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Оцiнювання параметру дифузiї стохастичного рiвняння теплопровiдностi з бiлим шумом

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Estimation of diffusion parameter for stochastic heat equation with white noise

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У роботi розглядається стохастичне диференцiальне рiвняння в часткових похiдних, а саме стохастичне диференцiальне рiвняння теплопровiдностi з бiлим шумом. Побудовано статистичну оцiнку параметру дифузiї, дослiджено властивостi отриманої оцiнки, а саме консистентнiсть та асимптотична нормальнiсть.

Ключовi слова: стохастичне диференцiальне рiвняння теплопровiдностi, оцiнка, асимптотична нормальнiсть, консистентнiсть.

This paper deals with stochastic differential heat equation which is the typical example of stochastic partial differential equations(SPDE). In particular , paper is devoted to the estimation of diffusion parameter σ for the random field which is the solution of stochastic differential heat equation for $\mathbb{R}^d, d = 1, 2, 3$. The estimtion of diffusion parameter was constructed in accordance with observations on the grid. It was shown that the constructed estimate is strictly consistent and asymptotically normal, the asymptotic variance was calculated.

Key Words: stochastic partial differential equations, stochastic differential heat equation, estimate, asymptotical normality, consistency.
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1 Introduction

Stochastic partial differential equations (SPDEs) appeared in literature in the mid-70s of the last century. One of the first publications to consider such was Walsh's work [1]. SPDEs had been introduced due to necessity of modeling and describing of random phenomena in physics, chemistry and biology (problems of filtration in different substances, wave propagation in random environment, diffusion processes under the various conditions, modeling of biology populations) on one hand and due to development of analysis and the theory of random processes, on another.

This paper deals with stochastic heat equation, which is a typical example of SPDE. Specifically, paper is devoted to the estimation of diffusion parameter σ for the random field being the solution of following stochastic heat equation with a Gaussian white noise $\dot{W}$ in dimension $d = 1, 2, 3$:

$$\left(\frac{\partial u}{\partial t} - \frac{1}{2}\Delta u\right)(t, x) = \sigma \dot{W}(x), t > 0, x \in \mathbb{R}^d,$$
$$u(0, x) = 0, x \in \mathbb{R}^d. \qquad (1)$$

According to [1], the solution to such equation is given by

$$u(t, x) = \sigma \int_0^t \int_{\mathbb{R}^d} G(t - s, x - y) W(dy) ds, \quad t \geqslant 0, x \in \mathbb{R}^d, \qquad (2)$$

where function G is the Green function defined as

$$G(t, x) = \begin{cases} (2\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{2t}\right), & \text{if } t > 0, \\ \delta_0(x) & \text{if } t = 0. \end{cases}$$

Statistical formulation of the problem is the following: for a fixed time $t > 0$ and a fixed step $\delta > 0$, the random field u given by (2) is observed on the grid

$$D_N = \left\{(i_1\delta, \dots, i_d\delta), i_1, \dots, i_d \in \{1, \dots, N\}\right\}.$$

Our aim is to construct an estimator for σ based on these observations and to study its asymptotic properties as $N \to \infty$. Our approach is based on the ergodicity, as a function of spatial variable x, of the random field u.

2 Properties of u

We first establish some properties of the solution u.

Theorem 2.1. *1. The field $u(t,x)$ given by (2) is well defined with*

$$E\big[u(t,x)^2\big] = \int_0^t\int_0^t \frac{1}{(2\pi(s_1+s_2))^{d/2}}ds_1ds_2 = \\ = \begin{cases} \frac{4(2-\sqrt{2})t^{3/2}}{3\sqrt{\pi}}, & d=1 \\ \frac{t\ln 2}{\pi}, & d=2 \\ \frac{2(-1+\sqrt{2})\sqrt{t}}{\pi^{3/2}}, & d=3. \end{cases} \quad (3)$$

