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SCALAR FIELD AT LARGE DISTANCES FROM SPHERICALLY SYMMETRIC STATIC CONFIGURATION

Stationary spherically symmetric space-time in the quasi-global coordinates is considered in presence of scalar field (SF) minimally coupled to gravity, with a monomial potential $V(\phi)=\phi^n$, $n>4$. We prove convergence of an iterative method to solve the joint system of Einstein – SF equations at sufficiently large distances from the center. The result can be used for a numerical solution for the metric and SF by means of backwards integration from large values of the radial variable to smaller ones.
Keywords**:** ***stationary spherically symmetric space-time, scalar field, monomial potential, numerical solution***

1. Introduction

Considerable attention in gravitational physics is paid to general relativistic configurations with various types of scalar field (SF) (see, e.g., [1,2]). In this field, there are a number of models of the spherically symmetric static configurations with non-linear SF that do not admit exact analytic solutions and, therefore, must be studied numerically. The most direct procedure involves a kind of the backwards problem: we use an approximate solution to derive conditions for ordinary initial value problem at sufficiently large distances r from the configuration and extend this solution by some numerical method for lower r. Usually asymptotic solution at spatial infinity is derived "on a physical level of rigor" without necessary justification. We propose such justification that involves a convergent iteration procedure in case of certain SF with monomial potential.

We consider non-linear SF ϕ described by the action

$$S=S_{GR}+\int d^4x\sqrt{|g|}\left[\frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}-V(\phi)\right], \tag{1}$$

where S_{GR} is the standard gravitational action of General Relativity, $V(\phi)$ is the scalar field self-interaction potential: $V(\phi)=\phi^n$, $n>4$.

Metric of static spherically symmetric space-time in the quasi-global coordinates is

$$ds^2=A(x)dt^2-\frac{1}{A(x)}dx^2-r^2(x)\left(d\theta^2+\sin^2(\theta)d\varphi^2\right). \tag{2}$$

The condition $n>4$ is essential because in this case we expect that $\phi(x)\approx\chi_0/x$ on large distances. For $n\le 4$ the asymptotics is different. The case $n=2$ (linear massive scalar field has been treated in [3]).

The non-trivial Einstein equations following from (1) are

$$\frac{d}{dx}\left(\frac{dA}{dx}r^2\right)=-2r^2V(\phi),\ \frac{d^2r}{dx^2}+\frac{1}{2}r\left(\frac{d\phi}{dx}\right)^2=0,\ A\frac{d^2r^2}{dx^2}-r^2\frac{d^2A}{dx^2}=2. \tag{3}$$

The equation for SF (covariant D'Alembert equation with the potential $V(\phi)$ in case of static field)

$$\frac{d}{dx}\left(r^2A\frac{d\phi}{dx}\right)=r^2\frac{dV}{d\phi}, \tag{4}$$

The equations (3, 4) are not independent; further we use the first and the third equations of (3) and equation (4).

2. Asymptotic conditions and transition to the integral equations

We consider asymptotic conditions at spatial infinity as follows

$$A(x)=1-\frac{2m}{x}+O\left(\frac{1}{x^2}\right), \tag{5a}$$

where $m>0$ is the configuration mass,

$$r^2(x)=x^2\left(1+\frac{C_r}{x}+\frac{D_r}{x^2}+\ldots\right); \tag{5b}$$

$$\phi(x)=\frac{\chi_0}{x}+O\left(\frac{1}{x^2}\right), \tag{5c}$$

where we put $C_r=0$, which is related to the choice of coordinates. In view of these conditions, we can pass to the system of integral equations that can be solved iteratively. Equations (3) yield

$$r^2\frac{dA}{dx}=2m+2\int_x^\infty dx\, r^2V(\phi),$$

$$A\frac{dr^2}{dx}-r^2\frac{dA}{dx}=2x+C.$$

In view of asymptotic conditions, the coefficient $C=-6m$ does not depend on x_0. Combining these equations we have

$$r^2(x)=\mathbf{R}(x,\phi,r), \tag{6a}$$

$$A(x)=\mathbf{A}(x,\phi,r), \tag{6b}$$

$$\phi(x)=\mathbf{\Phi}(x,\phi,r), \tag{6c}$$

where

$$\mathbf{A}(x,\phi,r)=1-\int_x^\infty dx'\frac{2m+2\int_{x'}^\infty dx''r^2V(\phi)}{r^2(x')}, \tag{7a}$$

$$\mathbf{R}^2(x,\phi,r)=r^2(x_0)+\int_{x_0}^x dx'\frac{1}{\mathbf{A}(x,\phi,r)}\left[2x'-4m+2\int_{x'}^\infty dx''\,r^2V(\phi)\right], \tag{7b}$$

$$\Phi(x,\phi,r)=\int_x^\infty dx'\frac{\chi_0+\int_{x'}^\infty dx''\,r^2(x'')\frac{dV}{d\phi}}{R^2(x')A(x',\phi,r)}. \tag{7c}$$

