed on the admissible set of controls and the cost criterion in different scenarios.

Keywords: ***a priory inequalities, linear hyperbolic equation, viscoelasticity, generalized solution, optimal control.***

AMS 2020 classification: 34K06, 34K10, 34K35, 35A01, 35A02.

Introduction

In the cylindrical domain $(x,t) \in Q = \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded simply connected domain with a smooth boundary $\partial\Omega$, we consider the initial-boundary value problem

$$Lu = f, \tag{1}$$

for the hyperbolic integro-differential operator

$$u := u_{tt} + Au + Bu, \tag{2}$$

with homogeneous initial and Dirichlet boundary conditions

$$u|_{\partial\Omega} = 0, \tag{3}$$

$$u|_{t=0} = u_t|_{t=0} = 0. \tag{4}$$

Here, the operator A does not depend on the variable t and is given by the second-order differential expression

$$Au = -\sum_{i,j=1}^n \left(a_{ij}(x)u_{x_j}\right)_{x_i} + \sum_{i=1}^n a_i(x)u_{x_i} + a(x)u, \tag{5}$$

while the Volterra-type integral operator B is given by

$$Bu = \int_0^t \sum_{i=1}^n (K_i(x,t,\tau)u_{x_i}(x,\tau))_{x_i} d\tau. \tag{6}$$

Integro-differential equations of this type have long been a standard subject of study in applied mathematics and often arise in the study of processes in viscoelastic media. In particular, it is known that the behavior of some viscoelastic materials (polymers, suspensions, emulsions) exhibits memory properties. Therefore, the forces arising within the body depend not only on the current values of deformation and/or velocity gradient but also on the entire time history of motion. Hence, many models for viscoelastic materials lead to equations of motion that take the form of a linear hyperbolic differential equation perturbed by a dissipative integral term of the Volterra type (Cannarsa, & Sforza, 2011). Thus, the problem (1)–(6) generalizes the model of evolution of a homogeneous isotropic linearly viscoelastic solid occupying a (bounded) volume Ω at rest. The unknown function u represents the displacement field relative to the reference configuration (Conti, Gatti, & Pata, 2008).

Examples of viscoelastic materials include amorphous polymers, semi-crystalline polymers, biopolymers, metals at very high temperatures, bituminous materials, etc. According to (Lakes, 1998), viscoelastic materials include discs in the human spine, skin tissues, wood, polymer foams used in seats, and so on. Materials that behave elastically at room temperature often exhibit significant viscoelastic properties when heated. For example, this occurs with metal turbine blades in jet engines. Viscoelasticity also plays a role in the motion and behavior of tectonic plates, which float and move independently on the mantle of the earth, and are responsible for earthquakes, volcanoes, etc. (Kelly, 2013). Detailed physical justification of considered model is given, for example, in the monographs (Renardy, Hrusa, & Nohel, 1987; Fabrizio, & Morro, 1992).

It is worth mentioning that in the literature, one can find many studies closely related to (1)–(6). For instance, in the work (Loretti, & Sforza 2010), an integro-differential equation of hyperbolic type is considered:

$$u_{tt}(t,x) - u_{xx}(t,x) + \beta \int_0^t e^{-\eta(t-s)} u_{xx}(s,x)ds = 0, t \in (0,T), x \in (0,\pi) \quad (7)$$

with zero initial conditions: $u(0,x) = u_t(0,x) = 0, x \in (0,\pi)$ and boundary conditions

$$u(t,x) = \begin{cases} 0, & x = 0, \\ g(t), & x = \pi, \end{cases}$$

where $g(t)$ is the control function. In the cited work, the problem of determining the control g is considered, for which the weak solution of equation (7), satisfying the given initial-boundary conditions, at the final point satisfies the conditions

$$u(T,x) = u_0(x),\ u_t(T,x) = u_1(x),\ x \in (0,\pi).$$

The proof was based on harmonic analysis and the Hilbert uniqueness method.

In the work (Diagana, 2014), an integro-differential equation is considered:

$$u_{tt} + Bu_t + Au = \int_{-\infty}^{t} C\,(t-s)u(s)ds + f(t,u),$$

where A, B are certain linear self-adjoint operators. Using methods of operator theory and Schauder's fixed-point principle, the existence of some generalized solutions was established.

The paper (Engler, 1991) investigates the equation

$$u_{tt}(x,t) = \mathrm{div}_x\left(g\big(\nabla u(x,t)\big) + \int_0^{\infty} a\,(s)h\big(\nabla u(x,t), \nabla u(x,t-s)\big)ds\right) + f(x,t),$$

with initial-boundary conditions

$$u(x,t)|_{\partial\Omega} = 0, u(x,0) = u_0,\ u_t(x,0) = u_1.$$

The author constructed global weak solutions of the given equation. It is shown that disturbances of steady states propagate with bounded speed, smoothing effects of the solution operator are revealed, and conditions for asymptotic stability of steady states are indicated.

