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ABOUT DESCENT PROPERTY FOR EXTRAGRADIENT ALGORITHM

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ПРО ВЛАСТИВІСТЬ СПУСКУ ДЛЯ ЕКСТРАГРАДІЄНТНОГО АЛГОРИТМУ

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Abstract. A new result on the descent property of the extragradient algorithm is proved, and for unconstrained problems, an estimate of the algorithm speed in terms of natural residuals is obtained under weaker conditions than known ones.
Keywords: variational inequality, operator equation, monotone operator, extragradient algorithm, rate of convergence.

Анотація. Доведено новий результат щодо властивості спуску екстраградієнтного алгоритму, а для задач без обмежень отримано оцінку швидкості екстраградієнтного алгоритму в термінах природної нев'язки за слабших умов, ніж відомі.
Ключові слова: варіаційна нерівність, операторне рівняння, монотонний оператор, екстраградієнтний алгоритм, швидкість збіжності.

1. Introduction

Variational inequalities provide a convenient general framework for formulating relevant problems in optimal control, operations research, and machine learning [1–5]. Certain non-smooth convex optimization problems can be efficiently solved by reformulating them as saddle-point problems and applying algorithms for solving variational inequalities [6]. Recently, this approach has been used to develop fast algorithms for convex programming problems: by

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leveraging duality theory, the problem is reformulated as a convex-concave saddle-point problem, and then extragradient algorithms (both standard and accelerated versions) for solving variational inequalities are applied [7].

The simplest method for solving variational inequalities is an analogue of the gradient descent method, which, in the case of saddle-point problems, is known as the gradient descent-ascent method (Arrow–Hurwicz method) [8]. However, this method may fail to converge for inequalities with a monotone operator (a typical example is the saddle-point problem for the Lagrangian of a linear programming problem).

A well-known modification of the projected gradient descent method for variational inequalities is the extragradient method by Korpelevich [8, 9], whose iteration requires two evaluations of the problem's operator and two metric projections onto the feasible set. «Computationally cheaper» variants of the extragradient algorithm, requiring only one metric projection onto the feasible set, have been proposed in [10–12].

Further attempts to reduce iteration complexity while preserving convergence properties led to the development of new methods, such as the «projected reflected gradient algorithm» and the «forward-reflected-backward algorithm» for solving variational inequalities and operator inclusions [13, 14]. These algorithms have certain advantages over the extragradient method regarding computational cost per iteration (one operator evaluation and one projection). Variants of this algorithm are known among specialists as «optimistic gradient descent ascent» [15, 16] and the «operator extrapolation algorithm» [17]. Combining the extragradient algorithm with the Halpern iteration scheme has enabled the development of accelerated algorithms for saddle-point problems [18, 19].

The objective of this work is to establish a descent property inequality for the extragradient algorithm, which allows justifying the existence of solutions to the problem and proving the algorithm's convergence under the assumption of boundedness of the generated sequence. Moreover, for unconstrained problems with the descent property, a convergence rate estimate for the natural residual follows.

The results obtained in this work are new and refine the existing ones [20, 21].

2. Preliminaries and problem statement

Let H be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Consider the variational inequality problem:

$$\text{find } x \in C : \quad \langle Ax, y - x \rangle \geq 0 \quad \forall y \in C, \tag{1}$$

where C is a non-empty subset of Hilbert space H, A is an operator mapping from H in H. Denote the solution set of (1) as S.

