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REEB GRAPH OF THE HEIGHT FUNCTION ON A PLANAR POLYGON

Height functions, which are Morse functions of the general position, are often used when studying the structure of manifolds. The structure of such functions is described using the Reeb graph. On two-dimensional oriented closed manifolds, as well as on compact domains with a smooth boundary, the Reeb graph is a complete topological invariant of a simple Morse function. Its vertices have degree 1 if the vertex corresponds to a local extremum, or 3 if the vertex corresponds to a saddle critical point. For the height function on a polygon, we consider a Reeb graph whose vertices coincide with the vertices of the polygon. In this case, in addition to vertices of degrees 1 and 3, the Reeb graph will also have vertices of degree 2, which correspond to the regular vertices of the polygon. We show that the Reeb graph of a polygon can be constructed in a time no less than O(n log n), which is the best possible for many computational geometry problems. In addition, the Reeb graph can be embedded as a rectilinear graph in a polygon. This allows you to construct a division of a polygon into monotone polygons with their subsequent triangulation. We have also estab-lished the connection between the Reeb graph of the polygon and the Reeb graph of the height function on the smoothed axis 3D thickening, which opens up the possibility of using these structures to build the skeleton of 3D models with its further use in computer graphics. We give an example of constructing a Reeb graph using the process of planar sweeping with a straight line and subsequent triangulation of a polygon. The obtained results can also be used to study the properties of Reeb graphs of combinations of polygons in three-dimensional space. It is also promising to find all possible Reeb graphs of polygons with a small number of vertices.

K e y w o r d s : ***Morse function, topological structure, trapezoidal map, rectilinear graph.***

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Introduction

Relevance of the research. To study the topological properties of functions on surfaces G. Reeb (Reeb, 1954) introduced a graph obtained by compressing each level component of the function to a point. More formally, the Reeb graph of a smooth function on a surface is the quotient space of the surface according to the equivalence relation, in which two points are equivalent if they belong to the same connectivity component of the level line of the function. For functions in the general position on a closed surface, which are simple Morse functions, the Reeb graph is a complete topological invariant, which means that it completely describes the topological structure of the division of the surface into layers of the function (Cole-McLaughlin et al., 2004). The direction of growth of the function gives rise to the orientation of the edges of the Reeb graph. The Reeb graph is also used to study mappings of a closed surface onto a circle (Prishlyak, 2002), as well as functions on compact surfaces with a boundary (Hladysh, & Prishlyak, 2019), in particular for functions with critical points on the boundary in (Hladysh, & Prishlyak, 2016). The properties of the Reeb graph on non-compact surfaces are described in (Polulyakh, 2016), as the function changes over time, so does its Reeb graph. Such typical changes are described in papers (Hladysh, & Prishlyak, 2020; Maksymenko, 2020). If an oblique gradient field (Hamiltonian dynamical system) is considered for a function on the surface, then its trajectories coincide with the layers of the function. Therefore, the Reeb graph is a topological invariant of such fields. Trajectory equivalence of Hamiltonian flows generates automorphisms of Reeb graphs (Kravchenko & Maksymenko, 2020). Also, the Reeb graph can be used for gradient flows, as well as for flows on a surface with both a gradient and a Hamiltonian structure (Prishlyak, & Loseva, 2020). When the dimension increases, the structure of functions and flows becomes significantly more complicated, and two-dimensional complexes are used instead of Reeb graphs, as, for example, in (Prishlyak, 2002) for Morse-Smale flows.

For a simple polygon, the height function is piecewise linear on the boundary of the polygon. After smoothing the boundary, we get a smooth function, which in general is a Morse function. Therefore, the topological properties of the Reeb graph of the height function on the polygon are similar to the topological properties of smooth functions. The possibility of using the Reeb graph in the problem of polygon triangulation and construction of the skeleton of a region or a three-dimensional body makes the task of describing the topological properties of such graphs relevant not only for topology, but also for computational geometry and computer graphics.