The work (Gan, & Yin, 2015) is devoted to the application of the finite element method to a linear integro-differential boundary value problem of the second order:

$$u_{tt}(x,t) - \nabla\cdot(A\nabla u) - \int_0^t \nabla\cdot\big(B\nabla u(s)\big)ds = f(x,t)$$

with initial-boundary conditions

$$u(x,t)|_{\partial\Omega} = 0, u(x,0) = u_0,\ u_t(x,0) = u_1$$

in a two-dimensional spatial domain Ω. Note that positive definiteness conditions are imposed on the operator B. Optimal error estimates in the L_2 and H_1 norms were obtained for a semi-discrete scheme. Through numerical experiments, the effectiveness of symmetric schemes was confirmed, and it was demonstrated that the convergence rate in the L_2 norm for fully discrete Euler and Crank-Nicolson inverse schemes is $O(h^2 + \tau)$ and $O(h^2 + \tau^2)$, respectively.

In (Karaa, Pani, & Yadav, 2015), the hp-continuous Galerkin method is applied to a class of linear hyperbolic integro-differential equations of the second order. The problem has the form

$$u_{tt} - \nabla\cdot\left(A(x)\nabla u + \int_0^t B\,(x,t,s)\nabla u(s)ds\right) = f(x,t),$$

$$u(x,t)|_{\partial\Omega} = 0,$$

$$u(x,0) = u_0,\ u_t(x,0) = u_1,\ x \in \Omega.$$

Based on the analysis of the extended Ritz-Volterra-type mixed projection, the authors obtained a priori estimates of the hp-error in the $L^{\infty}(L_2)$ norm for velocity and displacement. Convergence order results were obtained, and numerical experiments were presented.

In the paper (Saedpanah, 2014), the existence and uniqueness of solutions to the problem

$$\rho u_{tt}(x,t) + Au(x,t) - \int_0^t \beta\,(t-s)Au(x,s)ds = f(x,t)$$

with mixed homogeneous Dirichlet and non-homogeneous Neumann conditions are investigated:

$$u(x,t)\big|_{\Gamma_1} = 0, \sigma(u,x,t)\cdot n\big|_{\Gamma_2} = g(x,t),$$

$$u(x,0) = u_0,\ u_t(x,0) = v_0,\ x \in \Omega.$$

Note that in such formulations, the differential operators in the integral component and outside of it are the same, and the kernel of the integral operator depends only on the difference of arguments. The existence and uniqueness of the solution are proved using the Galerkin method. Regularity estimates are obtained. The proposed approach is also used to prove higher-order regularity for models with smooth kernels, under appropriate assumptions regarding the initial conditions.

The aim of the work (Cannarsa, & Sforza, 2011) is to study the damping effect of memory-related terms on the asymptotic behavior of solutions to second-order evolution equations in Hilbert spaces. The global existence and asymptotic behavior at infinity of solutions to the semi-linear integro-differential equation

$$u_{tt}(t) + Au(t) - \int_0^t g\,(t-s)Au(s)ds = \nabla F\big(u(t)\big) + f(t)$$

are considered. Exponential stability of weak solutions in the energy norm is obtained for kernels that exponentially decay at infinity, under certain assumptions.

In the work (Shen, Yang, & Liu, 2014), the authors study the problem of optimal control for systems of the form

$$u_{tt} + Au + \int_0^t C\,(t,s)u(s)ds = f + Bu,$$
$$u(x,t)|_{\partial\Omega} = 0,$$
$$u(x,0) = u_0,\ u_t(x,0) = u_1,\ x \in \Omega.$$

Here, A is a linear, self-adjoint, strongly elliptic second-order differential operator, B is a certain linear continuous operator, and C is a second-order differential operator.

The investigation in (Conti, Gatti, & Pata, 2008) establishes some results regarding exponential and polynomial decay of energy associated with a linear integro-differential Volterra equation

$$u_{tt} - \alpha\Delta u + \beta u_t + \int_0^t \mu\,(s)\Delta u(t-s)ds = 0,$$

and provides sufficient conditions for energy decay without using differential inequalities involving the convolution kernel μ.

The well-posedness of a problem with abstract hyperbolic Volterra-type equations is also addressed in (Dafermos, 1970). The existence and uniqueness of solutions to the equation

$$(\rho u_t)_t = C(t)u(t) - \int_0^t G\,(t,s)u(s)ds + f(t)$$

in certain spaces is proved, and their asymptotic behavior is studied.

It is worth outlying that one of the effective methods in studying generalized solutions and optimal control of linear systems is the method of a priori inequalities in negative norms. Estimates in negative norms were first obtained in the 1950s in the study of elliptic boundary value problems (Berezanskyi, 1968), and later used in the works of V.P. Didenko (Didenko, 1972), where such estimates were obtained for a wide class of second-order differential operators and a method for their construction.