Assume the following conditions hold:

- the set $C \subseteq H$ is convex and closed;
- the operator $A : H \to H$ is monotone on C, i.e.,

$$\langle Ax - Ay, x - y \rangle \geq 0 \quad \forall x, y \in C,$$

and Lipschitz continuous on C (with constant $L > 0$), i.e.,

$$\|Ax - Ay\| \leq C\|x - y\| \quad \forall x, y \in C.$$

For concreteness, let us consider an example of a variational inequality problem. A saddle-point (minimax) problem has the form:

$$\min_{x \in X} \max_{y \in Y} F(x, y),$$

where $X \subseteq R^{d_1}$, $Y \subseteq R^{d_2}$ is convex closed subsets, $F : X \times Y \to R$ is a smooth function. The saddle-point problem consists of finding a saddle point $\bar{z} = (\bar{x}, \bar{y}) \in C = X \times Y$ such that

$$F(\bar{x}, y) \leq F(\bar{x}, \bar{y}) \leq F(x, \bar{y}) \quad \forall x \in X \ \ \forall y \in Y.$$

The saddle point exists if, for instance, the function F is convex-concave, and the sets X, Y are compact. The point $\bar{z} \in C$ can be characterized as a solution to a variational inequality:

$$\langle A\bar{z}, z - \bar{z} \rangle \geq 0 \quad \forall z \in C,$$

where the monotone operator A (in the convex-concave case) is given by:

$$Az = \begin{pmatrix} \nabla_1 F(x, y) \\ -\nabla_2 F(x, y) \end{pmatrix}, \quad z = (x, y) \in C = X \times Y.$$

In the absence of constraints, (1) reduces to the operator equation

$$Ax = 0.$$

Let K be a nonempty, closed, and convex subset of a Hilbert space H. It is known [22], that for all $x \in H$, there exists a unique element $z \in C$ such that

$$\|z - x\| = \inf_{y \in C} \|y - x\|.$$

This element z is denoted as $P_K x$, and the corresponding operator $P_K : H \to K$ is called projection H onto K (metric projection) [22]. The projection is characterized as follows [22]:

$$z = P_K x \quad \Leftrightarrow \quad z \in K \ \text{ and } \ \langle z - x, y - z \rangle \geq 0 \quad \forall y \in K.$$

The last inequality is equivalent to the following one [22]:

$$\|y - P_K x\|^2 \leq \|y - x\|^2 - \|P_K x - x\|^2 \quad \forall y \in C.$$

The variational inequality (1) can be reformulated as a fixed-point problem [22]:

$$x = P_C(x - \lambda A x),$$

where $\lambda > 0$. This formulation is useful because it leads to the iterative scheme:

$$x_{n+1} = P_C(x_n - \lambda A x_n),$$

which is weakly convergent for inverse strongly monotone (cocoercive) operators $A : H \to H$ [22]. However, for general Lipschitz continuous monotone operators, this scheme may fail to converge. The most well-known modification of this scheme for variational inequalities is the extragradient method by Korpelevich [8, 9]. Its iteration requires two evaluations of the operator and two metric projections onto the feasible set. More details on the extragradient method follow below.

For the variational inequality (1) with a Lipschitz continuous and monotone operator, the extragradient algorithm is given by:

$$\begin{cases} y_n = P_C\left(x_n - \lambda A x_n\right), \\ x_{n+1} = P_C\left(x_n - \lambda A y_n\right), \end{cases}$$

where $x_0 \in C$, $\lambda \in \left(0, \frac{1}{L}\right)$ [8,9].

It is known that if the feasible set $C \subseteq H$ is bounded, an efficiency estimate in terms of the gap function holds:

$$\operatorname{gap}(z_N) = \sup_{y \in C} \langle Ay, z_N - y \rangle = O\left(\frac{1}{\sqrt{N}}\right),$$

where $z_N = \frac{1}{N} \sum_{n=1}^{N} y_n$ is averaged output of the algorithm [6]. Additionally, in [23], a variant of the algorithm using Bregman divergence and an adaptive selection of $\lambda > 0$ was proposed.

From the known inequality [9–11]

$$\|x_{n+1} - z\|^2 \leq \|x_n - z\|^2 - \left(1 - \lambda^2 L^2\right) \|y_n - x_n\|^2,$$

where $z \in S$, in the unconstrained case ($C = H$), the following estimate holds:

$$\min_{0 \leq i \leq N} \|A x_i\| \leq \frac{\|x_0 - z\|}{\lambda \sqrt{1 - \lambda^2 L^2} \sqrt{N+1}}, \quad N \geq 0,$$

where $z \in S$, $\lambda \left(0, \frac{1}{L}\right)$.