3. Definition of the class of functions and contraction mapping

We are looking for the solution $\{r(x),A(x),\phi(x)\}$ of integral equations (7) on $[x_0,\infty)$ with given χ_0, $m>0$ and $r^2(x_0)$ satisfying conditions $ax_0^2<r^2(x_0)<bx_0^2$, $0<a=1-\varepsilon<1$, $b>1$. We fix x_0 (that affects only the choice of the radial variable); x_0 must be sufficiently large.

In view of the asymptotic conditions (5), we consider the set S of continuously differentiable functions satisfying for $x\ge x_0$

$$1-\varepsilon_1\le A(x)\le 1+\varepsilon_1, \tag{8a}$$

$$\frac{a}{3}x^2<r^2(x)\le 3bx^2, \tag{8b}$$

$$|\phi(x)|\le\frac{c|\chi_0|}{x},\quad c>1, \tag{8c}$$

where

$$0<\varepsilon_1<1,\ \ 1<b(1-\varepsilon_1),\ c=3+\varepsilon_2. \tag{8d}$$

Here $\varepsilon,\varepsilon_1,\varepsilon_2$ will be assumed to be sufficiently small in view of the asymptotic conditions (4).

For integral equations (7a-c) we have to estimate the r.h.s. of (7) for functions satisfying (8). Now denote $\mathcal{A}=\{\mathbf{A},\mathbf{R},\Phi\}$ – operator of the r.h.s. of equations (7). We must show that $\mathcal{A}(S)\subset S$ and the mapping $\mathcal{A}:S\Rightarrow S$ is a contractor.

After some estimates on account of (8), we have ($x\ge x_0>>m$)

$$\mathbf{R}^2(x,\phi,r)<r^2(x_0)+\frac{x^2}{1-\varepsilon_1}\left[1+6\frac{bc^n\chi_0^n}{(n-3)\,x_0^{n-2}}\right]<bx^2+\frac{x^2}{1-\varepsilon_1}+\frac{x^2}{1-\varepsilon_1}\left[6\frac{bc^n\chi_0^n}{(n-3)\,x_0^{n-2}}\right],$$

$$\mathbf{R}^2(x,\phi,r)>ax_0^2+\frac{(x-x_0)^2}{(1+\varepsilon_1)}-(x-x_0)x_0\left[\frac{4m}{x_0}+\frac{2}{1-\varepsilon_1}\frac{3n-5}{n-1}\frac{1}{x_0^{n-2}}(c\chi_0)^n\,b\right],$$

whence for a sufficiently large x_0 we have $ax^2/3<\mathbf{R}^2(x,\phi,r)\le 3bx^2$.

Using (8) we have

$$|\mathbf{A}(x,\phi,r)-1|\le\frac{3}{ax_0}\cdot\left[2m+6\frac{bc^n\chi_0^n}{(n-3)\,x_0^{n-3}}\right],$$

which is smaller than any prescribed ε_1 in case of sufficiently large x_0. Thus $A(x)$ fulfills (8a).

We shall now estimate $|\mathbf{\Phi}|$ using (8)

$$|\mathbf{\Phi}(x,\phi,r)|\le\frac{3}{x}\left[|\chi_0|+\left|\frac{3bn}{(n-4)x_0^{n-4}}|c\chi_0|^{n-1}\right|\right].$$

Then, $|\mathbf{\Phi}|$ fulfills (8c,d) for $x>x_0$ provided that we choose

$$\left|\frac{9bn}{(n-4)x_0^{n-4}}|c\chi_0|^{n-1}\right|<\varepsilon_2; \tag{9}$$

this choice being possible for sufficiently large x_0.

Thus, the set of functions of $C[x_0,\infty)$ satisfying (8) is invariant with respect to the mapping $\mathcal{A}$ defined by the right hand sides of (7).

Now we estimate the contractive properties of $\mathcal{A}: S \Rightarrow S$. We denote $\delta f(x) \equiv f(x) - f'(x)$, where $f(x), f'(x) \in S$.