The object of research of this article is the topological properties of Reeb graphs on flat polygons. The aim of the paper is to describe an effective algorithm for constructing a Reeb graph in which the set of vertices coincides with the set of polygon vertices and to investigate the possibility of making this graph rectilinear. Also, the purpose of the work is to describe the possible application of the Reeb graph in the problem of dividing a polygon into monotone polygons.

To construct the Reeb graph, we use the method of planar sweeping with a straight line, which is effective in many problems of computational geometry. We also use methods of graph theory, in particular topological graph theory for nested graphs on a plane.

1. Main results

Let's fix the right rectangular Cartesian coordinate system x, y and the height function $h(x, y) = y$ on the plane. A polygon is in general position to a function h if the function takes different values at different vertices. Next, we consider just such polygons.

Let's divide all vertices of the polygon into five types:

- *start* vertices (start vertex) or V-vertices are the lowest points for the edges of the polygon at these points;
- *end* vertices (end vertex) or Λ-vertices are the highest points for the edges of the polygon at these points;
- *split vertex* or Y-vertex are the lowest points for two edges adjacent to it and are not the initial vertex;
- merge *vertex* or ⅄-vertex are the highest points for two edges adjacent to it and are not the final vertex;
- *regular vertices* – they are the lower end of one side and the upper end of the other side.

At the same time, regular vertices are divided into two types – left, if the surface lies to the right of them, and right, otherwise.

A *Reeb* graph can be defined as some graph (perhaps not rectilinear) that intersects each line component of the level $h^{-1}(c)$ at one point. At the same time, the vertices correspond to the vertices of the polygon. Here we consider that no more than one vertex of the polygon lies on one level of the function. If this is not the case, then we can turn the coordinate system clockwise by a small enough angle to satisfy this condition. Otherwise, this situation can be processed as follows: instead of the height function, we consider a discrete function, consistent with the lexicographic order of the vertices by the second coordinate.

In particular, if two vertices have the same height, belong to the same component of the level and there are no other vertices between them in this component, then they are connected by an edge on the Reeb graph and this edge is oriented from the left vertex to the right. Other edges of the graph are oriented according to the direction of growth of the function (from bottom to top).

The vertices of the Reeb graph, as well as the vertices of the polygon, are divided into 5 types:

- sources are vertices of degree 1 from which the edge originates. They correspond to the initial vertices of the polygon.
- sink are vertices of degree 1 into which the edge goes. They correspond to the final vertices of the polygon.
- Y-vertices are vertices of degree 3, which include one and two outgoing edges. They correspond to the peaks of the cleavage.
- ⅄-vertices are vertices of degree 3, in which two edges enter and one exits. They correspond to the vertices of fusion.
- regular vertices are vertices of degree 2 in which one edge enters and one edge exits. They correspond to the regular vertices of the polygon.

Theorem 1. *The Reeb graph can be constructed in O(n+ k log n) time, where k is the minimum number of merging vertices and splitting vertices that does not exceed O(n log n) time.*

Theorem 2. *A Reeb graph is embedded as a rectilinear graph in a polygon so that its vertices coincide with the vertices of the polygon.*

Theorem 1 can be applied in the problem of polygon triangulation.

A Y-monotone polygon is a polygon whose intersection with an arbitrary horizontal line is connected (that is, an empty set, a point, or a segment). An n-gon is Y-monotone if and only if it has no splitting vertices and no merging vertices. The triangulation of a Y-monotone n-gon can be performed in O(n) time (Preparata, & Shamos, 1985).

So, to split a polygon into Y-monotone polygons, you need to draw a diagonal from each merging vertex to the vertex above it, and for each splitting vertex to the vertex below it.

After constructing the Reeb graph, the diagonals are drawn as follows: for Y-vertices, we consider the edges included in them. We find pairs of vertices on the polygon corresponding to the ends of these edges and connect them with segments. Similarly, other segments on the polygon are constructed for ⅄-vertices and edges coming from them.