For a long time, an open question has been whether the best-iterate estimate

$$\min_{0 \leq i \leq N} \|A x_i\| = O\left(\frac{1}{\sqrt{N}}\right)$$

could be strengthened to a last-iterate estimate

$$\|A x_N\| = O\left(\frac{1}{\sqrt{N}}\right).$$

A key aspect is proving the monotonic behavior of the natural residual $\|A x_i\|$. Only in [20] was this question resolved positively, but under the assumption

$$\lambda \in \left(0, \frac{1}{\sqrt{2} L}\right).$$

Similar results for the extragradient and related algorithms were obtained in [21, 24].

Below, under the assumption $\lambda \in \left(0, \frac{1}{L}\right)$, we will prove the monotonicity of the residual with an estimate of its decrease at each iteration (Theorem 2). This follows from a new descent property inequality (Theorem 1), which we propose for analyzing the extragradient method without explicitly using solutions of the variational inequality.

These results refine the known ones [20, 21] and, in our view, are simpler.

3. A NOVEL DESCENT PROPERTY FOR MONOTONE VARIATIONAL INEQUALITIES

Consider the extragradient algorithm for the variational inequality (1)

$$\begin{cases} y_n = P_C\left(x_n - \lambda A x_n\right), \\ x_{n+1} = P_C\left(x_n - \lambda A y_n\right), \end{cases} \tag{2}$$

where $\lambda > 0$ [9].

The following inequalities hold

$$\langle y_n - x_n + \lambda A x_n, y - y_n\rangle \geq 0 \quad \forall y \in C, \tag{3}$$

$$\langle x_{n+1} - x_n + \lambda A y_n, y - x_{n+1}\rangle \geq 0 \quad \forall y \in C. \tag{4}$$

From inequalities (3) and (4), we obtain

$$\langle y_{n+1} - x_{n+1} + \lambda A x_{n+1}, x_{n+2} - y_{n+1}\rangle \geq 0, \tag{5}$$

$$\langle x_{n+1} - x_n + \lambda A y_n, y_{n+1} - x_{n+1}\rangle \geq 0, \tag{6}$$

$$\langle x_{n+2} - x_{n+1} + \lambda A y_{n+1}, x_{n+1} - x_{n+2}\rangle \geq 0. \tag{7}$$

Inequality (5) is obtained from (3) by substitution $y = x_{n+2}$ under $n \leftarrow n+1$, inequality (6) is obtained from (4) by substitution $y = y_{n+1}$, and inequality (7) is obtained from (4) by substitution $y = x_{n+1}$ under $n \leftarrow n+1$.

Summing (5), (6), and (7) and multiplying by 2

$$\begin{aligned} &2\langle y_{n+1} - x_{n+1}, x_{n+2} - y_{n+1}\rangle + 2\lambda\langle A x_{n+1}, x_{n+2} - y_{n+1}\rangle \\ &\quad + 2\langle x_{n+1} - x_n, y_{n+1} - x_{n+1}\rangle + 2\lambda\langle A y_n, y_{n+1} - x_{n+1}\rangle \\ &\quad\quad - 2\|x_{n+2} - x_{n+1}\|^2 + 2\lambda\langle A y_{n+1}, x_{n+1} - x_{n+2}\rangle \geq 0. \end{aligned} \tag{8}$$

Transform the expression

$$2\langle x_{n+1} - x_n, y_{n+1} - x_{n+1}\rangle + 2\lambda\langle A y_n, y_{n+1} - x_{n+1}\rangle$$

in (8)

$$\begin{aligned} &\langle x_{n+1} - x_n, y_{n+1} - x_{n+1}\rangle + 2\lambda\langle A y_n, y_{n+1} - x_{n+1}\rangle \\ &\quad = \|x_n - y_{n+1}\|^2 - \|x_n - x_{n+1}\|^2 - \|x_{n+1} - y_{n+1}\|^2 \\ &\quad\quad + 2\lambda\langle A y_n, y_{n+1} - x_{n+1}\rangle. \end{aligned} \tag{9}$$