2. For the fixed $t\in[0,T]$, the random field $u(t,\cdot)$ is stationary.

3. For $d=1$, the following is true:

$$\operatorname{cov}\big(u(t,0),u(t,x)\big) = \frac{2\exp\left(-\frac{x^2}{2t}\right)}{3\sqrt{2\pi}}t^{3/2} - \\ -\frac{2\sqrt{t}x^2\exp\left(-\frac{x^2}{2t}\right)}{3\sqrt{2\pi}} + \frac{2|x|^3}{3} - \frac{4|x|^3}{3}\Phi\left(\frac{|x|}{\sqrt{t}}\right) + \\ +4tx\Phi\left(\frac{x}{\sqrt{2t}}\right) - 2tx - \frac{1}{3}\sqrt{\frac{t}{\pi}}\exp\left(-\frac{x^2}{2t}\right)\times \\ \left[\left(5\sqrt{2}-8\exp\left(\frac{x^2}{4t}\right)\right)t + \left(\sqrt{2}-2\exp\left(\frac{x^2}{4t}\right)\right)x^2 \\ +6\exp\left(\frac{x^2}{4t}\right)\sqrt{\pi t}x\left(2\Phi\left(\frac{x}{\sqrt{t}}\right)-1\right)\right] + \\ +\frac{2|x|^3}{3}\Phi\left(\frac{|x|}{\sqrt{2t}}\right).$$

Proof. 1. The integral in (2) is well defined if the integrand is square integrable.

$$v^2 = E(u(t,x))^2 = \\ = \int_{\mathbb{R}^d}\left(\int_0^t (G(t-s,x-y))ds\right)^2 dy$$

Let introduce the substitution $t-s\to s$, in that case

$$v^2 = \int_{\mathbb{R}^d}\left(\int_0^t G(s,x-y)ds\right)^2 dy$$

At next steps we will use the properties of function G

$$v^2 = \int_{\mathbb{R}^d}\left(\int_0^t G(s,x-y)ds\right)^2 dy = \\ = \int_{\mathbb{R}^d}\left(\int_0^t s^{-d/2}G(1,\frac{x-y}{\sqrt{s}})ds\right)^2 dy = \\ = \int_0^t\int_0^t \frac{1}{(s_1s_2)^{d/2}(2\pi)^d}\int_{\mathbb{R}^d}\exp\left(-\frac{1}{2s_1}|x-y|^2\right)\times \\ \exp\left(-\frac{1}{2s_2}|x-y|^2\right)dy\,ds_1ds_2 = \\ = \int_0^t\int_0^t \frac{1}{(s_1s_2)^{d/2}(2\pi)^d}\times \\ \int_{\mathbb{R}^d}\exp\left(-\frac{|x-y|^2(s_1+s_2)}{2s_1s_2}\right)dy\,ds_1ds_2 = \\ = \int_0^t\int_0^t \frac{1}{(s_1s_2)^{d/2}(2\pi)^d}\left(\frac{2s_1s_2\pi}{s_1+s_2}\right)^{d/2}ds_1ds_2 = \\ = \int_0^t\int_0^t \frac{1}{(2\pi(s_1+s_2))^{d/2}}ds_1ds_2.$$

The last integral exists for $1\leqslant d\leqslant 3$. Computing it for each case $d=1,2,3$, we arrive at (3). Thereby, the field $u(t,x)$ given by formula (2) is well defined.

2. Let us transform the covariance:

$$\operatorname{cov}\big(u(t,z),u(t,z+x)\big) = \\ = \int_{\mathbb{R}^d}\int_0^t G(t-s_1,z-h)ds_1\times \\ \times\int_0^t G(t-s_2,z+x-h)ds_2\,dy = \\ = \int_{\mathbb{R}^d}\int_0^t G(u_1,-y)du_1\int_0^t G(u_2,x-y)du_2\,dy.$$

Since the expression does not depend on z, the field is stationary.

3. Let us evaluate the covariance:

$$\operatorname{cov}\big(u(t,0),u(t,x)\big) = \\ = \int_{\mathbb{R}}\int_0^t G(s_1,x-y)ds_1\int_0^t G(s_2,-y)ds_2\,dy = \\ = \int_{\mathbb{R}}\int_0^t G\Big(1,\frac{|x-y|}{\sqrt{s_1}}\Big)\frac{ds_1}{\sqrt{s_1}}\times \\ \times\int_0^t G\Big(1,\frac{|y|}{\sqrt{s_2}}\Big)\frac{ds_2}{\sqrt{s_2}}dy = \\ = \int_0^t\int_0^t\int_{\mathbb{R}}\exp\left(-\frac{|x-y|^2}{2s_1}-\frac{|y|^2}{2s_2}\right)\frac{dy\,ds_1ds_2}{2\pi\sqrt{s_1s_2}} = \\ = \int_0^t\int_0^t \exp\left(-\frac{|x|^2}{2(s_1+s_2)}\right)\frac{ds_1ds_2}{\sqrt{2\pi(s_1+s_2)}} = \\ = \int_0^t\sqrt{\frac{u}{2\pi}}\exp\left(-\frac{|x|^2}{2u}\right)du+$$

$$+\int_t^{2t}\frac{2t-u}{\sqrt{2\pi u}}\exp\left(-\frac{|x|^2}{2u}\right)du,$$

where in the last equality we have integrated over $s_1+s_2=u$. A direct computation then gives the required formula. □