The application of the theory of a priori inequalities in negative norms to various optimization problems of systems with distributed parameters is thoroughly described in the monograph (Lyashko, 2002) and partially in the works (Klyushin et al., 2011; Anikushyn, & Nomirovskii, 2011). The approach developed by S. I. Lyashko and his colleagues has proven to be quite effective in studying various issues of well-posedness, optimal control, controllability, numerical methids of systems with distributed parameters. Specifically, works such as (Lyashko, 2002; Klyushin et al., 2011; Sergienko et al., 2023; Lyashko, & Semenov, 2021; Nomirovskii, 2004; Lyashko, & Nomirovskii, 2003a; Lyashko, Nomirovskii, & Sergienko, 2001; Lyashko, & Nomirovskii, 2003b; Nomirovskii, 2006; Nomirovskii, 1999; Nomirovskii, 2007; Nomirovskii, & Vostrikov, 2016; Tymchyshyn, & Nomirovskii, 2021; Semenov, & Denisov, 2023) investigate various formulations for differential equations with partial derivatives.

It has been found that this approach can be successfully applied to Dirichlet problems for integro-differential equations with Volterra-type integral components. For instance, in (Anikushyn, 2010a), the problem with an elliptic type integro-differential equation was studied, while in (Anikushyn, & Hulianytskyi, 2014; Anikushyn, 2010b), problems of parabolic types were addressed. Some special types of equations with Volterra-type integral components are also considered in (Anikushyn, & Kosteeva, 2018; Anikushyn, 2014). The work (Anikushyn et al., 2015) provides a collection of results in this area.

We should underline that the works (Anikushyn, & Hulianytskyi, 2014; Anikushyn, 2010a) transfer results for elliptic and parabolic differential operators, obtained in (Lyashko, 2002), to the case of corresponding integro-differential operators.

In our work, using a priori inequalities for the differential part of the operator (2) from (Lyashko, 2002) and obtaining an estimate for the integral component, we will transfer the result of (Lyashko, 2002) to the case of hyperbolic integro-differential operator. Based on the obtained a priori inequalities, we will provide theorems of existence, uniqueness of generalized solutions, and a theorem on the continuous dependence of the solution on the right-hand side of the equation. Finally, we will use the obtained a priori inequalities to study the problem of optimal control.

1. Main notations and spaces. We start considering the linear operator L defined by the relation (2). As before, $u(x,t)$ represents the function describing the system's state in the domain $(t,x) \in Q = (0,T) \times \Omega$, where Ω is a bounded domain in n-dimensional space with a smooth boundary $\partial\Omega$. The operators A and B have the forms (5) and (6), respectively.

We assume that the kernels $K_i(x,t,\tau)$ are continuously differentiable on $\Omega \times [0,T] \times [0,T]$. Additionally, let $a(x)$ be continuous, and the coefficients $a_{ij}(x)$, $a_i(x)$ be continuously differentiable functions on $\overline{\Omega}$, with $a_{ij}(x) = a_{ji}(x)$, and the operator A being uniformly elliptic in the domain $\overline{\Omega}$, meaning that for any real numbers λ_i the following inequality holds:

$$\sum_{i,j=1}^{n} a_{ij}\,(x)\lambda_i\lambda_j \geq \alpha \sum_{i=1}^{n} \lambda_i^2,$$

where α is a positive constant independent of $x \in \overline{\Omega}$ and λ_i.

Additionally, we assume (as in (Lyashko, 2002)) that for any $x \in \overline{\Omega}$

$$a(x) \geq 0,\ a(x) \geq \textstyle\sum_{i=1}^{n}(\,a_i(x))_{x_i} \tag{8}$$

Let C_0^∞ denote the set of all functions $u \in C^\infty(\overline{Q})$ that satisfy the boundary conditions (3) and the initial conditions (4), and let C_T^∞ denote the set of functions $v \in C^\infty(\overline{Q})$ that satisfy the conditions (3) and

$$u|_{t=T} = u_t|_{t=T} = 0. \tag{9}$$

Let H_{BR}, H_{BR^+} denote the completions of the sets C_0^∞, C_T^∞, respectively, with respect to the norm

$$\| u \|_H^2 = \int_Q u_t^2 + \sum_{i=1}^{n} u_{x_i}^2\, dQ,$$

and let H_{RB}^-, $H_{BR^+}^-$ denote the corresponding negative spaces constructed with respect to $L_2(Q)$.

Also, we consider the adjoint operator to L, which we formally write as

$$L^*u = v_{tt} + A^*u + B^*u, \tag{10}$$

where

$$A^*v = -\sum_{i,j=1}^{n}\left(a_{ij}v_{x_j}\right)_{x_i} - \sum_{i=1}^{n}(a_i v)_{x_i} + av,$$

$$B^*v = \int_t^T \sum_{i=1}^{n} (K_i(x,\tau,t)v_{x_i}(x,\tau))_{x_i} d\tau.$$

We will consider the spaces C_0^∞, C_T^∞ as the domains of definition for the operators L and L^*, respectively.

