АЛГЕБРА, ГЕОМЕТРІЯ ТА ТЕОРІЯ ЙМОВІРНОСТЕЙ

UDC 519.2
DOI: https://doi.org/10.17721/1812-5409.2025/1.2

Oleksandr BORYSENKO, PhD (Phys. & Math.), Assoc. Prof.
ORCID ID: 0000-0002-6670-8605
e-mail: borysenko_oleksandr@knu.ua
Taras Shevchenko National University of Kyiv, Kyiv, Ukraine

Olga BORYSENKO, PhD (Phys. & Math), Assoc. Prof.
ORCID ID: 0000-0002-2099-2911
e-mail: oborisenko1373@gmail.com
National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute", Kyiv, Ukraine

STOCHASTIC SIR MODEL WITH POISSON NOISES

Due to the negative impact of infectious diseases on population growth, it is important to understand the dynamic behavior of such diseases. Mathematical deterministic SIR models are widely used to study the spread of infectious diseases. In real life, there is a lot of randomness and stochasticity such as environmental noise, so using stochastic models is more suitable. Suppose we want to take into account abrupt environmental perturbations, such as epidemics, fires, earthquakes, etc. in the considered models. In that case, we must introduce Poisson noises into the population models for describing such discontinuous systems. The existence and uniqueness of the global positive solution are proved for the system of stochastic differential equations describing a non-autonomous SIR model disturbed by white noise, centered and non-centered Poisson noises. In the deterministic case there is a threshold of the system for an epidemic to occur, so called the basic reproduction number. Depending on the value of the reproduction number there is the disease-free equilibrium, or there is an endemic equilibrium, which implies the disease always remains. In the case of the autonomous stochastic SIR model, we study the asymptotic behavior of the solution to the corresponding system of stochastic differential equations around these points of equilibrium.

Keywords: ***stochastic SIR model, environmental noise, centered and non-centered Poisson noises, global positive solution, disease-free equilibrium, endemic equilibrium, asymptotic behavior.***

AMS 2020 classification: 60H10, 92D30.

Introduction

Due to the negative impact of infectious diseases on population growth, it is important to understand the dynamic behavior of such diseases. Mathematical deterministic SIR models are widely used to study the spread of infectious diseases. Let $S(t)$ denote the number of individuals susceptible to the disease, $I(t)$ denote the number of infected members and $R(t)$ denote the recovered members, who have been removed from the possibility of infection through full immunity. One of the basic deterministic SIR models of Kermack and McKendrick is presented and studied in (Anderson, & May, 1979). This deterministic SIR model has a form

$$\begin{aligned} \frac{dS(t)}{dt} &= \left(\Lambda - \beta S(t)I(t) - \mu S(t)\right), \\ \frac{dI(t)}{dt} &= \beta S(t)I(t) - (\mu + \varepsilon + \rho)I(t), \\ \frac{dR(t)}{dt} &= \rho I(t) - \mu R(t), \end{aligned} \tag{1}$$

where all parameters are positive constants, Λ is the influx of individuals into the susceptibles, μ is the natural death rate, β is the average number of contacts per infective per day (so-called "transmission coefficient"), ε is the death rate of infective caused by the disease, ρ is the rate of recovery from infection.

If the basic reproduction number $R_0 = \frac{\beta\Lambda}{\mu(\mu+\varepsilon+\rho)}$ we have $R_0 \leq 1$, then model (1) has only equilibrium $E_0 = \left(\frac{\Lambda}{\mu}, 0, 0\right)$, which is globally asymptotical stable. In the case $R_0 > 1$ there exists a global asymptotically stable endemic equilibrium $E^* = \left(\frac{\mu+\varepsilon+\rho}{\beta}, \frac{\Lambda}{\mu+\varepsilon+\rho} - \frac{\mu}{\beta}, \frac{\Lambda\rho}{\mu(\mu+\varepsilon+\rho)} - \frac{\rho}{\beta}\right)$. This means that disease always remains.

In (Jiang et al., 2011) the autonomous stochastic SIR model was studied. This model is driven by a system of stochastic differential equations

$$\begin{aligned} dS(t) &= \left(\Lambda - \beta S(t)I(t) - \mu S(t)\right)dt + \sigma_1 S(t)dw_1(t), \\ dI(t) &= \left(\beta S(t)I(t) - (\mu + \varepsilon + \rho)I(t)\right)dt + \sigma_2 I(t)dw_2(t), \\ dR(t) &= \left(\rho I(t) - \mu R(t)\right)dt + \sigma_3 R(t)dw_3(t), \end{aligned} \tag{2}$$

where $w_i(t), i = 1, 2, 3$ are independent one-dimensional Wiener processes. The existence and uniqueness of a global positive solution of model (2) are proved. The authors study the asymptotic behavior of the solution to the system (2) around E_0 and E^* respectively. In the paper by (Zhang, & Wang, 2013) the analogous results obtained for the autonomous stochastic SIR model with jumps, generated by centered Poisson measure

$$dS(t) = \left(\Lambda - \beta S(t)I(t) - \mu S(t)\right)dt + \sigma_1 S(t)dw_1(t) + \int_Z C_1(z)S(t^-)\tilde{N}(dt, dz),$$

$$dI(t) = \left(\beta S(t)I(t) - (\mu + \varepsilon + \rho)I(t)\right)dt + \sigma_2 I(t)dw_2(t) + \int_Z C_2(z)I(t^-)\tilde{N}(dt, dz),$$

$$dR(t) = \left(\rho I(t) - \mu R(t)\right)dt + \sigma_3 R(t)dw_3(t) + \int_Z C_3(z)R(t^-)\tilde{N}(dt, dz),$$

where $X(t^-)$ means the left limit of $X(t)$, $\tilde{N}(dt,dz)$ is the centered Poisson measure independent on Wiener processes $w_i(t), i = 1,2,3$, $Z \subseteq [0,+\infty)$ is a measurable subset. So considered model takes into account only "small" jumps corresponding to the integral with respect to the centered Poisson measure.

The impact of centered and non-centered Poisson noises to the stochastic non-autonomous stochastic predator-prey model that depends on the population density of the predator is studied in the paper (Borysenko, O. D., & Borysenko, O. V., 2022). The authors proved the existence and uniqueness theorem for the positive, global (no explosions in the finite time) solution of the considered system is proved. It is obtained sufficient conditions of stochastic ultimate boundedness and stochastic permanence in the considered stochastic predator-prey model.

In this paper, we consider the non-autonomous SIR model, disturbed by white noise and jumps generated by centered and non-centered Poisson measures. So, the novelty of the considered model is the following: we investigate the non-autonomous stochastic SIR model and we take into account not only "small" jumps, corresponding to the centered Poisson measure but also the "large" jumps, corresponding to the non-centered Poisson measure.

This model is driven by the system of stochastic differential equations

$$\begin{aligned} dS(t) &= \big(\Lambda(t) - \beta(t)S(t)I(t) - \mu(t)S(t)\big)dt + \sigma_1(t)S(t)dw_1(t) + \\ &+ \int_{\mathbb{R}} \gamma_1(t,z)S(t^-)\tilde{\nu}_1(dt,dz) + \int_{\mathbb{R}} \delta_1(t,z)S(t^-)\nu_2(dt,dz), \\ dI(t) &= \Big(\beta(t)S(t)I(t) - \big(\mu(t)+\varepsilon(t)+\rho(t)\big)I(t)\Big)dt + \sigma_2(t)I(t)dw_2(t) + \\ &+ \int_{\mathbb{R}} \gamma_2(t,z)I(t^-)\tilde{\nu}_1(dt,dz) + \int_{\mathbb{R}} \delta_2(t,z)I(t^-)\nu_2(dt,dz), \\ dR(t) &= \big(\rho(t)I(t) - \mu(t)R(t)\big)dt + \sigma_3(t)R(t)dw_3(t) + \\ &+ \int_{\mathbb{R}} \gamma_3(t,z)R(t^-)\tilde{\nu}_1(dt,dz) + \int_{\mathbb{R}} \delta_3(t,z)R(t^-)\nu_2(dt,dz), \end{aligned} \tag{3}$$

where $\nu_i(t,A), i = 1,2$ are independent Poisson measures, which are independent on $w_i(t), i = 1,2,3$, $\tilde{\nu}_1(t,A) = \nu_1(t,A) - t\Pi_1(A)$, $\mathrm{E}[\nu_i(t,A)] = t\Pi_i(A), i = 1,2$, $\Pi_i(A), i = 1,2$ are finite measures on the Borel sets A in $\mathbb{R}$.