Theorem 3. *The set of segments constructed in this way are diagonals that do not cross each other and divide the polygon into monotone polygons*.

Theorem 4. If the polygon in the xz *plane is thickened by the y coordinate and smoothed by the Catmull – Clarke method, then the Reeb graph on the resulting surface can be obtained from the Reeb graph of the polygon by dissolving all vertices of degree 2.*

Here, by the dissolution of a vertex of degree 2, we mean the removal of this vertex and the replacement of two edges incident to it by one edge.

It follows from Theorem 2 that the Reeb graph is a skeleton of a polygon, and can also be used to construct a skeleton of smoothed thickening, which can be used in 2D and 3D animation.

2. Proofs

Proof of Theorem 1. The Reeb graph is constructed starting from the lowest points by the sweeping line method. At the same time, each of the vertices and edges included in it, except for Y-vertices, can be built in a constant time. For Y-vertices, it is necessary to decide between which vertices (edges) that have already been built, it is located. This takes O (log n) time. So, everything is needed O (n + k log n) time, where k is the minimum number of merging vertices and splitting vertices. This is no faster than O (n log n).

□

Proof of Theorem 2. Let us take the corresponding vertices of the polygon as the vertices of the Reeb graph. We will prove that if two vertices belong to the same edge, then the segment connecting them belongs to the polygon. Let's cut the polygon with horizontal straight lines passing through these vertices (See Fig. 1). By construction, the prototype of each Reeb is a trapezoid or a triangle (degenerate trapezium). Since the trapezoid and the triangle are convex sets, the segment connecting the two vertices belongs to them, and therefore it also belongs to the polygon.

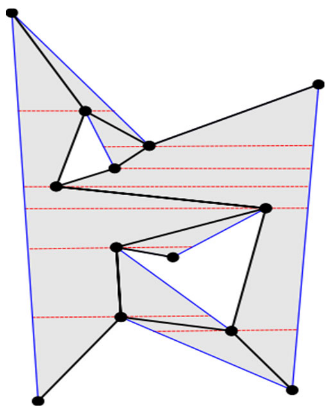


Fig. 1. Sweeping (dash red horizontal) line and Reeb graph (black)

□

Proof of Theorem 3. Consider the projection of a polygon onto a Reeb graph. Consider the preimage of an edge included in the Y - vertex. The Y- vertex is the inner point of the upper base. Therefore, the segment connecting them to the lower point lies inside the trapezoid, and therefore inside the polygon. Therefore, it is a diagonal. Since we chose different edges, the constructed diagonals lie in different trapezoids and, therefore, do not intersect. After diagonal cutting, the merging and splitting vertices in the resulting polygons will be regular vertices, so these polygons are monotonic.

□

Proof of Theorem 4. Let the thickening occur in both directions as multiplication by the segment $[-1,1]$. Then the plane y=0 is the plane of symmetry of both the thickening and the smoothed surface. Therefore, the formation of the critical point of the height function lies on it. Therefore, the directions of the coordinate axes are the main directions at the critical points. For the point of the local minimum (V-vertex), smoothing along the x- axis from V gives U. Smoothing along the y-axis from П will also give U. Therefore, we will get a single critical point on the surface — the point of the local minimum. Similarly, inverted V vertices give rise to local maximum points. For Y vertices, after smoothing the thickening in one direction, we get a U, and in the other – an inverted U. This corresponds to the saddle point of the height function on the surface.

The regular vertices of the polygon after smoothing the thickening will become regular points of the height function, so they will no longer be vertices of the Reeb graph. Therefore, the Reeb graph of the smoothed thickening is obtained by dissolving such vertices.

□

2. An example

Let's construct the Reeb graph, the division into Y-monotone polygons and the triangulation of the polygon with vertices $P_1(0,0)$, $P_2(4,-4)$, $P_3(10,0)$, $P_4(10,5)$, $P_5(8,3)$, $P_6(6,5)$, $P_7(8,1)$, $P_8(4,-3)$, $P_9(2,0)$, $P_{10}(4,-1)$, $P_{11}(6,1)$, $P_{12}(2,5)$. It is shown in Fig. 2.

