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CIRCULAR ORBITS AROUND STATIC SPHERICALLY SYMMETRIC CONFIGURATION WITH LINEAR MASSIVE SCALAR FIELD IN GENERAL RELATIVITY

Background. Theories with scalar fields are widely discussed in various generalizations of the Standard cosmological model dealing with the inflation of the Universe, dynamical dark energy, Hubble tension etc. Introduction of a scalar field in static spherically symmetric (SSS) asymptotically flat gravitational system causes the appearance of specific features in the distribution of stable circular orbits, which may lead to observational effects in the accretion disk structure. We study this distribution in the case of a massive scalar field in the framework of General Relativity.

Methods. We consider the Einstein equations with minimally coupled massive scalar field resulting in the ordinary differential system that is solved numerically (SSS case). This solution is applied to study the distribution of stable circular orbits of test bodies using properties of an effective potential for the test-body motion.

Results. There are two main regions around the asymptotically flat SSS configuration with scalar field: a weak-field region, where we have an approximate analytical description of the problem and usual distribution of circular orbits as in case of the Schwarzschild metric, and a scalarization region with a significant deviation of the metric from the Schwarzschild case. In the scalarization region, a numerical solution for the linear scalar field with the corresponding metric coefficients is derived. Analytical relations to study the test body circular orbits are obtained. We show that, for a wide range of masses of the linear scalar field and astrophysically relevant configuration masses, there either exists a region of weakly stable circular orbits near the center, or there are no circular orbits in the scalarization region at all.

Conclusions. The distribution of stable circular orbits is significantly different from the case of a Schwarzschild black hole. In case of sufficiently large masses of the scalar field and configuration masses, the circular orbits in the scalarization region will be virtually absent. This suggest a similar conclusion about the structure of accretion disks.

Keywords: relativistic objects, naked singularities, modified gravity.

Background

The standard Λ CDM cosmological model satisfactorily describes a large amount of astronomical observations. However, there are a number of facts, such as the Hubble constant tension (see, e.g., Dainotti et al., 2023) etc, which suggest that a new physics is required beyond the Λ CDM model to explain a number of astronomical phenomena. In this view, the scalar field (SF) theories are attracting considerable attention, especially in models of the dynamical dark energy (see e.g., Li et al, 2013; Mortonson, Weinberg, & White, 2013). Natural questions arise about the possible effects of SFs in relativistic astrophysical objects. For example, these may be non-connected ring-like structures in the distribution of circular orbits (cf., e.g., Stashko, Zhdanov & Alexandrov, 2021), which can manifest itself through the structure of thin accretion disks.

In this paper we study circular orbits around static spherically symmetric (SSS) configurations with minimally coupled scalar field in General Relativity. As distinct from the paper by Stashko, Zhdanov & Alexandrov (2021), which dealt with massless power-law SF potentials, here we consider the solutions describing massive SF.

Methods

The Einstein equations for the metric¹ $g_{\alpha\beta}$ with an additional equation for SF ϕ are

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = \kappa T_{\alpha\beta}; \quad \nabla_{\alpha}\nabla^{\alpha}\phi = -V(\phi); \quad \kappa = \frac{8\pi G}{c^4}, \quad (1)$$

where we assume $V(\phi) = \mu^2 w(\phi)$; $\mu = 1/l_{\mu}$ stands for the SF mass.