We prove the existence and uniqueness of the global positive solution to the non-autonomous system (3). For the autonomous system

$$\begin{aligned} dS(t) &= \big(\Lambda - \beta S(t)I(t) - \mu S(t)\big)dt + \sigma_1 S(t)dw_1(t) + \int_{\mathbb{R}} \gamma_1(z)S(t^-)\tilde{\nu}_1(dt,dz) + \int_{\mathbb{R}} \delta_1(z)S(t^-)\nu_2(dt,dz), \\ dI(t) &= \big(\beta S(t)I(t) - (\mu+\varepsilon+\rho)I(t)\big)dt + \sigma_2 I(t)dw_2(t) + \int_{\mathbb{R}} \gamma_2(z)I(t^-)\tilde{\nu}_1(dt,dz) + \int_{\mathbb{R}} \delta_2(z)I(t^-)\nu_2(dt,dz), \\ dR(t) &= \big(\rho I(t) - \mu R(t)\big)dt + \sigma_3 R(t)dw_3(t) + \int_{\mathbb{R}} \gamma_3(z)R(t^-)\tilde{\nu}_1(dt,dz) + \int_{\mathbb{R}} \delta_3(z)R(t^-)\nu_2(dt,dz) \end{aligned} \tag{4}$$

we study the asymptotic behavior of the solution to the system (4) around E_0 and E^* respectively.

1. Main results

Let $(\Omega,\mathcal{F},P)$ be a probability space, $w_i(t), i = 1,2,3, t \ge 0$, are mutually independent standard one-dimensional Wiener processes on $(\Omega,\mathcal{F},P)$, and $\nu_i(t,A), i = 1,2$, are mutually independent Poisson measures defined on $(\Omega,\mathcal{F},P)$ independent on $w_i(t), i = 1,2,3, t \ge 0$. Here $E[\nu_i(t,A)] = t\Pi_i(A), i = 1,2$, $\tilde{\nu}_i(t,A) = \nu_i(t,A) - t\Pi_i(A), i = 1,2$, $\Pi_i(A),\ i = 1,2$ are finite measures on the Borel sets in $\mathbb{R}$. On the probability space $(\Omega,\mathcal{F},P)$ we consider an increasing, right continuous family of complete sub-σ-algebras $\{\mathcal{F}_t\}_{t\ge0}$, where $\mathcal{F}_t = \sigma\{w_i(s), \nu_j(s,A), s \le t, i = 1,2,3, j = 1,2\}$.

1.1. Existence and uniqueness of a global positive solution

Assumption 1. It is assumed, that $\Lambda(t), \beta(t), \mu(t), \varepsilon(t), \rho(t)$ are positive, bounded, continuous functions, $\sigma_i(t), i = 1,2,3$, $\gamma_j(t,z), \delta_j(t,z), j = 1,2$ are continuous on t functions, $\sigma_i(t), i = 1,2,3, \ln(1+\gamma_j(t,z))$, $\ln(1+\delta_j(t,z)), j = 1,2,3$ are bounded, $\inf_{t\ge0}\big(\mu(t) + \varepsilon(t) + \rho(t)\big) > 0, \Pi_i(\mathbb{R}) < \infty, i = 1,2$.

Theorem 1. *If Assumption 1 holds, then there exists a unique global solution $X(t) = (S(t), I(t), R(t))$ to the system (3) for any initial value $X(0) \in \mathbb{R}^3_+ = \{x \in \mathbb{R}^3 \mid x_i > 0, i = 1,2,3\}$, and $P\{X(t) \in \mathbb{R}^3_+, t \ge 0\} = 1$.*

Proof. The coefficients of the system (3) are local Lipschitz continuous. Therefore, for any initial value $\big(S(0), I(0), R(0)\big) \in \mathbb{R}^3_+$ there exists a unique local solution $X(t) = \big(S(t), I(t), R(t)\big)$ on $[0,\tau_e)$, where $\sup_{t<\tau_e}|X(t)| = +\infty$ (cf. Gikhman, & Skorokhod, 1982, p. 246). To show this solution is global, we need to show that $\tau_e = +\infty$ a.s. At first, we prove that $S(t)$ and $I(t)$ do not explode to infinity in a finite time. Let $n_0 \in \mathbb{N}$ be sufficiently large for $S(0) \in \left[\frac{1}{n_0}, n_0\right]$ and $I(0) \in \left[\frac{1}{n_0}, n_0\right]$. For any $n \ge n_0$ we define the stopping time $\tau_n = \inf\left\{t \in [0,\tau_e) : S(t) \notin \left(\frac{1}{n}, n\right) \ or\ I(t) \notin \left(\frac{1}{n}, n\right)\right\}$.

Clearly, that τ_n is increasing as $n \to +\infty$. Set $\tau_\infty = \lim_{n\to\infty} \tau_n$, whence $\tau_\infty \le \tau_e$ a.s. If we prove that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. and $(S(t), I(t)) \in \mathbb{R}^2_+, t \ge 0$ a.s. If this statement is false, then there are constants $T > 0$ and $\varepsilon \in (0,1)$, such that $P\{\tau_\infty < T\} > \varepsilon$. So, there is $n_1 \ge n_0$ such that

$$P\{\tau_n < T\} > \varepsilon, \forall n \ge n_1. \tag{5}$$

Let us consider the non-negative function $V(S,I) = \left(S - a - a\ln\frac{S}{a}\right) + (I - 1 - \ln I), S > 0, I > 0$, where $a > 0$ is a constant, which will be defined later. If we denote

$$\kappa_i(t) = \frac{\sigma_i^2(t)}{2} + \int_{\mathbb{R}} [\gamma_i(t,z) - \ln(1+\gamma_i(t,z)]\Pi_1(dz) - \int_{\mathbb{R}} \ln(1+\delta_i(t,z)\,\Pi_2(dz), \qquad i = 1,2,$$

$$\Phi(t)=\Lambda(t)+a\mu(t)+\big(\mu(t)+\varepsilon(t)+\rho(t)\big)+a\kappa_1(t)+\kappa_2(t)-\big(\mu(t)+\beta(t)\big)S(t)-a\frac{\Lambda(t)}{S(t)}+$$
$$+\big[a\beta(t)-\big(\mu(t)+\varepsilon(t)+\rho(t)\big)\big]I(t),$$

then by the Itô formula and the definition of a stochastic integral with respect to the non-centered Poisson measure $\nu_2(du,dz)$ we derive

$$V\big(S(T\wedge\tau_n),I(T\wedge\tau_n)\big)=$$
$$=V\big(S(0),I(0)\big)+\int_0^{T\wedge\tau_n}\Phi(u)du+\int_0^{T\wedge\tau_n}(S(u)-a)\sigma_1(u)dw_1(u)+\int_0^{T\wedge\tau_n}(I(u)-1)\sigma_2(u)dw_2(u)+$$
$$+\int_0^{T\wedge\tau_n}\int_{\mathbb{R}}[\gamma_1(u,z)S(u^-)-a\ln(1+\gamma_1(u,z))+\gamma_2(u,z)I(u^-)-\ln(1+\gamma_2(u,z)]\tilde{\nu}_1(du,dz)+\tag{6}$$
$$+\int_0^{T\wedge\tau_n}\int_{\mathbb{R}}[\delta_1(u,z)S(u^-)-a\ln(1+\delta_1(u,z))+\delta_2(u,z)I(u^-)-\ln(1+\delta_2(u,z)]\tilde{\nu}_2(du,dz).$$