2. A priori inequalities

Lemma 1. *There exists a positive constant* c *such that for any functions* $u \in C_0^\infty$, $v \in C_T^\infty$, *the following inequalities hold:*

$$\| Lu \|_{H_{BR^+}^-} \leq c \| u \|_{H_{BR}},$$

$$\| L^*v \|_{H_{BR}^-} \leq c \| v \|_{H_{BR^+}}.$$

Proof. In (Lyashko, 2002), it is shown that for the differential part $L_D u = u_{tt} + Au$ of the operator (2), for any functions $u \in C_0^\infty$, $v \in C_T^\infty$, the following inequality holds:

$$|(L_D u, v)| \leq C \| u \|_{H_{BR}} \| v \|_{H_{BR^+}}. \tag{11}$$

We now consider the integral part of the operator. Due to the continuity of the kernels K_i, there exists a constant $c_K > 0$ such that $|K_i(x,t,\tau)| \leq c_K$ for $(x,t,\tau) \in \Omega \times (0,T) \times (0,T)$. For any $u \in C_0^\infty, v \in C_T^\infty$, we have

$$\left|(Bu,v)_{L_2(Q)}\right| = \left|\sum_{i=1}^{n}\int_Q \int_0^t \sum_{i=1}^{n}(K_i(x,t,\tau)u_{x_i}(x,\tau))_{x_i} d\tau v(x,t) dQ\right| =$$

$$= \left|\sum_{i=1}^{n}\int_Q \int_0^t K_i(x,t,\tau)u_{x_i}(x,\tau)d\tau v_{x_i}(x,t)dQ\right| =$$

$$\leq \sum_{i=1}^{n}\int_Q \int_0^t \left|K_i(x,t,\tau)u_{x_i}(x,\tau)v_{x_i}(x,t)\right| d\tau dQ =$$

$$\leq c_K \sum_{i=1}^{n}\int_Q \int_0^t \left|u_{x_i}(x,\tau)v_{x_i}(x,t)\right| d\tau dQ.$$

Utilizing the Cauchy-Schwarz inequality twice, we have

$$\int_Q \int_0^t \left|u_{x_i}(x,\tau)v_{x_i}(x,t)\right| d\tau dQ \leq \int_Q \int_0^T \left|u_{x_i}(x,\tau)\right| d\tau \left|v_{x_i}(x,t)\right| dQ \leq$$

$$\leq \int_Q \left(\int_0^T u_{x_i}^2(x,\tau)d\tau \int_0^T d\tau\right)^{\frac{1}{2}} \left|v_{x_i}(x,t)\right| dQ =$$

$$= T^{\frac{1}{2}} \int_Q \left(\int_0^T u_{x_i}^2(x,\tau)d\tau\right)^{\frac{1}{2}} \left|v_{x_i}(x,t)\right| dQ \leq$$

$$\leq T^{\frac{1}{2}} \left(\int_Q \int_0^T u_{x_i}^2(x,\tau)d\tau dQ \int_Q v_{x_i}^2(x,t)dQ\right)^{\frac{1}{2}} =$$

$$= T\left(\int_Q u_{x_i}^2(x,t)dQ \int_Q v_{x_i}^2(x,t)dQ\right)^{\frac{1}{2}}.$$

Thus,

$$\left|(Bu,v)_{L_2(Q)}\right| \leq c \| u \|_{H_{BR}} \| v \|_{H_{BR^+}},$$

for some constant $c > 0$. Using (11), we obtain

$$\left|((L_D + B)u,v)_{L_2(Q)}\right| \leq \left|(L_D u,v)_{L_2(Q)}\right| + \left|(Bu,v)_{L_2(Q)}\right| \leq C \| u \|_{H_{BR}} \| v \|_{H_{BR^+}}.$$

Dividing by $\| v \|_{H_{BR^+}}$, taking the supremum over all functions $v \in C_T^\infty$, and considering their density in the space H_{BR^+}, we obtain

$$\| Lu \|_{H_{BR^+}^-} = \sup_{v \in C_T^\infty} \frac{(Lu,v)_{L_2(Q)}}{\| v \|_{H_{BR^+}}} \leq C \| u \|_{H_{BR}}.$$

The second inequality is proven similarly.

□

Remark 1. The inequalities proved in Lemma 1 allow us to extend the operators L and L^* from the sets C_0^∞, C_T^∞ to the spaces H_{BR}, H_{BR^+}, respectively, by continuity. We will keep the same notation L,L^* for the extended operators, and note that the inequalities stated in Lemma 1 are now hold for all functions $u \in H_{BR}$, $v \in H_{BR^+}$.

Lemma 2. *There exists a positive constant* c *such that for any functions* $u \in H_{BR}$, $v \in H_{BR^+}$, *the following inequalities hold:*

$$c^{-1} \| u \|_{L_2} \leq \| Lu \|_{H_{BR^+}^-},$$

$$c^{-1} \| v \|_{L_2} \leq \| L^*v \|_{H_{BR}^-}.$$

Proof. Let's prove the first inequality. First, assume that $u \in C_0^\infty \subset H_{BR}$. Due to Lemma 1, we have $Lu \in H_{BR}^-$, thus, Lu defines a linear continuous functional $(Lu,\cdot)_{H_{BR}}$ on the space H_{BR}. Consider the value of this functional on an element v defined as follows:

$$v(x,t) = \int_t^T e^{-M\tau}\, u(x,\tau)\, d\tau. \tag{12}$$

We will choose the value of the positive constant M later. Note that the function v defined in this way belongs to the space H_{BR^+} and $u = -e^{Mt} v_t$. From the proof of Lemma 2 in Section 6.1 of (Lyashko, 2002), it follows that under the assumptions (8) and (12), for sufficiently large M, the following estimate holds:

$$(L_D u, v)_{L_2(Q)} \ge \frac{M}{4}\int_Q e^{Mt}\, v_t^2\, dQ + \frac{\alpha M}{4}\sum_{i=1}^n \int_Q e^{Mt}\, v_{x_i}^2\, dQ + \frac{\alpha}{2}\int_\Omega \sum_{i=1}^n v_{x_i}^2\,|_{t=0}\, d\Omega. \tag{13}$$

For the integral component Bu, we further examine $(Bu,v)_{L_2(Q)}$.