We will use the straight-line sweeping method to construct the Reeb graph. First, we arrange the vertices according to the second coordinate lexicographically: P_2, P_8, P_{10}, P_1, P_9, P_3, P_{11}, P_7, P_5, P_{12}, P_6, P_4. In this order, we will enter these vertices into the queue of events.

We denote the edge P_iP_{i+1} by e_i if $P_i < P_{i+1}$ and by $e_{\ i}$ otherwise.

We define the type of vertices: 1) regular vertices – P_0, P_3, P_7, P_{11}; 2) V-vertices – P_2, P_{10}; 3) Λ-vertices – P_4, P_6, P_{12}; 4) Y-vertices – P_8, P_5; 5) ʎ-vertex – P_9.

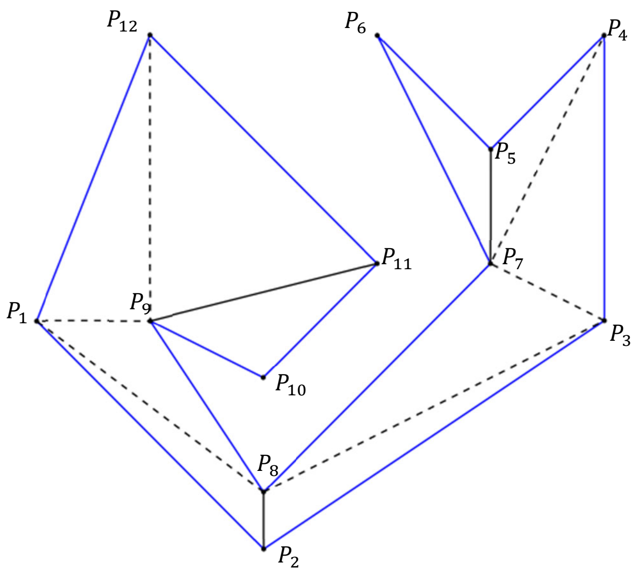
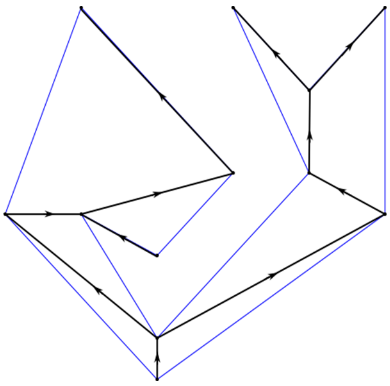


Fig. 2. Triangulation of the polygon

Status *S* contains an ordered set of edges crossed by a horizontal line. After completing the events (the vertex), the following status changes will occur:



P_2: $S = e_1, e_2$ enter the vertex v_2;
P_8: $S = e_1, e_8, e_7, e_2$, introduce the edge $v_2 \to v_8$;
P_{10}: $S = e_1, e_8, e_9, e_{10}, e_7, e_2$, enter the vertex v_{10};
P_1: $S = e_{12}, e_8, e_9, e_{10}, e_7, e_2$, introduce the edge $v_8 \to v_1$;
P_9: $S = e_{12}, e_{10}, e_7, e_2$, $v_1 \to v_9$, $v_{10} \to v_9$;
P_3: $S = e_{12}, e_{10}, e_7, e_3$, $v_8 \to v_3$;
P_{11}: $S = e_{12}, e_{11}, e_7, e_3$, $v_9 \to v_{11}$;
P_7: $S = e_{12}, e_{11}, e_6, e_3$, $v_3 \to v_7$;
P_5: $S = e_{12}, e_{11}, e_6, e_5, e_4, e_3$, $v_7 \to v_5$;
P_{12}: $S = e_6, e_5, e_4, e_3$, $v_{11} \to v_{12}$;
P_6: $S = e_4, e_3$, $v_5 \to v_{6;}$;
P_4: $S = \emptyset$, $v_5 \to v_4$.