From the definition of stopping time τ_n it follows that for $0\le u\le T\wedge\tau_n$ we have $\frac{1}{n}<S(u)<n,\frac{1}{n}<I(u)<n$. Hence under condition on the coefficients of the system (3), there is a constant $K>0$ such that for $0\le u\le T\wedge\tau_n$ we obtain

$$\Phi(u)\le\sup_{t\ge0}\{\Lambda(t)+a\mu(t)+\big(\mu(t)+\varepsilon(t)+\rho(t)\big)+a\kappa_1(t)+\kappa_2(t)\}+$$
$$+\Big[a\sup_{t\ge0}\beta(t)-\inf_{t\ge0}\big(\mu(t)+\varepsilon(t)+\rho(t)\big)\Big]I(u)\le K.$$

If we choose $a=\frac{\inf_{t\ge0}(\mu(t)+\varepsilon(t)+\rho(t))}{\sup_{t\ge0}\beta(t)}$, then taking expectations, we derive from (6)

$$\mathrm{E}\big[V\big(S(T\wedge\tau_n),I(T\wedge\tau_n)\big)\big]\le V\big(S(0),I(0)\big)+KT.$$

Let $\Omega_n=\{\omega\in\Omega\,|\tau_n(\omega)\le T\}$, for $n\ge n_1$. Then by (5) we have $P(\Omega_n)>\varepsilon$, for $n\ge n_1$. Note that for every $\omega\in\Omega_n$ at least one of $S(\tau_n,\omega)$ and $I(\tau_n,\omega)$ equals either n or $1/n$. So

$$V\big(S(\tau_n,\omega),I(\tau_n,\omega)\big)\ge\min\Big(n-1-\ln n,\frac{1}{n}-1+\ln n\Big).$$

Therefore,

$$V\big(S(0),I(0)\big)+KT\ge E\big[\mathbb{I}_{\Omega_n}V\big(S(\tau_n),I(\tau_n)\big)\big]\ge\varepsilon\min\Big(n-1-\ln n,\frac{1}{n}-1+\ln n\Big),$$

where $\mathbb{I}_{\Omega_n}$ is the indicator function of Ω_n.

Letting $n\to\infty$ leads to the contradiction $\infty>V\big(S(0),I(0)\big)+KT=\infty$. Therefore, it implies $S(t)$ and $I(t)$ will not explode in a finite time almost surely.

For the process $\zeta(t)=R_0(t)R(t)$ we have by the Itô formula $d\zeta(t)=d(R_0(t)R(t))=R_0(t)\rho(t)I(t)dt$, where

$$R_0(t)$$
$$=\exp\Big\{\int_0^t\Big[\mu(s)+\frac{\sigma_3^2(s)}{2}+\int_{\mathbb{R}}[\gamma_3(s,z)-\ln(1+\gamma_3(s,z))]\Pi_1(dz)\Big]ds-\int_0^t\sigma_3(s)dw_3(s)-$$
$$-\int_0^t\int_{\mathbb{R}}\ln\big(1+\gamma_3(s,z)\big)\tilde{\nu}_1(ds,dz)-\int_0^t\int_{\mathbb{R}}\ln\big(1+\delta_3(s,z)\big)\nu_2(ds,dz)\Big\},$$

$R(t)$ satisfies the third equation in (3). Therefore

$$R(t)=\frac{R(0)+\int_0^t R_0(s)\rho(s)I(s)ds}{R_0(t)}.$$

Since $I(t)$ has been proven to be global and positive, $R(t)$ is also global and positive. This completes the proof of the theorem.

1.2. Asymptotic behavior around the disease-free equilibrium of the autonomous deterministic model.

As we mentioned in the Introduction if $R_0\le1$, where $R_0=\frac{\beta\Lambda}{\mu(\mu+\varepsilon+\rho)}$, then model (1) has only the disease-free equilibrium $E_0=\big(\frac{\Lambda}{\mu},0,0\big)$, which is globally asymptotical stable. While for the autonomous stochastic system (4), E_0 is no longer the disease-free equilibrium. In this section, we will study the asymptotic behavior of the solution to the system (4) around E_0.

Assumption 2. It is assumed, that $\Lambda,\beta,\mu,\varepsilon,\rho$ are positive constants, $\ln(1+\gamma_j(z))$, $\ln(1+\delta_j(z))$, $j=1,2,3$ are bounded, $\Pi_i(\mathbb{R})<\infty, i=1,2$.

Theorem 2. *Let Assumption 2 be fulfilled and $(S(t),I(t),R(t))$ be the solution of system (4) with any initial value $\big(S(0),I(0),R(0)\big)\in\mathbb{R}_+^3$. If*

$$R_0=\frac{\beta\Lambda}{\mu(\mu+\varepsilon+\rho)}\le1,$$
$$2\mu>\max\Big\{\Big[2\sigma_1^2+3\int_{\mathbb{R}}\gamma_1^2(z)\Pi_1(dz)+\int_{\mathbb{R}}[3\delta_1^2(z)+4|\delta_1(z)|+|\delta_2(z)|]\Pi_2(dz)\Big],\ 2\int_{\mathbb{R}}\delta_3(z)\Pi_2(dz)\Big\},$$
$$2(\mu+\varepsilon+\rho)>\sigma_2^2+3\int_{\mathbb{R}}\gamma_2^2(z)\Pi_1(dz)+\int_{\mathbb{R}}[3\delta_2^2(z)+4|\delta_2(z)|+2|\delta_1(z)|]\Pi_2(dz),$$

then

$$\limsup_{t\to\infty}\frac{1}{t}\,E\left[\int_0^t\left(\left(S(v)-\frac{\Lambda}{\mu}\right)^2+I^2(v)+R(v)\right)dv\right]\le$$
$$\le\frac{1}{K_1}\left(\frac{\Lambda}{\mu}\right)^2\left(2\sigma_1^2+3\int_{\mathbb{R}}\gamma_1^2(z)\Pi_1(dz)+\int_{\mathbb{R}}\left[3\delta_1^2(z)+4|\delta_1(z)|+\frac{(\mu c_1)^2}{4\Lambda^2}|\delta_2(z)|\right]\Pi_2(dz)\right),$$

where

$$K_1=\min\left\{\left[2\mu-2\sigma_1^2-3\int_{\mathbb{R}}\gamma_1^2(z)\,\Pi_1(dz)-\int_{\mathbb{R}}[3\delta_1^2(z)+4|\delta_1(z)|+|\delta_2(z)|]\Pi_2(dz)\right],\right.$$
$$\left.\left[2(\mu+\varepsilon+\rho)-\sigma_2^2-3\int_{\mathbb{R}}\gamma_2^2(z)\,\Pi_1(dz)-\int_{\mathbb{R}}[3\delta_2^2(z)+4|\delta_2(z)|+2|\delta_1(z)|]\Pi_2(dz)\right],c_2\left(\mu-\int_{\mathbb{R}}\delta_3(z)\Pi_2(dz)\right)\right\},$$
$$c_1=\frac{2}{\beta}(2\mu+\varepsilon+\rho),\qquad c_2=c_1\frac{\Lambda\beta}{\mu\rho}\left(\frac{1-R_0}{R_0}\right).$$

Proof. If $(t)=S(t)-\frac{\Lambda}{\mu}$, then from the system (4), we derive

$$\begin{gathered}dU(t)=\left(-\mu U(t)-\beta U(t)I(t)-\beta\frac{\Lambda}{\mu}I(t)\right)dt+\sigma_1\left(U(t)+\frac{\Lambda}{\mu}\right)dw_1(t)+\\+\int_{\mathbb{R}}\gamma_1(z)\left(U(t^-)+\frac{\Lambda}{\mu}\right)\tilde{\nu}_1(dt,dz)+\int_{\mathbb{R}}\delta_1(z)\left(U(t^-)+\frac{\Lambda}{\mu}\right)\nu_2(dt,dz),\\dI(t)=\left(\beta\left(U(t)+\frac{\Lambda}{\mu}\right)I(t)-(\mu+\varepsilon+\rho)I(t)\right)dt+\sigma_2I(t)dw_2(t)+\\+\int_{\mathbb{R}}\gamma_2(z)I(t^-)\tilde{\nu}_1(dt,dz)+\int_{\mathbb{R}}\delta_2(z)I(t^-)\nu_2(dt,dz),\\dR(t)=(\rho I(t)-\mu R(t))dt+\sigma_3R(t)dw_3(t)+\int_{\mathbb{R}}\gamma_3(z)R(t^-)\tilde{\nu}_1(dt,dz)+\int_{\mathbb{R}}\delta_3(z)R(t^-)\nu_2(dt,dz).\end{gathered}\tag{7}$$

Let us consider the non-negative function $V(U,I,R)=(U+I)^2+c_1I+c_2R,\ U\in\mathbb{R},\ I>0,\ R>0,$ where constants $c_i>0,i=1{,}2$ will be defined later.