Let $c_K > 0$ majorize $|K_i(x,t,\tau)|$ and $\left|\left(K_i(x,t,\tau)\right)'_\tau\right|$ in the domain $\Omega \times (0,T) \times (0,T)$. Using (14), we have

$$\left|(Bu,v)_{L_2(Q)}\right| = \left|\sum_{i=1}^n \int_Q \int_0^t K_i\,(x,t,\tau) u_{x_i}(x,\tau)d\tau v_{x_i}(x,t)dQ\right| = \left|\sum_{i=1}^n \int_Q \int_0^t K_i\,(x,t,\tau)e^{M\tau} v_{\tau x_i}(x,\tau)d\tau v_{x_i}(x,t)\, dQ\right|$$

Integrating by parts (here and below K_i' denotes the derivative with respect to τ):

$$\int_0^t K_i\,(x,t,\tau)e^{M\tau} v_{\tau x_i}(x,\tau)d\tau = K_i(x,t,\tau)e^{M\tau} v_{x_i}(x,\tau)\Big|_0^t - \int_0^t \left(K_i(x,t,\tau)e^{M\tau}\right)'_\tau v_{x_i}(x,\tau)d\tau =$$

$$= K_i(x,t,t)e^{Mt} v_{x_i}(x,t) - K_i(x,t,0)v_{x_i}(x,0) - \int_0^t \left(K_i'(x,t,\tau)e^{M\tau} + MK_i(x,t,\tau)e^{M\tau}\right) v_{x_i}(x,\tau)d\tau.$$

Thus,

$$\left|(Bu,v)_{L_2(Q)}\right| = \left|\sum_{i=1}^n \int_Q \int_0^t K_i\,(x,t,\tau)e^{M\tau} v_{\tau x_i}(x,\tau)d\tau v_{x_i}(x,t)\, dQ\right| \le$$

$$\le \sum_{i=1}^n \left|\int_Q \left(K_i(x,t,t)e^{Mt} v_{x_i}(x,t) - K_i(x,t,0)v_{x_i}(x,0)\right) v_{x_i}(x,t)\, dQ\right| +$$

$$+ \sum_{i=1}^n \left|\int_Q \int_0^t \left(K_i'(x,t,\tau)e^{M\tau} + MK_i(x,t,\tau)e^{M\tau}\right) v_{x_i}(x,\tau)d\tau v_{x_i}(x,t)\, dQ\right| \le$$

$$\le c_K \sum_{i=1}^n \int_Q e^{Mt}\, v_{x_i}^2\, dQ + c_K \sum_{i=1}^n \left|\int_Q v_{x_i}\,|_{t=0} v_{x_i}\, dQ\right| + c_K \sum_{i=1}^n \int_Q \int_0^t \left(e^{M\tau} + Me^{M\tau}\right)\left|v_{x_i}(x,\tau)\right| d\tau \left|v_{x_i}(x,t)\right|\, dQ.$$

For the second term, let's apply the inequality $|AB| \le \frac{\varepsilon}{2}A^2 + \frac{1}{2\varepsilon}B^2$:

$$\left|\int_Q v_{x_i}\,|_{t=0} v_{x_i}\, dQ\right| \le \int_Q \frac{\alpha}{4c_K T} v_{x_i}^2|_{t=0} + \frac{c_K T}{\alpha} v_{x_i}^2\, dQ = \frac{\alpha}{4c_K}\int_\Omega v_{x_i}^2\,|_{t=0}\, d\Omega + \frac{c_K T}{\alpha}\int_\Omega v_{x_i}^2\, dQ.$$

Thus,

$$c_K\left|\int_Q v_{x_i}\,|_{t=0} v_{x_i}\, dQ\right| \le \frac{\alpha}{4}\int_\Omega v_{x_i}^2\,|_{t=0}\, d\Omega + \frac{c_K^2 T}{\alpha}\int_\Omega v_{x_i}^2\, dQ. \tag{14}$$