Theorem 2.2. *For fixed $t>0$, the random field $\{u(t,x), x\in\mathbb{R}^d\}$ is ergodic.*

Proof. Since the field $\{u(t,x), x\in\mathbb{R}^d\}$ is centered Gaussian, it suffices to prove that

$$r(x)=\operatorname{cov}\big(u(t,0),u(t,x)\big)=$$

$$=Eu(t,0)u(t,x)\to 0, x\to\infty.$$

Using the definition of the function G, we have

$$r(x)=\int_{\mathbb{R}^d}\int_0^t G(t-s_1,x-y)ds_1\times$$

$$\times\int_0^t G(t-s_2,-y)ds_2dy=$$

$$=\int_{\mathbb{R}^d}\int_0^t G\left(1,\frac{|x-y|}{\sqrt{s_1}}\right)\frac{ds_1}{(s_1)^{d/2}}\times$$

$$\times\int_0^t G\left(1,\frac{|-y|}{\sqrt{s_2}}\right)\frac{ds_2}{(s_2)^{d/2}}dy=$$

$$=\int_0^t\int_0^t\int_{\mathbb{R}^d}\exp\left(-\frac{|x-y|^2}{2s_1}-\frac{|y|^2}{2s_2}\right)\times$$

$$\times\frac{dy\,ds_1ds_2}{(2\pi)^d(s_1s_2)^{d/2}}=$$

$$=\int_0^t\int_0^t\exp\left(-\frac{|x|^2}{s_1+s_2}\right)\frac{ds_1ds_2}{\big(2\pi(s_1+s_2)\big)^{d/2}}\to$$

$$\to 0, x\to\infty.\quad\square$$

3 Main results

By the results of the previous section, the field $u(t,x_k), k\in\mathbb{Z}^d$ is strictly stationary and ergodic. Therefore, for any Borel function $g:\mathbb{R}^d\to\mathbb{R}$ such that $E|g(u(t,0))|<\infty$, thanks to ergodic theorem, almost surely it holds that

$$\frac{1}{N^d}\sum_{x_k\in D_N}g(u(t,x_k))\to Eg(u(t,x)), N\to\infty. \tag{4}$$

This gives the idea to consider the following estimator for σ^2:

$$\hat{\sigma}^2=\frac{1}{N^dv^2}\sum_{x_k\in D_N}u(t,x_k)^2,$$

where

$$v^2=\int_0^t\int_0^t\frac{1}{(2\pi(s_1+s_2))^{d/2}}ds_1ds_2=$$

$$=\begin{cases}\frac{4(2-\sqrt{2})t^{3/2}}{3\sqrt{\pi}}, & d=1,\\ \frac{t\ln 2}{\pi}, & d=2,\\ \frac{2(-1+\sqrt{2})\sqrt{t}}{\pi^{3/2}}, & d=3.\end{cases}$$

Taking into account (4), we we have the following theorem.

Theorem 3.1. *$\hat{\sigma}^2$ is a strictly consistent estimator for the parameter σ^2 for $N\to\infty$, it means*

$$\hat{\sigma}^2\to\sigma, N\to\infty, \text{ a.s.}$$

Now let us investigate the asymptotic normality of the estimator $\hat{\sigma}^2$. Consider the case $d=1$; for other cases the proofs are similar.

Denote

$$\widetilde{V_N}=\frac{V_N-E[V_N]}{D(V_N)},$$

where

$$V_N=\sum_{k=1}^N u(T,x_k)^2.$$

Since $\widetilde{V_N}$ is an element of the second Wiener chaos (w.r.t. the noise W), we may use the Nualart–Peccati Fourth moment theorem [4].

Consider first the variance of V_N.