Perform similar transformations for the other terms from (8)

$$\begin{aligned} &2\langle y_{n+1} - x_{n+1}, x_{n+2} - y_{n+1}\rangle + 2\lambda\langle A x_{n+1}, x_{n+2} - y_{n+1}\rangle \\ &\quad - 2\|x_{n+2} - x_{n+1}\|^2 + 2\lambda\langle A y_{n+1}, x_{n+1} - x_{n+2}\rangle \\ &= 2\lambda\langle A y_{n+1} - A x_{n+1}, x_{n+1} - x_{n+2}\rangle + 2\lambda\langle A x_{n+1}, x_{n+1} - y_{n+1}\rangle \\ &\quad - \|x_{n+2} - x_{n+1}\|^2 - \|y_{n+1} - x_{n+1}\|^2 - \|x_{n+2} - y_{n+1}\|^2. \end{aligned} \tag{10}$$

Using (9) and (10), rewrite inequality (8) in the following form

$$\begin{aligned} &\|x_{n+2} - x_{n+1}\|^2 + \|x_{n+2} - y_{n+1}\|^2 + 2\lambda\langle A x_{n+1} - A y_{n+1}, x_{n+1} - x_{n+2}\rangle \\ &\quad \leq \|x_n - y_{n+1}\|^2 - \|x_n - x_{n+1}\|^2 - 2\|x_{n+1} - y_{n+1}\|^2 \\ &\quad\quad + 2\lambda\langle A y_n - A x_{n+1}, y_{n+1} - x_{n+1}\rangle. \end{aligned} \tag{11}$$

Let us estimate the last term in (11) from above. It follows from the identity

$$\|a+b\|^2 = \|a\|^2 + \|b\|^2 - \|a-b\|^2,$$

the Lipschitz continuity, and the monotonicity of operator A that

$$\begin{aligned}
2\lambda\langle Ay_n - Ax_{n+1}, y_{n+1} - x_{n+1}\rangle &= \underbrace{2\lambda\langle Ax_n - Ax_{n+1}, x_{n+1} - x_n\rangle}_{\leq 0} \\
&+ 2\lambda\langle Ay_n - Ax_n, x_{n+1} - x_n\rangle + 2\lambda\langle Ay_n - Ax_{n+1}, y_{n+1} + x_n - 2x_{n+1}\rangle \\
&\leq 2\lambda\langle Ay_n - Ax_n, x_{n+1} - x_n\rangle + \lambda^2\|Ay_n - Ax_{n+1}\|^2 + \|y_{n+1} + x_n - 2x_{n+1}\|^2 \\
&\leq 2\lambda\langle Ay_n - Ax_n, x_{n+1} - x_n\rangle + \lambda^2L^2\|y_n - x_{n+1}\|^2 \\
&+ 2\|x_n - x_{n+1}\|^2 + 2\|y_{n+1} - x_{n+1}\|^2 - \|x_n - y_{n+1}\|^2. \qquad (12)
\end{aligned}$$

Applying (12) to (11), we obtain the inequality

$$\begin{aligned}
&\|x_{n+2} - x_{n+1}\|^2 + \|x_{n+2} - y_{n+1}\|^2 + 2\lambda\langle Ax_{n+1} - Ay_{n+1}, x_{n+1} - x_{n+2}\rangle \\
&\quad\leq \|x_{n+1} - x_n\|^2 + \|x_{n+1} - y_n\|^2 + 2\lambda\langle Ax_n - Ay_n, x_n - x_{n+1}\rangle \\
&\qquad - (1 - \lambda^2L^2)\|y_n - x_{n+1}\|^2.
\end{aligned}$$

Thus, we obtain the following result (Descent Property).