For the third term, let's apply the Cauchy-Schwarz inequality:

$$\int_Q \int_0^t e^{M\tau}\left|v_{x_i}(x,\tau)\right| d\tau \left|v_{x_i}(x,t)\right|\, dQ \le \int_Q \left(\int_0^t e^{M\tau}\, d\tau \int_0^t e^{M\tau}\, v_{x_i}^2(x,\tau)d\tau\right)^{\frac{1}{2}} \left|v_{x_i}(x,t)\right|\, dQ \le$$

$$\le \int_Q \left(\frac{e^{Mt}-1}{M}\int_0^T e^{M\tau}\, v_{x_i}^2(x,\tau)d\tau\right)^{\frac{1}{2}} \left|v_{x_i}(x,t)\right|\, dQ \le \frac{1}{\sqrt{M}}\int_Q e^{Mt/2}\left(\int_0^T e^{M\tau}\, v_{x_i}^2(x,\tau)d\tau\right)^{\frac{1}{2}} \left|v_{x_i}(x,t)\right|\, dQ \le$$

$$\le \frac{1}{\sqrt{M}}\left(\int_Q e^{Mt}\, v_{x_i}^2(x,t)\, dQ \int_Q \int_0^T e^{M\tau}\, v_{x_i}^2(x,\tau)d\tau\, dQ\right)^{\frac{1}{2}} = \sqrt{\frac{T}{M}}\left(\int_Q e^{Mt}\, v_{x_i}^2(x,t)\, dQ \int_Q e^{M\tau}\, v_{x_i}^2(x,\tau)\, dQ\right)^{\frac{1}{2}}.$$

Thus,

$$c_K(1+M)\sum_{i=1}^n \int_Q \int_0^t e^{M\tau}\left|v_{x_i}(x,\tau)\right| d\tau \left|v_{x_i}(x,t)\right|\, dQ \le \frac{c_K(1+M)\sqrt{T}}{\sqrt{M}}\sum_{i=1}^n \int_Q e^{Mt}\, v_{x_i}^2(x,t)\, dQ. \tag{15}$$

Using (13), (14), and (15), we have

$$(Lu,v)_{L_2(Q)} \ge (L_D u, v)_{L_2(Q)} - \left|(Bu,v)_{L_2(Q)}\right| \ge$$

$$\ge \frac{M}{4}\int_Q e^{Mt}\, v_t^2\, dQ + \frac{\alpha M}{4}\sum_{i=1}^n \int_Q e^{Mt}\, v_{x_i}^2\, dQ + \frac{\alpha}{2}\int_\Omega \sum_{i=1}^n v_{x_i}^2\,|_{t=0}\, d\Omega - c_K\sum_{i=1}^n \int_Q e^{Mt}\, v_{x_i}^2\, dQ -$$

$$- \frac{\alpha}{4}\int_\Omega v_{x_i}^2\,|_{t=0}\, d\Omega - \frac{c_K^2 T}{\alpha}\int_\Omega v_{x_i}^2\, dQ - \frac{c_K(1+M)\sqrt{T}}{\sqrt{M}}\sum_{i=1}^n \int_Q e^{Mt}\, v_{x_i}^2(x,t)\, dQ.$$

Thus,

$$(Lu,v)_{L_2(Q)} \ge \left(\frac{M}{4} - c_K - \frac{c_K^2 T}{\alpha}\right)\int_Q e^{Mt}\, v_t^2\, dQ + \left(\frac{\alpha M}{4} - \frac{c_K(1+M)\sqrt{T}}{\sqrt{M}}\right)\sum_{i=1}^n \int_Q e^{Mt}\, v_{x_i}^2\, dQ.$$

Considering that for sufficiently large M, the coefficients in the last inequality will be positive, we have that there exists some $C > 0$ such that the inequality

$$(Lu, v)_{L_2(Q)} \geq C \parallel v \parallel^2_{H_{BR^+}}$$

holds. Similarly to (Lyashko, 2002), applying the Schwarz inequality to the left side and considering that for some $c_0 > 0$, $\parallel v \parallel_{H_{BR^+}} \geq c_0 \parallel u \parallel_{L_2}$, we have

$$\parallel Lu \parallel_{H^-_{BR^+}} \parallel v \parallel_{H_{BR^+}} \geq (Lu, v)_{L_2(Q)} \geq C \parallel v \parallel^2_{H_{BR^+}} \geq C \parallel v \parallel_{H_{BR^+}} \parallel u \parallel_{L_2},$$

from which the first inequality of the lemma for $u \in C_0^\infty$ follows.

For the remaining functions in H_{BR}, the assertion is established passing to the limit. Let $u \in H_{BR}$ (not necessarily smooth). Due to the density of C_0^∞ in H_{BR}, the element u can be approximated by a sequence of elements $u_m \in C_0^\infty$. That is, $\parallel u - u_m \parallel_{H_{BR}} \to 0$, as $m \to \infty$. Due to Lemma 1, we have the inequality

$$\parallel Lu - Lu_m \parallel_{H^-_{BR^+}} \leq c_1 \parallel u - u_m \parallel_{H_{BR}}.$$

Hence, $Lu_m \to Lu$ in the space $H^-_{BR^+}$. Furthermore, for each u_m, the inequality

$$c^{-1} \parallel u_m \parallel_{L_2} \leq \parallel Lu_m \parallel_{H^-_{BR^+}}$$

holds. It remains to note that from the convergence in the space H_{BR} it follows the convergence in L_2, and thus $\parallel u - u_m \parallel_{L_2} \to 0$. Now, passing to the limit as $m \to \infty$ in the last inequality, we obtain the desired result.