After cumbersome estimations on account of (8), we get

$$\left|\mathbf{A}(x,\phi',r') - \mathbf{A}(x,\phi,r)\right| \le s_1 \left\| x^{-2}\delta r^2(x)\right\| + s_2\chi_0^{-1}\left\| x\delta\phi(x)\right\|, \tag{10a}$$

where dimensionless $s_1 = O\left(\frac{m}{x_0}\right) + O\left[|\chi_0|^2\left(\frac{\chi_0}{x_0}\right)^{n-2}\right]$, $s_2 = O\left[\left[|\chi_0|^2\left(\frac{\chi_0}{x_0}\right)^{n-2}\right]\right]$ tend to zero for $x_0 \to \infty$ (dimensions: $\chi_0 \sim x;\quad w \sim x^{-2}$); $\left\|f(x)\right\| \equiv \sup\{|f(x)|,\ x \ge x_0\}$.

Next, using (8) we get

$$x^{-2}\left|\mathbf{R}^2(x,r^2,\Phi) - \mathbf{R}^2(x,r'^2,\Phi')\right| \le \left\{ s_3\left\| x^{-2}\delta r^2(x)\right\| + s_4\chi_0^{-1}\left\| x\delta\phi(x)\right\|\right\}, \tag{10b}$$

$s_i = s_i(x_0) \to 0$ for $x_0 \to \infty$.

Analogously for $\Phi(x,\phi,r)$, we get

$$\left|\Phi(x,\phi,r) - \Phi(x,\phi',r')\right| \le \frac{\chi_0}{x}\left\{ s_5\left\|\frac{\delta r^2}{x^2}\right\| + s_6\chi_0^{-1}\left\| x\delta\phi\right\|\right\} \le$$

$$\le \frac{\chi_0}{x}\max(s_5,s_6)\left\{\left\|\frac{\delta r^2}{x^2}\right\| + \chi_0^{-1}\left\| x\delta\phi\right\|\right\}, \tag{10c}$$

where $s_i = s_i(x_0) \to 0,\quad x_0 \to \infty$.

Thus, we see from (10) that the mapping $\mathcal{A}$ is contractive in the norm

$$\left\| A(x)\right\| + \left\|\frac{r^2(x)}{x^2}\right\| + \chi_0^{-1}\left\| x\phi(x)\right\|, \tag{11}$$

where $x_0 \to \infty$. This finishes the consideration of convergence of the iteration sequence obtained with operator $\mathcal{A}$.

By the end, we note that consideration with the norm can be expected to work in much wider situation, e.g., in case of a superposition of several SFs.

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СКАЛЯРНЕ ПОЛЕ НА ВЕЛИКИХ ВІДСТАНЯХ ВІД СФЕРИЧНО СИМЕТРИЧНОЇ СТАТИЧНОЇ КОНФІГУРАЦІЇ

Розглянуто нелінійне скалярне поле з дією

$$S = S_{GR} + \int d^4x\sqrt{|g|}\left[\frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(\phi)\right],$$

де S_{GR} ***– стандартна гравітаційна дія загальної теорії відносності, потенціал самодії скалярного поля є мономним:*** $V(\phi) = \phi^{2n}$, $n > 2$. ***Умова на степінь*** n ***пов'язана із асимптотикою поля на великих відстанях.***

Розглянуто сферично симетричні розв'язки системи рівнянь Ейнштейна та скалярного поля у просторі-часі у квазіглобальних координатах із метрикою

$$ds^2 = A(x)dt^2 - A^{-1}(x)dx^2 - r^2(x)\left(d\theta^2 + \sin^2(\theta)d\varphi^2\right).$$

Система – це три звичайні диференціальні рівняння; її зведено до інтегральних рівнянь, зручних для проведення ітераційної процедури, яка не виводить із множини S ***неперервно диференційовних функцій, що задовольняють певні обмеження, зокрема*** $1-\varepsilon_1 \le A(x) \le 1+\varepsilon_1$, $ax^2/3 < r^2(x) \le 3bx^2$, $|\phi(x)| \le c|\chi_0|/x$, ***де константи*** $\varepsilon_1, a, b, c, \chi_0$ ***задовольняють певні вимоги. На основі принципу стискаючих відображень показано збіжність ітераційної процедури, яка дає розв'язок на достатньо великих відстанях від центра. Результат може бути використано для числового знаходження метрики та скалярного поля шляхом оберненого інтегрування від великих до менших значень радіальної змінної.***

Ключові слова: сферично симетричний простір-час, скалярне поле, мономний потенціал, числове інтегрування