Lemma 1. *The following convergence is true:*

$$\frac{D(V_N)}{N} \to 2\sigma^4 + 4\sum_{i=1}^{\infty}\rho(i)^2 = = 2\sum_{i=-\infty}^{\infty}\rho(i)^2, N\to\infty, \quad (5)$$

where

$$\sigma^2 = D(u(T,x_1))$$
$$\rho(i) = \operatorname{cov}\big(u(T,x_i),u(T,0)\big).$$

Proof. Transform

$$D(V_N) = D\bigg(\sum_{k=1}^{N}u(T,x_k)^2\bigg) = \sum_{k=1}^{N}D\big(u(T,x_k)^2\big)+ +2\sum_{k=j}\operatorname{cov}\big(u(T,x_k)^2,u(T,x_j)^2\big) = = N\sigma^2 + 2\sum_{i=1}^{N-1}(N-i)\operatorname{cov}\big(u(T,x_i)^2,u(T,0)^2\big) = = 2N\sigma^4 + 4\sum_{i=1}^{N-1}(N-i)\rho(i)^2.$$

Therefore,

$$\frac{D(V_N)}{N} = 2\sigma^4 + 4\sum_{i=1}^{N-1}\Big(1-\frac{i}{N}\Big)\rho(i)^2 = = 2\sigma^4 + 4\sum_{i=1}^{\infty}\Big(1-\frac{i}{N}\Big)^{+}\rho(i)^2.$$

Since $(1-\frac{i}{N})^+\rho(i)^2 \uparrow \rho(i)^2, N\to\infty$ and $\sum_{i=1}^{\infty}\rho(i)^2<\infty$, then, thanks to the dominated convergence,

$$\frac{D(V_N)}{N} \to 2\sigma^4 + 4\sum_{i=1}^{\infty}\rho(i)^2, N\to\infty. \qquad \square$$

Lemma 2. *The following convergence is true:*

$$\frac{E(V_N-EV_N)^4}{N^2} \to 12\sigma^8 + +48\sigma^4\sum_{m=1}^{\infty}\rho(m)^2 + 48\left(\sum_{m=1}^{\infty}\rho(m)^2\right)^2 = = 12\left(\sum_{m=-\infty}^{\infty}\rho(m)^2\right)^2, N\to\infty. \quad (6)$$

Proof. Write

$$E(V_N-EV_N)^4 = E\left(\sum_{k=1}^{N}\big(u(T,x_k)^2-\sigma^2\big)\right)^4.$$

Define $u(T,x_k)^2-\sigma^2 = Z_k$. Since $u(T,x_k)\sim N_k(0,\sigma^2)$, we have $u(T,x_k)-\rho(0)\sim\sigma^2(N_k(0,1)-1) =: \sigma^2(N_k-1)$. Now write

$$E(V_N-EV_N)^4 = E\left(\sum_{k=1}^{N}Z_k\right)^4 = = \sum_{k=1}^{N}EZ_k^4 + 4\sum_{i_1\neq i_2}EZ_{i_1}^3Z_{i_2} + 3\sum_{i_1\neq i_2}EZ_{i_1}^2Z_{i_2}^2 + +6\sum_{i_1\neq i_2\neq i_3}EZ_{i_1}^2Z_{i_2}Z_{i_3} + \sum_{i_1\neq i_2\neq i_3\neq i_4}EZ_{i_1}Z_{i_2}Z_{i_3}Z_{i_4},$$

where $\sum_{i_1\neq\cdots\neq i_j}$ denotes the some over all j-tuples $(i_1,\ldots,i_j)$ of distinct indices from $\{1,\ldots,N\}$.

We now consider each term separately.

$$EZ_k^4 = \sigma^8E(N_k^2-1)^4 = = \sigma^8(EN_k^8 - 4EN_k^6 + 6EN_k^4 - 4EN_k^2 + 1) = = \sigma^8(7!! - 4\cdot 5!! + 6\cdot 3!! - 4\cdot 1!! + 1) = 70\sigma^8.$$

Further,

$$EZ_{i_1}^3Z_{i_2} = \sigma^8E((N_{i_1}^2-1))^3(N_{i_2}^2-1).$$

Denoting by H_j the jth Hermite polynomial, we have

$$(x^2-1)^3 = H_6(x) + 12H_4(x) + 30H_2(x) + 8,$$

therefore, using the Isserlis formula [6],

$$EZ_{i_1}^3Z_{i_2} = \sigma^8E\big((N_{i_1}^2-1)\big)^3(N_{i_2}^2-1) = = \sigma^8E\big(H_6(N_{i_1}) + 12H_4(N_{i_1})+ +30H_2(N_{i_1}) + 8\big)H_2(N_{i_2}) = = 30\sigma^8E(H_2(N_{i_1})H_2(N_{i_2})) = = 60\sigma^8[E(N_{i_1}N_{i_2})]^2 = 60\sigma^4\rho(i_1-i_2)^2.$$