The second inequality of the lemma is proved analogously. □

Thus, using Lemmas 1–2, we conclude the following result:

Theorem 1. *For all $u \in H_{BR}$ and $v \in H_{BR^+}$, the following inequalities hold:*

$$\|u\|_{L_2} \leq c_1 \|Lu\|_{H^-_{BR^+}} \leq c_2 \|u\|_{H_{BR}}, \tag{16}$$

$$\|v\|_{L_2} \leq c_1 \|L^* v\|_{H^-_{BR}} \leq c_2 \|v\|_{H_{BR^+}}. \tag{17}$$

3. Generalized solvability. Consider the problem

$$Lu = f,\ f \in H^-_{BR^+}. \tag{18}$$

We will understand solutions of this problem in the following senses (Lyashko, 2002):

Definition 1. A solution to problem (18) with the right-hand side $f \in H^-_{BR^+}$ is an element $u \in H_{BR}$ for which there exists a sequence of functions $u_i \in C_0^\infty$ such that

$$\parallel u - u_i \parallel_{H_{BR}} \to 0, \parallel Lu_i - f \parallel_{H^-_{BR^+}} \to 0,\ i \to \infty.$$

Definition 2. A strong solution to problem (18) with the right-hand side $f \in H^-_{BR^+}$ is a function $u \in H_{BR}$ such that $Lu = f$ in the space $H^-_{BR^+}$.

We can consider the definition of the solution to problem (18) in the broader class H_{BR} as well.

Definition 3. A solution to problem (18) with the right-hand side $f \in H^-_{BR^+}$ is a function $u \in L_2$, for which there exists a sequence of functions $u_i \in C^\infty_{BR}$, $i = 1,2,\ldots$ such that

$$\parallel u - u_i \parallel_{L_2} \underset{i \to \infty}{\to} 0, \quad \parallel Lu_i - f \parallel_{H^-_{BR^+}} \underset{i \to \infty}{\to} 0.$$

Definition 4. A weak solution to problem (18) with the right-hand side $f \in H^-_{BR^+}$ is a function $u \in L_2$ such that the equality

$$(u, L^* v)_{L_2} = (f, v)_{H_{BR^+}},$$

holds for any functions $v \in H_{BR^+}$ such that $L^* v \in L_2$.

Similarly, we will define solutions to the adjoint problem. Based on the inequalities (16)–(17) and according to (Lyashko, 2002), we formulate the theorems of generalized solvability.

Theorem 2. *Definitions 1 and 2 are equivalent. Definitions 3 and 4 are equivalent.*

Theorem 3. *For any element $f \in L_2$, there exists a unique solution to problem (18) in the sense of definitions 1 and 2. For any element $f \in H^-_{BR^+}$, there exists a unique solution to problem (18) in the sense of definitions 3 and 4. These solutions depend continuously on the corresponding right-hand sides.*

Remark 2. Similar theorems can be formulated for the adjoint problem.

4. Optimal control. Now we consider the problem of optimal control for a system whose evolution is described by the equation

$$Lu = f + \mathcal{A}h. \tag{19}$$

Here h is the control belonging to some set of admissible controls $\mathcal{U}$ from the control space $\mathcal{H}$, and $\mathcal{A}$ is some operator. A functional $J(h) = \Phi\big(u(h)\big)$ is defined on the solutions of equation (19), which needs to be minimized under the condition $h \in \mathcal{U}$.

Let's consider two examples of operators $\mathcal{A}$ and indicate the corresponding control spaces for them.

$$\mathcal{A}_1 h = \textstyle\sum_{k=1}^{d} \delta\,(t - t_k) \otimes \varphi_k(x), t, t_k \in [0, T], \varphi_k(x) \in L_2(\Omega). \tag{20}$$

Here

$$h = \{\big(t_k, \varphi_k(x)\big)\}_{k=1}^{d},\ \mathcal{H} = \big([0, T] \times L_2(\Omega)\big)^d.$$

By $\mathcal{A}_1 h$, we mean a functional that acts on smooth functions in $\overline{Q}$ as follows

$$(\mathcal{A}_1 h)(v) = \sum_{k=1}^{d} \int_\Omega v\,(t_k, x) \varphi_k(x)\, d\Omega.$$

Next, for simplicity, let's assume that the domain Ω is cylindrical in the variable x_1, that is, $\Omega = [a, b] \times \Omega'$.

$$\mathcal{A}_2 h = \textstyle\sum_{k=1}^{d} \delta\,(x_1 - x_{1,k}) \otimes \varphi_k(t, x_2, \ldots, x_n), \tag{21}$$

$$x_1, x_{1,k} \in [a, b], \varphi_k(t, x_2, \ldots, x_n) \in L_2\big((0, T) \times \Omega'\big).$$

Here

$$h = \{\big(x_{1,k}, \varphi_k(t, x_2, \ldots, x_n)\big)\}_{k=1}^{d},\ \mathcal{H} = \Big([a, b] \times L_2\big((0, T) \times \Omega'\big)\Big)^d.$$

By $\mathcal{A}_2 h$, we mean a functional that acts on smooth functions in $\overline{Q}$ as follows

$$(\mathcal{A}_2 h)(v) = \sum_{k=1}^{d} \int_{[0,T]\times\Omega'} v\left(t, x_{1,k}, x_2, \dots, x_n\right) \varphi_k(t, x_2, \dots, x_n)\, d\Omega' dt.$$