Using the Itô formula, system (7), and the definition of a stochastic integral with respect to the non-centered Poisson measure $\nu_2(dt,dz)$ we derive

$$\begin{gathered}dV(U(t),I(t),R(t))=\{2(U(t)+I(t))[-\mu U(t)-(\mu+\varepsilon+\rho)I(t)]+\sigma_1^2\left(U(t)+\frac{\Lambda}{\mu}\right)^2+\sigma_2^2I^2(t)+\\+c_1\left[\beta U(t)I(t)-\left(\mu+\varepsilon+\rho-\beta\frac{\Lambda}{\mu}\right)I(t)\right]+c_2(\rho I(t)-\mu R(t)+\int_{\mathbb{R}}\left[\gamma_1(z)\left(U(t)+\frac{\Lambda}{\mu}\right)+\gamma_2(z)I(t)\right]^2\Pi_1(dz)+\\+\int_{\mathbb{R}}\left\{\left[\delta_1(z)\left(U(t)+\frac{\Lambda}{\mu}\right)+\delta_2(z)I(t)\right]^2+2(U(t)+I(t))\left[\delta_1(z)\left(U(t)+\frac{\Lambda}{\mu}\right)+\delta_2(z)I(t)\right]+\right.\\\left.+c_1\delta_2(z)I(t)+c_2\delta_3(z)R(t)\right\}\Pi_2(dz)\}dt+dJ_{stoch}(t)=\\=\left\{-U^2(t)\left[2\mu-\sigma_1^2-\int_{\mathbb{R}}\gamma_1^2(z)\,\Pi_1(dz)-\int_{\mathbb{R}}[\delta_1^2(z)+2\delta_1(z)]\Pi_2(dz)\right]-\right.\\-I^2(t)\left[2(\mu+\varepsilon+\rho)-\sigma_2^2-\int_{\mathbb{R}}\gamma_2^2(z)\,\Pi_1(dz)-\int_{\mathbb{R}}[\delta_2^2(z)+2\delta_2(z)]\Pi_2(dz)\right]-c_2\mu R(t)+\\+U(t)I(t)\left[-2(2\mu+\varepsilon+\rho)+c_1\beta+2\int_{\mathbb{R}}\gamma_1(z)\gamma_2(z)\Pi_1(dz)+2\int_{\mathbb{R}}[\delta_1(z)\delta_2(z)+\delta_1(z)+\delta_2(z)]\Pi_2(dz)\right]+\\+I(t)\left[-c_1\beta\frac{\Lambda}{\mu}\left(\frac{1-R_0}{R_0}\right)+c_2\rho+2\frac{\Lambda}{\mu}\left[\int_{\mathbb{R}}\gamma_1(z)\gamma_2(z)\Pi_1(dz)+\int_{\mathbb{R}}(\delta_1(z)\delta_2(z)+\delta_1(z))\Pi_2(dz)\right]+\right.\\\left.+c_1\int_{\mathbb{R}}\delta_2(z))\Pi_2(dz)\right]+2\frac{\Lambda}{\mu}U(t)\left[\sigma_1^2+\int_{\mathbb{R}}\gamma_1^2(z)\Pi_1(dz)+\int_{\mathbb{R}}(\delta_1^2(z)+\delta_1(z))\Pi_2(dz)\right]+\\\left.+c_2R(t)\int_{\mathbb{R}}\delta_3(z)\Pi_2(dz)+\left(\frac{\Lambda}{\mu}\right)^2\left(\sigma_1^2+\int_{\mathbb{R}}\gamma_1^2(z)\Pi_1(dz)+\int_{\mathbb{R}}\delta_1^2(z)\Pi_2(dz)\right)\right\}dt+dJ_{stoch}(t)=\\=LF(U(t),I(t),R(t))dt+dJ_{stoch}(t),\end{gathered}\tag{8}$$

where

$$\begin{gathered}dJ_{stoch}(t)=2(U(t)+I(t))\sigma_1\left(U(t)+\frac{\Lambda}{\mu}\right)dw_1(t)+\big(2(U(t)+I(t))+c_1\big)\sigma_2I(t)dw_2(t)+c_2\sigma_3R(t)dw_3(t)+\\+\int_{\mathbb{R}}\left\{\left[\gamma_1(z)\left(U(t^-)+\frac{\Lambda}{\mu}\right)+\gamma_2(z)I(t^-)\right]^2+2(U(t^-)+I(t^-))\left[\gamma_1(z)\left(U(t^-)+\frac{\Lambda}{\mu}\right)+\gamma_2(z)I(t^-)\right]+\right.\\\left.+c_1\gamma_2(z)I(t^-)+c_2\gamma_3(z)R(t^-)\right\}\tilde{\nu}_1(dt,dz)+\\+\int_{\mathbb{R}}\left\{\left[\delta_1(z)\left(U(t^-)+\frac{\Lambda}{\mu}\right)+\delta_2(z)I(t^-)\right]^2+2(U(t^-)+I(t^-))\left[\delta_1(z)\left(U(t^-)+\frac{\Lambda}{\mu}\right)+\delta_2(z)I(t^-)\right]+\right.\\\left.+c_1\delta_2(z)I(t^-)+c_2\delta_3(z)R(t^-)\right\}\tilde{\nu}_2(dt,dz).\end{gathered}$$

If we choose $c_1 = \frac{2}{\beta}(2\mu+\varepsilon+\rho) > 0, c_2 = c_1\frac{\Lambda\beta}{\mu\rho}\left(\frac{1-R_0}{R_0}\right) > 0$, then using inequality $a^2+b^2 \geq 2ab$ we obtain the estimate

$$\begin{aligned}
LF(U,I,R) = -U^2\left[2\mu - \sigma_1^2 - \int_{\mathbb{R}} \gamma_1^2(z)\,\Pi_1(dz) - \int_{\mathbb{R}} [\delta_1^2(z)+2\delta_1(z)]\Pi_2(dz)\right] - \\
-I^2\left[2(\mu+\varepsilon+\rho) - \sigma_2^2 - \int_{\mathbb{R}} \gamma_2^2(z)\,\Pi_1(dz) - \int_{\mathbb{R}} [\delta_2^2(z)+2\delta_2(z)]\Pi_2(dz)\right] - c_2R\left(\mu - \int_{\mathbb{R}} \delta_3(z)\Pi_2(dz)\right) + \\
+2UI\left[\int_{\mathbb{R}} \gamma_1(z)\gamma_2(z)\Pi_1(dz) + \int_{\mathbb{R}} [\delta_1(z)\delta_2(z)+\delta_1(z)+\delta_2(z)]\Pi_2(dz)\right] + \\
+2I\left[\frac{\Lambda}{\mu}\left(\int_{\mathbb{R}} \gamma_1(z)\gamma_2(z)\Pi_1(dz) + \int_{\mathbb{R}} (\delta_1(z)\delta_2(z)+\delta_1(z))\Pi_2(dz)\right) + \frac{(2\mu+\varepsilon+\rho)}{\beta}\int_{\mathbb{R}} \delta_2(z))\Pi_2(dz)\right] + \\
+2\frac{\Lambda}{\mu}U\left[\sigma_1^2 + \int_{\mathbb{R}} \gamma_1^2(z)\Pi_1(dz) + \int_{\mathbb{R}} (\delta_1^2(z)+\delta_1(z))\Pi_2(dz)\right] + \left(\frac{\Lambda}{\mu}\right)^2\left(\sigma_1^2 + \int_{\mathbb{R}} \gamma_1^2(z)\Pi_1(dz) + \int_{\mathbb{R}} \delta_1^2(z)\Pi_2(dz)\right) \leq \\
\leq -U^2\left[2\mu - 2\sigma_1^2 - 3\int_{\mathbb{R}} \gamma_1^2(z)\,\Pi_1(dz) - \int_{\mathbb{R}} [3\delta_1^2(z)+4|\delta_1(z)|+|\delta_2(z)|]\Pi_2(dz)\right] - \\
-I^2\left[2(\mu+\varepsilon+\rho) - \sigma_2^2 - 3\int_{\mathbb{R}} \gamma_2^2(z)\,\Pi_1(dz) - \int_{\mathbb{R}} [3\delta_2^2(z)+4|\delta_2(z)|+2|\delta_1(z)|]\Pi_2(dz)\right] - \\
-c_2R\left(\mu - \int_{\mathbb{R}} \delta_3(z)\Pi_2(dz)\right) + \left(\frac{\Lambda}{\mu}\right)^2\left(2\sigma_1^2 + 3\int_{\mathbb{R}} \gamma_1^2(z)\Pi_1(dz) + \int_{\mathbb{R}} [3\delta_1^2(z)+4|\delta_1(z)| + \frac{(\mu c_1)^2}{4\Lambda^2}|\delta_2(z)|]\Pi_2(dz)\right).
\end{aligned} \tag{9}$$