Fig. 3. Reeb graph of the polygon

So, we obtain Reeb graph as a union of oriented edges (see Fig. 3).

Next, we find the edges included in the Y-vertex: $v_2 \to v_8$, $v_7 \to v_5$, and the edge coming from the ʎ-vertex -- $v_9 \to v_{11}$. Therefore, we draw the following diagonals: P_2P_8, P_7P_5, P_9P_{11}. They split the initial polygon into four Y-monotone polygons: 1) $P_1P_2P_8P_9P_{11}P_{12}$, 2) $P_2P_3P_3P_5P_7P_8$, 3) $P_5P_6P_7$, 4) $P_9P_{10}P_{11}$ (see Fig. 2). We triangulate these polygons by cutting off the ears from the bottom up, draw the following diagonals: 1) P_8P_1, P_1P_9, P_9P_{12}, 2) P_8P_3, P_3P_7, P_7P_4, and in the third and fourth diagonals do not need to be drawn, because they are already triangles.

Discussion and conclusions

The topological properties of the Reeb graph of the height function on flat polygons was described. They have certain differences from the Reeb graph of Morse functions on closed surfaces. The proposed method of constructing triangulations based on Reeb graphs can be an alternative to already known methods (Tereshchenko et al., 2013; Tereshchenko, V., & Tereshchenko, Y., 2017). The results are useful for building visual models (Tereshchenko, 2009). In the future, we consider it promising to solve the following problems: 1) finding all Reeb graphs of polygons with a small number of vertices, 2) describing the properties of Reeb graphs of combining polygons, 3) describing the necessary and sufficient conditions under which a Reeb graph defines a trapezoidal map.

Authors' contribution: Alexandr Prishlyak – conceptualization; methodology; empirical research; Vasyl Tereshchenko – empirical data collection and validation; software; analysis of sources, preparation of literature review and theoretical foundations of research.

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ГРАФ РІБА ФУНКЦІЇ ВИСОТИ НА ПЛАНАРНОМУ ПОЛІГОНІ

Під час дослідження структури многовидів часто використовують функції висоти, що є функціями Морса загального положення. Це функції, у яких усі критичні точки є невиродженими. Структуру таких функцій описують за допомогою графа Ріба. На двовимірних орієнтованих замкнених многовидах, так само як і на компактних областях із гладкою межею, граф Ріба є повним топологічним інваріантом простої функції Морса. Його вершини мають степінь 1, якщо вершина відповідає локальному екстремуму, або 3, якщо вершина відповідає сідловій критичній точці. У запропонованому дослідженні для функції висоти на багатокутнику розглянуто граф Ріба, вершини якого збігаються з вершинами багатокутника. У цьому випадку у графа Ріба, крім вершин степенів 1 та 3, будуть ще вершини степеня 2, що відповідають регулярним вершинам багатокутника. Показано, що граф Ріба багатокутника може бути побудований за час не менший ніж O(n log n), що є найкращим можливим для багатьох задач обчислювальної геометрії. Крім того, граф Ріба можна вкласти як прямолінійний граф у багатокутник. Це дає змогу побудувати розбиття багатокутника на монотонні багатокутники з подальшою їхньою тріангуляцією. Також нами встановлено зв'язок графа Ріба багатокутника з графом Ріба функції висоти на згладженому 3D-потовщенні, що відкриває можливість застосування цих конструкцій для побудови кістяка 3D-моделей з подальшим його використанням у комп'ютерній графіці. Наведено приклад побудови графа Ріба за допомогою процесу плоского замітання прямою, та подальшої тріангуляції багатокутника. Отримані результати також можуть бути використані для вивчення властивостей графів Ріба об'єднань багатокутників у тривимірному просторі. Також перспективним є знаходження всіх можливих графів Ріба багатокутників з невеликим числом вершин.

К л ю ч о в і с л о в а : ***функція Морса, топологічна структура, трапецеїдальна карта, прямолінійний граф.***

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