For a SSS space-time we use the Schwarzschild (curvature) coordinates with

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (2)$$

After introduction of a new radial variable $x = \mu r$ the parameter μ disappears from the Einstein equations and field equations yielding

$$\frac{d}{dx} \left(\frac{\nu + \lambda}{2} \right) = \frac{\kappa}{2} x \left(\frac{\partial \phi}{\partial x} \right)^2, \quad (3)$$

$$\frac{d}{dx} \left(\frac{\nu - \lambda}{2} \right) = -\frac{1}{x} + \frac{e^{\lambda}}{x} [1 - \kappa x^2 w(\phi)], \quad (4)$$

¹ We use the metric signature (+ - - -); Riemann tensor is $R^{\alpha}_{\beta\gamma\delta} = \partial_{\gamma}\Gamma^{\alpha}_{\beta\delta} - \dots$, Ricci tensor is $R_{\beta\delta} = R^{\alpha}_{\beta\alpha\delta}$. In the numerical calculations we put $c = G = 1, \kappa = 8\pi$. Note that in the geometrized units μM is a dimensionless quantity.

$$\frac{e^{-(\nu+\lambda)/2}}{x^2} \frac{\partial}{\partial x} \left[e^{(\nu-\lambda)/2} x^2 \frac{\partial \phi}{\partial x} \right] = w'(\phi), \quad w'(\phi) = \frac{dw}{d\phi}, \quad (5)$$

where the unknown functions $\nu(x), \lambda(x), \phi(x)$ depend only on radial variable. Equations (3),(4) are obtained from the "00" and "11" components of the Einstein equations in case of the SSS metric (2); equation (5) follows from the Klein-Gordon equation. The other Einstein equations are not independent from (4),(5) and (6).

Approximations and initial conditions. Now we focus on the linear massive SF with $w(\phi) = \phi^2/2$. In case of this potential one can use the results of Zhdanov & Stashko (2020), which show that non-trivial solution of (3)–(5) exists and is regular for all $x > 0$ and there is a naked singularity at the center $x = 0$. Moreover ϕ and $\phi'(x)$ do not change their signs; therefore we can confine ourselves to the case $\phi > 0$ and $\phi'(x) < 0$ for all x .

In case of an asymptotically flat isolated configuration we expect that at large distances from the center the field decays exponentially $\phi(x) \propto e^{-x}$ and we have approximately the Schwarzschild metric. Let M be a mass of an isolated static system; we denote $X_g = \mu r_g = 2\mu M$. Note that for typical SF models and astrophysically relevant masses the value of $\mu M \equiv M/l_\mu \gg 1$ is very large; this will be assumed below.

For a sufficiently large $x \gg X_g$

$$e^\nu = 1 - \frac{X_g}{x} = e^{-\lambda}. \quad (6)$$

Taking into account (6), the asymptotic solution of (5) in terms of x – variable (cf. Asanov, 1968; Rowan, & Stephenson, 1976; Stashko, & Zhdanov, 2019) is

$$\phi(x) = Q \left(\frac{X_g}{x} \right)^{1+\mu M} e^{-x}, \quad \frac{d\phi}{dx} = -\phi, \quad x \gg X_g, \quad (7)$$

where the constant Q will be called scalar "charge". Let $r_0 \gg r_g$ is a size of the "scalarization" region where we expect a significant deviations from (6),(7), and for $x > X_0 \equiv \mu r_0 \gg X_g$ the approximation (6),(7) is valid. The exact value of r_0 will be specified later.

Evidently, in the weak field region the effect of SF on the metric must be small, so we demand $|\phi(x)| \ll 1$. Moreover, one must estimate a next-order correction to (6). An analogous problem has been considered in Appendix A of (Zhdanov, 2025). Similar considerations show that corrections to e^ν, e^λ are of the order $\kappa x \phi^2$. Therefore, in order that approximate formulas (6),(7) be valid, we demand a stronger condition so that in the weak-field region $x > X_0$

$$\kappa x \phi^2 \ll 1. \quad (8)$$

Having the approximate formulas (6),(7), one can extend the solution to the scalarization region by means of numerical methods. We consider initial conditions at $x = X_0$

$$\nu(X_0) = \nu_0, \quad \lambda(X_0) = \lambda_0, \quad \phi(X_0) = \phi_0 = \frac{1}{X_0}, \quad \left. \frac{d\phi}{dx} \right|_{X_0} = -\phi_0,$$

where ν_0, λ_0, ϕ_0 satisfy (6)–(8). We assumed in the calculations $X_0 = (10^2 \div 10^5) X_g$.