The obtained a priori inequalities (16), (17) allow us to claim, similarly to (Lyashko, 2002), that the specified operators (20), (21) can be considered as mappings acting from the respective control spaces $\mathcal{H}$ to $H^-_{BR^+}$, are weakly continuous, and the following theorem holds:

Theorem 4. *Let the state of the system be described by the solution of problem (19) with one of the provided operators* $\mathcal{A}$ *and the corresponding control space* $\mathcal{H}$*. Assume that the set of admissible controls* $\mathcal{U}$ *is closed, convex, and bounded in* $\mathcal{H}$*, and the cost criterion* $\Phi(\cdot)\colon H^+_{BR} \to \mathbb{R}$ *is weakly lower semi-continuous with respect to the system state* $u(x,t,h)$ *and bounded from below. Then there exists an optimal control for the system (19).*

Discussion and conclusions

The proven a priori inequalities create a basis for investigation of the qualitative characteristics of hyperbolic systems. The paper presents the application of these inequalities to the existence and uniqueness of the generalized solution, its continuous dependence on the right-hand side, and the existence of optimal control. Moreover, the chosen method of investigation allows, among other aspects, through the proven inequalities (16), (17) to also demonstrate certain differential properties of the quality criterion, the properties of the regularized problem, to construct and to prove the convergence of the numerical methods for evaluation generalized solutions and optimal control, etc.

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Андрій АНІКУШИН, канд. фіз.-мат. наук, доц.
ORCID ID: 0000-0003-1109-6290
e-mail: anik_andrii@univ.kiev.ua
Київський національний університет імені Тараса Шевченка, Київ, Україна

Віктор ЛЯШКО, асп.
ORCID ID: 0000-0003-2647-1188
e-mail: lyashko91@gmail.com
Інститут кібернетики імені В. М. Глушкова НАН України, Київ, Україна

Олександр САМОСЬОНОК, канд. фіз.-мат. наук, наук. співроб.
ORCID ID: 0009-0007-6239-4753
e-mail: samosyonok@gmail.com
Інститут кібернетики імені В. М. Глушкова НАН України, Київ, Україна

ОПТИМАЛЬНЕ КЕРУВАННЯ СИСТЕМАМИ, ЩО ОПИСУЮТЬСЯ ЛІНІЙНИМИ ГІПЕРБОЛІЧНИМИ ІНТЕГРО-ДИФЕРЕНЦІАЛЬНИМИ РІВНЯННЯМИ

Присвячено вивченню гіперболічних інтегро-диференціальних рівнянь із частинними похідними. Інтегро-диференціальні рівняння такого типу давно стали стандартним об'єктом вивчення прикладної математики і часто зустрічаються під час дослідження процесів у в'язкопружних середовищах (аморфні полімери, напівкристалічні полімери, біополімери, метали за дуже високих температур, бітумні матеріали тощо).

Основна мета дослідження полягає у доведенні існування та єдиності узагальнених розв'язків відповідної початково-крайової задачі, а також вивченню існування оптимального керування системами, що описуються відповідними моделями. Основні результати щодо коректності постановки початково-крайової задачі та існування оптимального керування отримано за допомогою методу апріорних нерівностей у негативних нормах. Зокрема, було одержано нерівності в негативних нормах для інтегро-диференціального оператора в деяких спеціальних просторах. Використовуючи результати С. Ляшка та базуючись на доведених апріорних нерівностях, у дослідженні сформульовано різні означення узагальненого розв'язку і наведено результат щодо їх еквівалентності. Наведено теорему про коректність постановки, тобто про існування, єдиність і неперервну залежність узагальненого розв'язку від правої частини рівняння.

Робота починається оглядом релевантних результатів з подібними постановками. Надано посилання на роботи із фізичним обґрунтуванням моделі. Наведено посилання щодо застосування методики апріорних нерівностей для диференціальних та інтегро-диференціальних рівнянь. Також описано постановку задачі, обмеження на параметри рівняння й уведено функціональні простори, необхідні для дослідження. У частині 2 доведено апріорні нерівності, що є центральною частиною роботи. Частина 3 містить означення узагальнених розв'язків та теореми, що описують їхні властивості, зокрема встановлюється коректність постановки початково-крайової задачі. У частині 4 наведено задачу оптимального керування. Керування відбувається через праву частину рівняння за допомогою певного спеціального оператора. У роботі розглянуто деякі приклади таких операторів керування (що ілюструють різноманітні механізми керування) та відповідні простори функцій. Використовуючи доведені нерівності та ґрунтуючись на загальних теоремах з теорії апріорних нерівностей у негативних нормах, у роботі встановлено умови, за яких існують оптимальні керування. Зокрема, накладаються обмеження на допустимий набір керувань і критерій якості в різних сценаріях.

К л ю ч о в і с л о в а : ***апріорні нерівності, лінійне гіперболічне рівняння, в'язкопружність, узагальненні розв'язки, оптимальне керування.***

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