Let $n_0 \in \mathbb{N}$ be sufficiently large for $U(0)+\frac{\Lambda}{\mu} \in \left[\frac{1}{n_0}, n_0\right]$, $I(0) \in \left[\frac{1}{n_0}, n_0\right]$ and $R(0) \in \left[\frac{1}{n_0}, n_0\right]$. For any $n \geq n_0$ we define the stopping time $\tau_n = \inf\left\{t \geq 0 : X(t) \notin \left(\frac{1}{n}, n\right)^3\right\}$, where $X(t) = (U(t)+\frac{\Lambda}{\mu}, I(t), R(t))$, $(U(t), I(t), R(t))$ is the solution to the system (7). According to Theorem 1, the system (4) has a unique global (no explosion in the finite time interval) solution $(S(t), I(t), R(t))$, therefore the solution to the system (7) is also global and $\lim_{n\to\infty}\tau_n = +\infty$ a.s. From (8), applying the properties of stochastic integrals with respect to Wiener processes and with respect to centered Poisson measures, we derive

$$0 \leq E[V(U(t\wedge\tau_n), I(t\wedge\tau_n), R(t\wedge\tau_n))] = V(U(0), I(0), R(0)) + E\left[\int_0^{t\wedge\tau_n} LF(U(v), I(v), R(v))dv\right]. \tag{10}$$

If $n \to \infty$, then from (10) we have

$$0 \leq E[V(U(t), I(t), R(t))] = V(U(0), I(0), R(0)) + E\left[\int_0^t LF(U(v), I(v), R(v))dv\right]. \tag{11}$$

Therefore, applying the estimate (9), we obtain from (11)

$$\begin{aligned}
K_1 E\left[\int_0^t (U^2(v)+I^2(v)+R(v))dv\right] \leq V(U(0), I(0), R(0)) + \\
+\left(\frac{\Lambda}{\mu}\right)^2\left(2\sigma_1^2 + 3\int_{\mathbb{R}} \gamma_1^2(z)\Pi_1(dz) + \int_{\mathbb{R}} \left[3\delta_1^2(z)+4|\delta_1(z)| + \frac{(\mu c_1)^2}{4\Lambda^2}|\delta_2(z)|\right]\Pi_2(dz)\right)t,
\end{aligned}$$

where

$$\begin{aligned}
K_1 = \min\left\{\left[2\mu - 2\sigma_1^2 - 3\int_{\mathbb{R}} \gamma_1^2(z)\,\Pi_1(dz) - \int_{\mathbb{R}} [3\delta_1^2(z)+4|\delta_1(z)|+|\delta_2(z)|]\Pi_2(dz)\right],\right. \\
\left.\left[2(\mu+\varepsilon+\rho) - \sigma_2^2 - 3\int_{\mathbb{R}} \gamma_2^2(z)\,\Pi_1(dz) - \int_{\mathbb{R}} [3\delta_2^2(z)+4|\delta_2(z)|+2|\delta_1(z)|]\Pi_2(dz)\right], c_2\left(\mu - \int_{\mathbb{R}} \delta_3(z)\Pi_2(dz)\right)\right\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\limsup_{t\to\infty}\frac{1}{t}E\left[\int_0^t \left(\left(S(v)-\frac{\Lambda}{\mu}\right)^2 + I^2(v) + R(v)\right)dv\right] \leq \\
\leq \frac{1}{K_1}\left(\frac{\Lambda}{\mu}\right)^2\left(2\sigma_1^2 + 3\int_{\mathbb{R}} \gamma_1^2(z)\Pi_1(dz) + \int_{\mathbb{R}} \left[3\delta_1^2(z)+4|\delta_1(z)| + \frac{(\mu c_1)^2}{4\Lambda^2}|\delta_2(z)|\right]\Pi_2(dz)\right).
\end{aligned}$$

The proof of the theorem is completed.

Remark 1. If $\gamma_i(z) \equiv 0, \delta_i(z) \equiv 0, i = 1,2,3$, then Theorem 2 is Theorem 3.1 from (Jiang et al., 2011).

1.3. Asymptotic behavior around the endemic equilibrium of the autonomous deterministic model

As we mentioned in the Introduction, if $R_0 = \frac{\beta\Lambda}{\mu(\mu+\varepsilon+\rho)} > 1$, then there exists a global asymptotically stable endemic equilibrium $E^* = \left(\frac{\mu+\varepsilon+\rho}{\beta}, \frac{\Lambda}{\mu+\varepsilon+\rho} - \frac{\mu}{\beta}, \frac{\Lambda\rho}{\mu(\mu+\varepsilon+\rho)} - \frac{\rho}{\beta}\right)$ in the deterministic model (1). While for the autonomous stochastic system (4), E^* is no longer the endemic equilibrium. In this section, we will study the asymptotic behavior of the solution to the system (4) around E^*.

Theorem 3. *Let Assumption 2 be fulfilled and* $(S(t), I(t), R(t))$ *be the solution of system (4) with any initial value* $(S(0), I(0), R(0)) \in \mathbb{R}^3_+$. *If*

$$R_0 = \frac{\beta\Lambda}{\mu(\mu+\varepsilon+\rho)} > 1,$$
$$2\mu > \sigma_1^2 + 2\int_{\mathbb{R}} \gamma_1^2(z)\Pi_1(dz) + \int_{\mathbb{R}} (2\delta_1^2(z) + 3|\delta_1(z)| + |\delta_2(z)|)\Pi_2(dz),$$
$$2(\mu+\varepsilon+\rho) > \sigma_2^2 + 2\int_{\mathbb{R}} \gamma_2^2(z)\,\Pi_1(dz) + \int_{\mathbb{R}} [2\delta_2^2(z) + (3+a)|\delta_2(z)| + |\delta_1(z)|]\Pi_2(dz), \quad (12)$$
$$\mu > \sigma_3^2 + \int_{\mathbb{R}} \gamma_3^2(z)\Pi_1(dz) + \int_{\mathbb{R}} (\delta_3^2(z) + 2|\delta_3(z)|)\Pi_2(dz),$$

then

$$\limsup_{t\to\infty} \frac{1}{t} E\left[\int_0^t ((S(v) - m_1S^*)^2 + (I(v) - m_2I^*)^2 + (R(v) - m_3R^*)^2)dv\right] \le \frac{M(\sigma,\gamma,\delta)}{K(\sigma,\gamma,\delta)},$$