Denote $q = \ln Q / X_g$, this parameter is related to X_0 as follows

$$q = \frac{\ln \phi_0}{X_g} + \frac{X_0}{X_g} + \left(\frac{1}{X_g} + \frac{1}{2} \right) \ln \left(\frac{X_0}{X_g} \right) \approx \frac{X_0}{X_g} + \frac{1}{2} \ln \left(\frac{X_0}{X_g} \right).$$

From numerical calculations we see that the plots of ν, ϕ in the scalarization region against $x/X_0 \leq 1$ are practically unchanged for different $M\mu = 10^2 \div 10^{20}$ and $q = 10^2 \div 10^5$ (Fig.1). Moreover, there is an approximation formula

$$\nu(x) \approx 2 \ln(x / X_0). \quad (9)$$

To compare the approximate formula (9) with the numerical solution over a wide range of parameters q and $M\mu$, we use the r.m.s. deviation Δ . Figure 2 shows the result of the comparison between the approximation formula and the numerical solution.

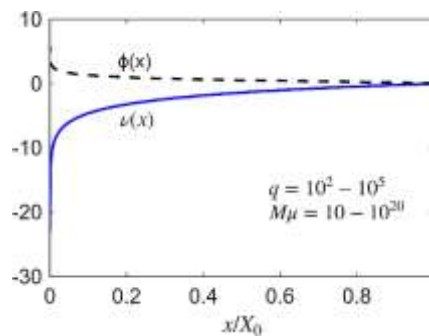


Fig. 1. The left panel shows $\phi(x)$ and $\nu(x)$ that are practically indistinguishable for $M\mu = 10^5 \div 10^{20}$ and $q = 10^3 \div 10^5$. The right panel demonstrates the coincidence of the numerical solution and approximate formula (9)

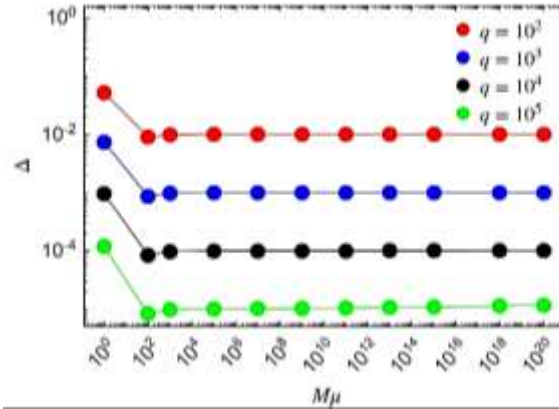


Fig. 2. Estimate of r.m.s. deviation Δ of the approximation (9) from the numerical solution as a function of $M\mu$ for different values of q

We observe that the analytic formula (9) agrees better with the numerical result as q increases. For a fixed value of q , the discrepancy between the numerical and approximate solutions exhibits a weak dependence on $M\mu > 10^2$.

Circular orbits. Motion of massive test bodies in case of SSS metric (2) is reduced to the problem of one-dimensional particle motion in the effective potential (see, e.g., Stashko, Zhdanov, & Alexandrov, 2021 for details)

$$U_{eff}(x) = e^v \left(1 + \frac{L^2}{x^2} \right),$$

where $L = x^2 d\varphi/d\tau = \text{const}$ is the angular momentum, τ is the proper time. Minimum of this potential correspond to a stable circular orbit (SCO). Denote

$$F(x) = \frac{x^3 v'}{2 - xv'}, \text{ where } v' \equiv \frac{dv}{dx}. \tag{10}$$

Simple investigation yields two conditions to have SCO:

$$L^2 = F(x) > 0 \text{ and } \frac{dF}{dx} > 0. \tag{11}$$

Evidently, in the weak field region $x > X_0 \gg X_g$ we have usual SCO in the Schwarzschild space-time.