Theorem 1. *Let $C \subseteq H$ be a nonempty, convex, and closed set, and $A : C \to H$ be a monotone and Lipschitz continuous operator (with constant $L > 0$). Then, for the sequences (x_n) and (y_n) generated by the extragradient algorithm (2), the following inequality holds:*

$$\begin{aligned}
&\|x_{n+2} - x_{n+1}\|^2 + \|x_{n+2} - y_{n+1}\|^2 + 2\lambda\langle Ax_{n+1} - Ay_{n+1}, x_{n+1} - x_{n+2}\rangle \\
&\quad\leq \|x_{n+1} - x_n\|^2 + \|x_{n+1} - y_n\|^2 + 2\lambda\langle Ax_n - Ay_n, x_n - x_{n+1}\rangle \\
&\qquad - (1 - \lambda^2L^2)\|y_n - x_{n+1}\|^2.
\end{aligned}$$

Remark 1. From Theorem 1, it follows that under the condition $\lambda \in \left(0, \frac{1}{L}\right)$ the numerical sequence

$$W_n = \|x_{n+1} - x_n\|^2 + \|x_{n+1} - y_n\|^2 + 2\lambda\langle Ax_n - Ay_n, x_n - x_{n+1}\rangle$$

is non-decreasing. Moreover, it is nonnegative, since

$$\begin{aligned}
W_n &= \|x_{n+1} - x_n\|^2 + \|x_{n+1} - y_n\|^2 + 2\lambda\langle Ax_n - Ay_n, x_n - x_{n+1}\rangle \\
&= \|x_{n+1} - x_n\|^2 + \|x_{n+1} - y_n\|^2 + 2\lambda\langle Ax_{n+1} - Ay_n, x_n - x_{n+1}\rangle \\
&\quad + 2\lambda\underbrace{\langle Ax_n - Ax_{n+1}, x_n - x_{n+1}\rangle}_{\geq 0} \\
&\geq \|x_{n+1} - x_n\|^2 + \|x_{n+1} - y_n\|^2 - 2\lambda\|Ax_{n+1} - Ay_n\|\|x_n - x_{n+1}\| \\
&\geq \|x_{n+1} - x_n\|^2 + \|x_{n+1} - y_n\|^2 - 2\lambda L\|x_{n+1} - y_n\|\|x_n - x_{n+1}\| \\
&\geq (1 - \lambda L)\left(\|x_{n+1} - x_n\|^2 + \|x_{n+1} - y_n\|^2\right) \geq 0.
\end{aligned}$$

Note that W_n can play the role of the Lyapunov function for a system with the dynamics described by the algorithm (2). It is obvious that, $W_n = 0 \Leftrightarrow x_n = y_n$.

4. A LAST-ITERATE CONVERGENCE FOR MONOTONE OPERATOR EQUATIONS

In the absence of constraints ($C = H$), the algorithm takes the form

$$\begin{cases} y_n = x_n - \lambda A x_n, \\ x_{n+1} = x_n - \lambda A y_n, \end{cases} \tag{13}$$

where $\lambda > 0$.

From Theorem 1, we obtain the following result.

Theorem 2. *Let $A : H \to H$ be a monotone and Lipschitz continuous operator (with constant $L > 0$). Then, for the sequences (x_n) and (y_n) generated by the extragradient algorithm* (13)*, the following inequality holds:*

$$\|Ax_{n+1}\|^2 \le \|Ax_n\|^2 - (1 - \lambda^2 L^2)\|Ay_n - Ax_n\|^2. \tag{14}$$

Proof. Rewrite the inequality from Theorem 1 for (13)

$$\begin{aligned} \lambda^2\|Ay_{n+1}\|^2 + \lambda^2\|Ay_{n+1} - Ax_{n+1}\|^2 + 2\lambda^2\langle Ax_{n+1} - Ay_{n+1}, Ay_{n+1}\rangle \\ \le \lambda^2\|Ay_n\|^2 + \lambda^2\|Ay_n - Ax_n\|^2 + 2\lambda^2\langle Ax_n - Ay_n, Ay_n\rangle \\ - (1 - \lambda^2 L^2)\lambda^2\|Ay_n - Ax_n\|^2 \end{aligned} \tag{15}$$