where

$$K(\sigma,\gamma,\delta) = \min\left\{\frac{k_1}{2}, \frac{k_2}{2\mu}, \frac{bk_3}{2}\right\},\ \ a = \frac{(2\mu+\varepsilon+\rho)}{\beta},$$
$$0 < b < \frac{\mu}{\rho^2}\left(2(\mu+\varepsilon+\rho) - \sigma_1^2 - 2\int_{\mathbb{R}} \gamma_1^2(z)\Pi_1(dz) - \int_{\mathbb{R}} (2\delta_1^2(z) + 3|\delta_1(z)| + |\delta_2(z)|)\Pi_2(dz)\right),$$
$$k_1 = 2\mu - \sigma_1^2 - 2\int_{\mathbb{R}} \gamma_1^2(z)\Pi_1(dz) - \int_{\mathbb{R}} (2\delta_1^2(z) + 3|\delta_1(z)| + |\delta_2(z)|)\Pi_2(dz),$$
$$m_1 = \frac{1}{k_1}\left(2\mu - \int_{\mathbb{R}} (|\delta_1(z)| + |\delta_2(z)|)\Pi_2(dz)\right),$$
$$k_2 = 2\mu(\mu+\varepsilon+\rho) - \mu\sigma_2^2 - 2\mu\int_{\mathbb{R}} \gamma_2^2(z)\Pi_1(dz) - \mu\int_{\mathbb{R}} (2\delta_2^2(z) + (3+a)|\delta_2(z)| + |\delta_1(z)|)\Pi_2(dz) - b\rho^2,$$
$$m_2 = \frac{2\mu}{k_2}\left(\mu+\varepsilon+\rho - \frac{1}{2}\int_{\mathbb{R}} (|\delta_1(z)| + (1+a)|\delta_2(z)|)\Pi_2(dz) - \frac{b\rho^2}{2\mu}\right)$$
$$k_3 = \mu - \sigma_3^2 - \int_{\mathbb{R}} \gamma_3^2(z)\Pi_1(dz) - \int_{\mathbb{R}} (\delta_3^2(z) + 2|\delta_3(z)|)\Pi_2(dz),\ m_3 = \frac{1}{k_3}\left(\mu - \int_{\mathbb{R}} |\delta_3(z)|\Pi_2(dz)\right),$$
$$M(\sigma,\gamma,\delta) = \frac{k_1}{2}m_1(m_1-1)S^{*2} + \frac{k_2}{2\mu}m_2(m_2-1)I^{*2} + \frac{bk_3}{2}m_1(m_1-1)R^{*2} +$$
$$+a\left(\frac{1}{2}\sigma_2^2 + \int_{\mathbb{R}} (\gamma_2(z) - \ln(1+\gamma_2(z)))\,\Pi_1(dz) + \int_{\mathbb{R}} (\delta_2(z) - \ln(1+\delta_2(z)))\,\Pi_2(dz)\right)I^* + \frac{a}{2}\int_{\mathbb{R}} |\delta_2(z)|\Pi_2(dz).$$

Proof. Let us consider the function $F(S,I,R) = F_1(S,I) + F_2(R) \ge 0, (S,I,R) \in \mathbb{R}^3_+$, where

$$F_1(S,I) = \frac{1}{2}(S - S^* + I - I^*)^2 + a\left(I - I^* - I^*\ln\frac{I}{I^*}\right), \qquad F_2(R) = \frac{b}{2}(R - R^*)^2.$$

Here constants $a > 0, b > 0$ to be defined later.

By the Itô formula, system (4), and the definition of a stochastic integral with respect to the non-centered Poisson measure $\nu_2(dt,dz)$ we derive

$$dF_1(S(t),I(t)) = LF_1(S(t),I(t))dt + (S(t) - S^* + I(t) - I^*)\sigma_1S(t)dw_1(t) +$$
$$+(S(t) - S^* + I(t) - I^*)\sigma_2I(t)dw_2(t) + a(I(t) - I^*)\sigma_2dw_2(t) +$$
$$+\int_{\mathbb{R}} \left[(S(t^-) - S^* + I(t^-) - I^*)(\gamma_1(z)S(t^-) + \gamma_2(z)I(t^-)) + \frac{1}{2}(\gamma_1(z)S(t^-) + \gamma_2(z)I(t^-))^2 +\right.$$
$$+a(\gamma_2(z)I(t^-) - I^*\ln(1+\gamma_2(z))]\tilde{\nu}_1(dt,dz) + \int_{\mathbb{R}} [(S(t^-) - S^* + I(t^-) - I^*)(\delta_1(z)S(t^-) + \delta_2(z)I(t^-)) +$$
$$+\frac{1}{2}(\delta_1(z)S(t^-) + \delta_2(z)I(t^-))^2 + a(\delta_2(z)I(t^-) - I^*\ln(1+\delta_2(z))]\tilde{\nu}_2(dt,dz). \quad (13)$$
$$dF_2(R(t)) = LF_2(R(t),I(t))dt + b(R(t) - R^*)\sigma_3R(t)dw_3(t) +$$
$$+\int_{\mathbb{R}} \left[b(R(t^-) - R^*)\gamma_3(z)R(t^-) + \frac{1}{2}b\gamma_3^2(z)R^2(t^-)\right]\tilde{\nu}_1(dt,dz) +$$
$$+\int_{\mathbb{R}} \left[b(R(t^-) - R^*)\delta_3(z)R(t^-) + \frac{1}{2}b\delta_3^2(z)R^2(t^-)\right]\tilde{\nu}_2(dt,dz),$$

where

$$LF_1(S,I) = (S - S^* + I - I^*)(\Lambda - \mu S - (\mu+\varepsilon+\rho)I) + a(I - I^*)(\beta S - (\mu+\varepsilon+\rho)) +$$
$$+\frac{1}{2}(\sigma_1^2S^2 + \sigma_2^2I^2) + \frac{a}{2}\sigma_2^2I^* + \int_{\mathbb{R}} \left[\frac{1}{2}(\gamma_1(z)S + \gamma_2(z)I)^2 + a(\gamma_2(z) - \ln(1+\gamma_2(z)))I^*\right]\Pi_1(dz) + \quad (14)$$
$$+\int_{\mathbb{R}} \left[(S - S^* + I - I^*)(\delta_1(z)S + \delta_2(z)I) + \frac{1}{2}(\delta_1(z)S + \delta_2(z)I)^2 + a(\delta_2(z)I - I^*\ln(1+\delta_2(z)))\right]\Pi_2(dz)$$
$$\le -\mu(S - S^*)^2 - (\mu+\varepsilon+\rho)(I - I^*)^2 +$$

$$+\big(a\beta-(2\mu+\varepsilon+\rho)\big)(S-S^*)(I-I^*)+\frac{1}{2}(\sigma_1^2S^2+\sigma_2^2I^2)+\frac{a}{2}\sigma_2^2I^*+$$
$$+S^2\int_{\mathbb{R}}\gamma_1^2(z)\Pi_1(dz)+I^2\int_{\mathbb{R}}\gamma_2^2(z)\Pi_1(dz)+aI^*\int_{\mathbb{R}}\big(\gamma_2(z)-\ln\big(1+\gamma_2(z)\big)\big)\Pi_1(dz)+$$
$$+\frac{1}{2}(S-S^*)^2\int_{\mathbb{R}}(|\delta_1(z)|+|\delta_2(z)|)\Pi_2(dz)+\frac{1}{2}(I-I^*)^2\int_{\mathbb{R}}(|\delta_1(z)|+|\delta_2(z)|)\Pi_2(dz)+$$
$$+S^2\int_{\mathbb{R}}(\delta_1^2(z)+|\delta_1(z)|)\Pi_2(dz)+I^2\int_{\mathbb{R}}(\delta_2^2(z)+|\delta_2(z)|)\Pi_2(dz)+\frac{a}{2}\int_{\mathbb{R}}|\delta_2(z)|\Pi_2(dz)+$$
$$+\frac{a}{2}(I-I^*)^2\int_{\mathbb{R}}|\delta_2(z)|\Pi_2(dz)+aI^*\int_{\mathbb{R}}\big(\delta_2(z)-\ln\big(1+\delta_2(z)\big)\big)\Pi_2(dz).$$