In the scalarization region, on account of (9) we have that $U_{eff} \approx (x^2 + L^2)/X_0^2$ is monotonically increasing function. However, formula (9) is approximate; the specificity of the problem for large μM is that the denominator of $F(x)$ is becomes small and more accurate investigation shows the existence of shallow minima for $x/X_0 < 0.8$.

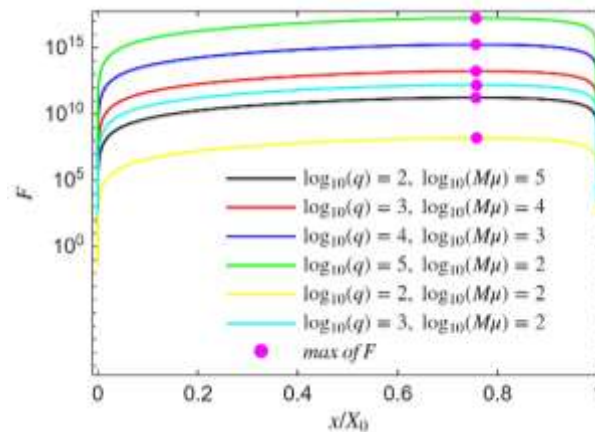


Fig. 3. The behavior of $F(x)$ for different $q, M\mu$. Regions of growth of $F(x)$ correspond stable circular orbits. The transition from stable to unstable circular orbits occurs around $x/X_0 \approx 0.76$

Examples of $F(x)$ in the scalarization region are shown in Fig. 3. Outside the scalarization region ($x > X_0$), the metric becomes approximately Schwarzschild, and stable circular orbits exist in this region. The minima of the potential $U_{eff}(x)$ are very shallow, i.e. the difference $\max(U_{eff}) - \min(U_{eff})$ is very small; in order to make them visible in Fig. 4 we use the normalized value

$$\tilde{U}_{eff}(x) = \frac{U_{eff}(x) - \min(U_{eff}(x))}{\max(U_{eff}(x)) - \min(U_{eff}(x))}.$$

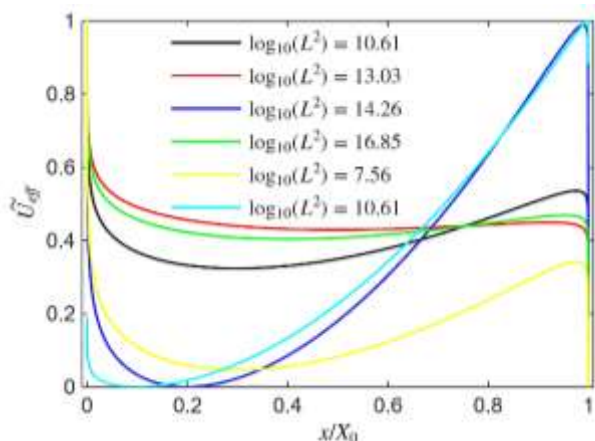


Fig. 4. Examples of the effective potential for different values of L^2 corresponding to (11).

The regions where $F(x)$ decreases in Fig. 3 correspond to local maxima of $U_{eff}(x)$ before the end of the scalarization region.

The values of q and $M\mu$ for each color are the same as the corresponding values for the same color in Fig. 3

Results

We investigated SSS configurations in General Relativity in presence of a minimally coupled linear massive scalar field with potential $V(\phi) = \mu^2 \phi^2 / 2$. The focus is on the scalar field with a sufficiently large value of $M\mu$, typical for astrophysical problems. Distribution of circular orbits around SSS configuration has been investigated for the configuration parameters $M\mu = 10^2 \div 10^{20}$ and $q = 10^3 \div 10^5$, where $q \approx r_0 / r_g$, r_0 being radius of the scalarization region, where there is a significant difference of the space-time metric from that of Schwarzschild. It turns out that, in this parameter region, the dependences of the scalar field $\phi(r)$ and metric coefficient $\exp[\nu(r)]$ on the radial variable are practically unnoticeable. We found that, for sufficiently large $M\mu$ the stable circular orbits in the scalarization region either do not exist or they are very shallow. More precisely, for some parameters there are minima of the effective potential that ensure the existence of circular orbits near the center. However, they are shallow: in this case, particles are unlikely to remain in these "stable" orbits in the presence of minor external influences. Thus, it can be argued that the accretion disk should be practically absent in the scalarization region. This may be an important observational signature to distinguish the configuration with scalar field from a regular black hole.