After expanding the squared norms of differences in (15), simplifying, and dividing by λ^2, we obtain inequality (14). □

Remark 2. From Theorem 2, it follows that the numerical sequence $(\|Ax_n\|)$ is nondecreasing under the condition $\lambda \in \left(0, \frac{1}{L}\right)$. This fact was established in [20] under the condition $\lambda \in \left(0, \frac{1}{\sqrt{2}L}\right)$.

Assume that

$$A^{-1}0 = \{z \in H : Az = 0\} \ne \emptyset.$$

For the extragradient algorithm (13), the estimate is known

$$\begin{aligned} \|x_{n+1} - z\|^2 &\le \|x_n - z\|^2 - \left(1 - \lambda^2 L^2\right)\|y_n - x_n\|^2 \\ &= \|x_n - z\|^2 - \left(1 - \lambda^2 L^2\right)\lambda^2\|Ax_n\|^2, \end{aligned} \tag{16}$$

where $z \in A^{-1}0$ [20]. Sum inequalities (16) over n from 0 to $N \ge 0$ and regroup

$$\left(1 - \lambda^2 L^2\right)\lambda^2 \sum_{n=0}^{N} \|Ax_n\|^2 \le \|x_0 - z\|^2 - \|x_{N+1} - z\|^2 \le \|x_0 - z\|^2.$$

Assume that

$$\lambda \in \left(0, \frac{1}{L}\right).$$

Using the monotonicity of the sequence $(\|Ax_n\|)$, which follows from Theorem 2, we obtain

$$\left(1 - \lambda^2 L^2\right)\lambda^2 (N+1)\|Ax_N\|^2 \le \|x_0 - z\|^2.$$

Thus, the following theorem holds.

Theorem 3. *Let $A : H \to H$ be a monotone and Lipschitz continuous operator (with constant $L > 0$), $A^{-1}0 \neq \emptyset$. Assume that $\lambda \in \left(0, \frac{1}{L}\right)$. Then, for the sequence (x_n) generated by algorithm* (13)*, the inequality holds.*

$$\|Ax_N\|^2 \le \frac{D^2}{(1-\lambda^2 L^2)\lambda^2 (N+1)}, \quad N \ge 0, \tag{17}$$

where $D = d(x_0, A^{-1}0) = \min_{z\in A^{-1}0} \|x_0 - z\|$.

Remark 3. If we choose $\lambda = \frac{1}{\sqrt{2}L}$ in algorithm (13), then (17) takes the form

$$\|Ax_N\| \le \frac{2LD}{\sqrt{N+1}}, \quad N \ge 0. \tag{18}$$

Remark 4. In future work, it is planned to obtain results like Theorems 1, 2, and 3 for the extragradient algorithm with Bregman divergence [6, 23], the Extrapolation from the Past algorithm [24–26], and the Operator Extrapolation algorithm [17]. It is also interesting to obtain similar results for algorithms for variational inequalities in Banach spaces.

5. Conclusions

The paper considers monotone variational inequalities and operator equations that arise as first-order optimality conditions in convex optimization or convex-concave minimax problems and the extragradient algorithm for their approximate solution.

A new result on the descent property of the extragradient algorithm is established. For unconstrained problems, a convergence rate estimate is obtained regarding the natural residual under weaker conditions than previously known.

These results refine the existing ones [20, 21], and we believe their proofs are more straightforward.

In future work, we plan to obtain results like Theorems 1, 2, and 3 for a variant of the extragradient algorithm with Bregman divergence [6, 23], the Extrapolation from the Past algorithm [24–26], and the Operator Extrapolation algorithm [17].

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