Here we use equalities $\Lambda=I^*(\mu+\varepsilon+\rho)+\mu S^*, \beta S^*=\mu+\varepsilon+\rho$, and basic estimates $2cd\le c^2+d^2,\ \ (c+d)^2\le 2(c^2+d^2)$.
Applying the equality $\rho I^*=\mu R^*$ and estimate $2cd\le c^2+d^2$ we obtain

$$LF_2(R,I)=b(R-R^*)(\rho I-\mu R)+\frac{1}{2}b\sigma_3^2R^2+\frac{b}{2}R^2\int_{\mathbb{R}}\gamma_3^2(z)\Pi_1(dz)+b(R-R^*)R\int_{\mathbb{R}}\delta_3(z)\Pi_2(dz)+$$
$$+\frac{b}{2}R^2\int_{\mathbb{R}}\delta_3^2(z)\Pi_2(dz)\le\ b(R-R^*)\rho(I-I^*)-b\mu(R-R^*)^2+\frac{b}{2}\bigg(\sigma_3^2+\int_{\mathbb{R}}\gamma_3^2(z)\Pi_1(dz)+$$
$$+\int_{\mathbb{R}}(\delta_3^2(z)+|\delta_3(z)|)\Pi_2(dz)\bigg)R^2+\frac{b}{2}(R-R^*)^2\int_{\mathbb{R}}|\delta_3(z)|\Pi_2(dz)\le\frac{b}{2}\bigg[\frac{\rho^2}{\mu}(I-I^*)^2-\qquad(15)$$
$$-\bigg(\mu-\int_{\mathbb{R}}|\delta_3(z)|\Pi_2(dz)\bigg)(R-R^*)^2+\bigg(\sigma_3^2+\int_{\mathbb{R}}\gamma_3^2(z)\Pi_1(dz)+\int_{\mathbb{R}}(\delta_3^2(z)+|\delta_3(z)|)\Pi_2(dz)\bigg)R^2\bigg].$$

Choose $a=\frac{(2\mu+\varepsilon+\rho)}{\beta}$ and taking together (14) and (15), we get

$$LF(S,I,R)=LF_1(S,I)+LF_2(R,I)\le-\bigg(\mu-\frac{1}{2}\int_{\mathbb{R}}(|\delta_1(z)|+|\delta_2(z)|)\Pi_2(dz)\bigg)(S-S^*)^2-$$
$$-\bigg(\mu+\varepsilon+\rho-\frac{1}{2}\int_{\mathbb{R}}(|\delta_1(z)|+(1+a)|\delta_2(z)|)\Pi_2(dz)-\frac{b\rho^2}{2\mu}\bigg)(I-I^*)^2-\frac{b}{2}\bigg(\mu-\int_{\mathbb{R}}|\delta_3(z)|\Pi_2(dz)\bigg)(R-R^*)^2$$
$$+\bigg(\frac{1}{2}\sigma_1^2+\int_{\mathbb{R}}\gamma_1^2(z)\Pi_1(dz)+\int_{\mathbb{R}}(\delta_1^2(z)+|\delta_1(z)|)\Pi_2(dz)\bigg)S^2+\bigg(\frac{1}{2}\sigma_2^2+\int_{\mathbb{R}}\gamma_2^2(z)\Pi_1(dz)+\qquad(16)$$
$$+\int_{\mathbb{R}}(\delta_2^2(z)+|\delta_2(z)|)\Pi_2(dz)\bigg)I^2+\frac{b}{2}\bigg(\sigma_3^2+\int_{\mathbb{R}}\gamma_3^2(z)\Pi_1(dz)+\int_{\mathbb{R}}(\delta_3^2(z)+|\delta_3(z)|)\Pi_2(dz)\bigg)R^2+$$
$$+a\bigg(\frac{1}{2}\sigma_2^2+\int_{\mathbb{R}}\big(\gamma_2(z)-\ln\big(1+\gamma_2(z)\big)\big)\Pi_1(dz)+\int_{\mathbb{R}}\big(\delta_2(z)-\ln\big(1+\delta_2(z)\big)\big)\Pi_2(dz)\bigg)I^*+\frac{a}{2}\int_{\mathbb{R}}|\delta_2(z)|\Pi_2(dz)=$$
$$=-\frac{k_1}{2}(S-m_1S^*)^2-\frac{k_2}{2\mu}(I-m_2I^*)^2-\frac{bk_3}{2}(R-m_2R^*)^2+M(\sigma,\gamma,\delta),$$

where

$$k_1=2\mu-\sigma_1^2-2\int_{\mathbb{R}}\gamma_1^2(z)\Pi_1(dz)-\int_{\mathbb{R}}(2\delta_1^2(z)+3|\delta_1(z)|+|\delta_2(z)|)\Pi_2(dz),$$
$$m_1=\frac{1}{k_1}\bigg(2\mu-\int_{\mathbb{R}}(|\delta_1(z)|+|\delta_2(z)|)\Pi_2(dz)\bigg),$$
$$k_2=2\mu(\mu+\varepsilon+\rho)-\mu\sigma_2^2-2\mu\int_{\mathbb{R}}\gamma_2^2(z)\Pi_1(dz)-\mu\int_{\mathbb{R}}(2\delta_2^2(z)+(3+a)|\delta_2(z)|+|\delta_1(z)|)\Pi_2(dz)-b\rho^2,$$
$$m_2=\frac{2\mu}{k_2}\bigg(\mu+\varepsilon+\rho-\frac{1}{2}\int_{\mathbb{R}}(|\delta_1(z)|+(1+a)|\delta_2(z)|)\Pi_2(dz)-\frac{b\rho^2}{2\mu}\bigg),$$
$$k_3=\mu-\sigma_3^2-\int_{\mathbb{R}}\gamma_3^2(z)\Pi_1(dz)-\int_{\mathbb{R}}(\delta_3^2(z)+2|\delta_3(z)|)\Pi_2(dz),m_3=\frac{1}{k_3}\bigg(\mu-\int_{\mathbb{R}}|\delta_3(z)|\Pi_2(dz)\bigg),$$
$$M(\sigma,\gamma,\delta)=\frac{k_1}{2}m_1(m_1-1)S^{*2}+\frac{k_2}{2\mu}m_2(m_2-1)I^{*2}+\frac{bk_3}{2}m_1(m_1-1)R^{*2}+$$
$$+a\bigg(\frac{1}{2}\sigma_2^2+\int_{\mathbb{R}}\big(\gamma_2(z)-\ln\big(1+\gamma_2(z)\big)\big)\Pi_1(dz)+\int_{\mathbb{R}}\big(\delta_2(z)-\ln\big(1+\delta_2(z)\big)\big)\Pi_2(dz)\bigg)I^*+\frac{a}{2}\int_{\mathbb{R}}|\delta_2(z)|\Pi_2(dz).$$

Due to conditions (12) we can choose

$$0<b<\frac{\mu}{\rho^2}\bigg(2(\mu+\varepsilon+\rho)-\sigma_1^2-2\int_{\mathbb{R}}\gamma_1^2(z)\Pi_1(dz)-\int_{\mathbb{R}}(2\delta_1^2(z)+3|\delta_1(z)|+|\delta_2(z)|)\Pi_2(dz)\bigg),$$

and we obtain that $k_i>0, m_i>1, i=1,2,3$.
From (13) we derive

$$dF\big(S(t),I(t),R(t)\big)=dF_1\big(S(t),I(t)\big)+dF_2\big(R(t)\big)=LF\big(S(t),I(t),R(t)\big)dt+dJ_{stoch}(t),\qquad(17)$$

where

$$dJ_{stoch}(t) = (S(t) - S^* + I(t) - I^*)\sigma_1 S(t)dw_1(t) + (S(t) - S^* + I(t) - I^*)\sigma_2 I(t)dw_2(t) + a(I(t) - I^*)\sigma_2 dw_2(t) +$$
$$+\int_{\mathbb{R}} \left[(S(t^-) - S^* + I(t^-) - I^*)\big(\gamma_1(z)S(t^-) + \gamma_2(z)I(t^-)\big) + \frac{1}{2}\big(\gamma_1(z)S(t^-) + \gamma_2(z)I(t^-)\big)^2 +\right.$$
$$\left.+a(\gamma_2(z)I(t^-) - I^* \ln(1+\gamma_2(z))\right]\tilde{\nu}_1(dt,dz) + \int_{\mathbb{R}} \left[(S(t^-) - S^* + I(t^-) - I^*)\big(\delta_1(z)S(t^-) + \delta_2(z)I(t^-)\big) +\right.$$
$$\left.+\frac{1}{2}\big(\delta_1(z)S(t^-) + \delta_2(z)I(t^-)\big)^2 + a(\delta_2(z)I(t^-) - I^* \ln(1+\delta_2(z))\right]\tilde{\nu}_2(dt,dz) + b(R(t) - R^*)\sigma_3 R(t)dw_3(t) +$$
$$+\int_{\mathbb{R}} \left[b(R(t^-) - R^*)\gamma_3(z)R(t^-) + \frac{1}{2}b\gamma_3^2(z)R^2(t^-)\right]\tilde{\nu}_1(dt,dz) +$$
$$+\int_{\mathbb{R}} \left[b(R(t^-) - R^*)\delta_3(z)R(t^-) + \frac{1}{2}b\delta_3^2(z)R^2(t^-)\right]\tilde{\nu}_2(dt,dz).$$