Discussion and conclusions

The distribution of the stable circular orbits around static spherically symmetric configuration of General Relativity in presence of a massive minimally coupled linear scalar field satisfying the conditions of asymptotic flatness is significantly different from the case of a Schwarzschild black hole. In case of sufficiently large masses of the scalar field and configuration masses, the circular orbits in the scalarization region will be virtually absent. This suggests a similar conclusion about the structure of accretion disks around such hypothetical configurations.

Authors' contribution: Valery Zhdanov – statement of the problem, formal analysis and methodology, testing of the numerical calculations, discussion of the results, writing the text and editing; Denis Tvardovsky – formal analysis and methodology, realization of numerical algorithms, numerical simulations, discussion of the results.

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КОЛОВІ ОРБИТИ НАВКОЛО СТАТИЧНОЇ СФЕРИЧНО-СИМЕТРИЧНОЇ КОНФІГУРАЦІЇ З ЛІНІЙНИМ МАСИВНИМ СКАЛЯРНИМ ПОЛЕМ У ЗАГАЛЬНІЙ ТЕОРІЇ ВІДНОСНОСТІ

Вступ. Теорії зі скалярними полями широко обговорюють в узагальненнях Стандартної космологічної моделі, що стосуються інфляції Всесвіту, динамічної темної енергії, *Hubble Tension* тощо. Введення скалярного поля в статичну сферично-симетричну (SSS) асимптотично плоску гравітаційну систему зазвичай призводить до особливостей розподілу стабільних колових орбіт, які можуть бути пов'язані зі спостережними ефектами. Ми вивчаємо цей розподіл у випадку масивного скалярного поля в межах загальної теорії відносності.

Методи. Ми розглядаємо рівняння Ейнштейна з мінімально зв'язаним масивним скалярним полем, що призводить до звичайної диференціальної системи, яка розв'язується чисельно (випадок SSS). Цей розв'язок застосовують для вивчення розподілу стабільних колових орбіт пробних тіл на основі властивостей ефективного потенціалу.

Результати. Навколо асимптотично плоскої конфігурації SSS є дві основні області: область слабого поля, де ми маємо наближений аналітичний опис розв'язку та звичайний розподіл колових орбіт, як у випадку метрики Шварцшильда, та область скаляризації зі значним відхиленням метрики від метрики Шварцшильда. В області скаляризації отримано числові розв'язки для скалярного поля з відповідними метричними коефіцієнтами. Отримано аналітичні співвідношення для дослідження колових орбіт пробного тіла. Ми показуємо, що для широкого діапазону мас скалярного поля й астрофізично релевантних конфігураційних мас або існує область слабкостабільних кругових орбіт поблизу центра, або кругові орбіти в області скаляризації взагалі відсутні.

Висновки. Розподіл стабільних колових орбіт суттєво відрізняється від випадку чорної діри Шварцшильда. Для достатньо великих мас скалярного поля та конфігураційних мас колові орбіти в області скаляризації будуть практично відсутні. Це дозволяє зробити аналогічний висновок щодо структури акреційних дисків.

Ключові слова: релятивістські об'єкти, голі сингулярності, модифікована гравітація.

Автори заявляють про відсутність конфлікту інтересів. Спонсори не брали участі в розробленні дослідження; у зборі, аналізі чи інтерпретації даних; у написанні рукопису; в рішенні про публікацію результатів.

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