Let $n_0 \in \mathbb{N}$ be sufficiently large for $S(0) \in \left[\frac{1}{n_0}, n_0\right]$, $I(0) \in \left[\frac{1}{n_0}, n_0\right]$ and $R(0) \in \left[\frac{1}{n_0}, n_0\right]$. For any $n \geq n_0$ we define the stopping time $\tau_n = \inf\left\{t \geq 0 : X(t) \notin \left(\frac{1}{n}, n\right)^3\right\}$, where $X(t) = (S(t), I(t), R(t))$, $(S(t), I(t), R(t))$ is the solution to the system (4). According to Theorem 1, the system (4) has a unique global (no explosion in the finite time interval) solution $(S(t), I(t), R(t))$, and $\lim_{n\to\infty} \tau_n = +\infty$ a.s. Integrating both sides of (17) from 0 to $t \wedge \tau_n$, then taking expectations, and applying inequality (16) we obtain

$$0 \leq F\big(S(t\wedge\tau_n), I(t\wedge\tau_n), R(t\wedge\tau_n)\big) \leq F\big(S(0), I(0), R(0)\big) -$$
$$-E\left[\int_0^{t\wedge\tau_n} \left(\frac{k_1}{2}(S(v) - m_1 S^*)^2 + \frac{k_2}{2\mu}(I(v) - m_2 I^*)^2 + \frac{bk_3}{2}(R - m_2 R^*)^2\right) dv\right] + M(\sigma,\gamma,\delta)t \leq \tag{18}$$
$$\leq F\big(S(0), I(0), R(0)\big) - K(\sigma,\gamma,\delta)E\left[\int_0^{t\wedge\tau_n} ((S(v) - m_1 S^*)^2 + (I(v) - m_2 I^*)^2 + (R - m_2 R^*)^2) dv\right] + M(\sigma,\gamma,\delta)t$$

where $K(\sigma,\gamma,\delta) = \min\left\{\frac{k_1}{2}, \frac{k_2}{2\mu}, \frac{bk_3}{2}\right\}$.

If $n \to \infty$, then from (18) we have

$$\frac{1}{t}E\left[\int_0^t ((S(v) - m_1 S^*)^2 + (I(v) - m_2 I^*)^2 + (R - m_2 R^*)^2) dv\right] \leq \frac{1}{K(\sigma,\gamma,\delta)t}\left[F\big(S(0), I(0), R(0)\big) + M(\sigma,\gamma,\delta)t\right]. \tag{19}$$

If $t \to \infty$, then we have from (19)

$$\limsup_{t\to\infty} \frac{1}{t} E\left[\int_0^t \{(S(v) - m_1 S^*)^2 + (I(v) - m_2 I^*)^2 + (R(v) - m_3 R^*)^2\} dv\right] \leq \frac{M(\sigma,\gamma,\delta)}{K(\sigma,\gamma,\delta)}.$$

The proof of the theorem is completed.

Remark 2. If $\gamma_i(z) \equiv 0, \delta_i(z) \equiv 0, i = 1,2,3$, then Theorem 3 is Theorem 4.1 from (Jiang et al., 2011).

Discussion and conclusions

The existence and uniqueness of the global positive solution are proved for the system of stochastic differential equations describing a non-autonomous SIR model disturbed by white noise, centered and non-centered Poisson noises. In the deterministic case there is a threshold of the system for an epidemic to occur, so called the basic reproduction number. Depending on the value of the reproduction number there is the disease-free equilibrium, or there is an endemic equilibrium, which implies the disease always remains. In the case of the autonomous stochastic SIR model, we study the asymptotic behavior of the solution to the corresponding system of stochastic differential equations around these points of equilibrium.

Authors' contribution: Oleksandr Borysenko – conceptualization, writing and editing; Olga Borysenko – methodology, writing-viewing and editing.

References

Anderson, R. M., & May, R. M. (1979). Population biology of infectious diseases. *Nature,* 280 (1), 361–367.

Jiang, D., Yua, J., Ji, Ch., & Shi, N. (2011). Asymptotic behavior of global positive solution to a stochastic SIR model. *Mathematical and Computer Modelling,* 54, 221–232. https://doi.org/10.1016/j.mcm.2011.02.004

Zhang, X., & Wang, Ke. (2013). Stochastic SIR model with jumps. *Applied Mathematics Letters,* 26, 867–874. https://doi.org/10.1016/j.aml.2013.03.013

Borysenko, O. D., & Borysenko, O. V. (2022). A stochastic predator-prey model that depends on the population density of the predator. *Bulletin of Taras Shevchenko National University of Kyiv, Series: Physics & Mathematics,* 4, 11–17. https://doi.org/10.17721/1812-5409.2022/4.1

Gikhman, I.I., & Skorokhod, A.V. (1982). *Stochastic Differential Equations and its Applications.* Naukova Dumka [in Russian]. [Гихман, И. И., & Скороход А. В. (1982). *Стохастические дифференциальные уравнения и их приложения.* Наукова думка].

Отримано редакцією журналу / Received: 17.12.24
Прорецензовано / Revised: 10.03.25
Схвалено до друку / Accepted: 14.03.25

Олександр БОРИСЕНКО, канд. фіз.-мат. наук, доц.
ORCID ID: 0000-0002-6670-8605
e-mail: borysenko_oleksandr@knu.ua
Київський національний університет імені Тараса Шевченка, Київ, Україна

Ольга БОРИСЕНКО, канд. фіз.-мат. наук, доц.
ORCID ID: 0000-0002-2099-2911
e-mail: oborisenko1373@gmail.com
Національний технічний університет України "Київський політехнічний інститут імені Ігоря Сікорського", Київ, Україна

СТОХАСТИЧНА SIR-МОДЕЛЬ ІЗ ПУАССОНІВСЬКИМИ ШУМАМИ

Ураховуючи негативний вплив інфекційних захворювань на зростання популяції, важливо зрозуміти динаміку поведінки таких захворювань. У вивченні розповсюдження інфекційних захворювань широко використовують детерміновані математичні SIR-моделі. У реальному світі є багато випадкових чинників, що впливають на динаміку поширення інфекційних захворювань, тому використання стохастичних моделей є більш виправданим. Якщо під час побудови математичної моделі ми хочемо брати до уваги такі збурення, яким піддається довкілля, як пожежі, землетруси тощо, то ми маємо вводити в популяційну модель пуассонівські шуми, щоб описувати такі системи зі стрибками. Ми довели існування і єдиність глобального додатного розв'язку системи стохастичних диференціальних рівнянь, що описує неавтономну SIR-модель, збурену білим шумом, центрованим і нецентрованим пуассонівськими шумами. У детермінованому автономному випадку існує поріг системи, так зване основне репродуктивне число, за якого відбувається епідемія. Залежно від величини репродуктивного числа існує рівновага, вільна від захворювання, або ендемічна рівновага, яка означає, що захворювання залишається назавжди. Щодо автономної стохастичної SIR-моделі зі стрибками, ми вивчили асимптотичну поведінку розв'язку відповідної системи стохастичних диференціальних рівнянь навколо цих точок рівноваги.

К л ю ч о в і с л о в а : ***стохастична SIR-модель, шум довкілля, центрований і нецентрований пуассонівські шуми, глобальний додатний розв'язок, рівновага вільна від захворювання, ендемічна рівновага, асимптотична поведінка.***

Автори заявляють про відсутність конфлікту інтересів. Спонсори не брали участі в розробленні дослідження; у зборі, аналізі чи інтерпретації даних; у написанні рукопису; в рішенні про публікацію результатів.

The authors declare no conflicts of interest. The funders had no role in the design of the study; in the collection, analyses or interpretation of data; in the writing of the manuscript; in the decision to publish the results.