Further,

$$EZ_{i_1}^2Z_{i_2}^2 = \sigma^8E\big((N_{i_1}^2-1)^2(N_{i_2}^2-1)^2\big)$$

Since

$$(x^2-1)^2 = x^4 - 2x^2 + 1 = H_4(x) + 4H_2(x) + 2,$$

we have

$$EZ_{i_1}^2Z_{i_2}^2 = \sigma^8E((N_{i_1}^2-1))^2(N_{i_2}^2-1)^2 = = \sigma^8E\big(H_4(N_{i_1}) + 4H_2(N_{i_1}) + 2\big)\times \times\big(H_4(N_{i_2}) + 4H_2(N_{i_2}) + 2\big) = = \sigma^8E(H_4(N_{i_1})H_4(N_{i_2}) + 16H_2(N_{i_1})H_2(N_{i_2}) + 4) =$$

$$= \sigma^8 E(H_4(N_{i_1})H_4(N_{i_2})) + 32\sigma^4\rho(i_1-i_2)^2 + 4\sigma^8$$

Let us calculate $E(H_4(N_{i_1})H_4(N_{i_2}))$ separately:

$$E\big(H_4(N_{i_1})H_4(N_{i_2})\big) = \\ = E\big(N_{i_1}^4 - 6N_{i_1}^2+3\big)\big(N_{i_2}^4-6N_{i_2}^2+3\big) = \\ = E(N_{i_1}^4N_{i_2}^4 - 6N_{i_1}^4N_{i_2}^2 + 3N_{i_1}^4 - 6N_{i_1}^2N_{i_2}^4 + \\ +36N_{i_1}^2N_{i_2}^2 - 18N_{i_1}^2 + 3N_{i_2}^4 - 18N_{i_2}^2 + 9)$$

Using the Isserlis formula, we obtain

$$EN_{i_1}^4N_{i_2}^4 = 9(EN_{i_1}^2)^2(EN_{i_2}^2)^2 + \\ +72EN_{i_1}^2EN_{i_2}^2(EN_{i_1}N_{i_2})^2 + 24(EN_{i_1}N_{i_2})^4 = \\ = 9 + 72(EN_{i_1}N_{i_2})^2 + 24(EN_{i_1}N_{i_2})^4$$

and

$$EN_{i_1}^4N_{i_2}^2 = 3EN_{i_2}^2(EN_{i_1}^2)^2 + \\ +12EN_{i_1}^2(EN_{i_1}N_{i_2})^2 = 3 + 12(EN_{i_1}N_{i_2})^2.$$

Substituting this result into initial formula, we get

$$E\big(H_4(N_{i_1})H_4(N_{i_2})\big) = \\ = E(N_{i_1}^4N_{i_2}^4 - 6N_{i_1}^4N_{i_2}^2 + 3N_{i_1}^4 - 6N_{i_1}^2N_{i_2}^4 + \\ +36N_{i_1}^2N_{i_2}^2 - 18N_{i_1}^2 + 3N_{i_2}^4 - 18N_{i_2}^2 + 9) = \\ = 9 + 72(EN_{i_1}N_{i_2})^2 + 24(EN_{i_1}N_{i_2})^4 - \\ -6\big(3+12(EN_{i_1}N_{i_2})^2\big) + 3\cdot 3 - \\ -6\big(3+12(EN_{i_1}N_{i_2})^2\big) + 36\big(1+2(EN_{i_1}N_{i_2})^2\big) - \\ -18 + 3\cdot 3 - 18 + 9 = \\ = 24(EN_{i_1}N_{i_2})^4.$$

Consequently,

$$EZ_{i_1}^2Z_{i_2}^2 = \sigma^8E(H_4(N_{i_1})H_4(N_{i_2})) + \\ +32\sigma^4\rho(i_1-i_2)^2 + 4\sigma^8 = \\ = 24\rho(i_1-i_2)^4 + 32\sigma^4\rho(i_1-i_2)^2 + 4\sigma^8.$$

Now

$$EZ_{i_1}^2Z_{i_2}Z_{i_3} = \sigma^8E(N_{i_1}^2-1)^2(N_{i_2}^2-1)\times \\ \times(N_{i_3}^2-1) = \sigma^8E\big(N_{i_1}^4N_{i_2}^2N_{i_3}^2 - N_{i_1}^4N_{i_2}^2 - \\ -N_{i_1}^4N_{i_3}^2 + N_{i_1}^4 - 2N_{i_1}^2N_{i_2}^2N_{i_3}^2 + 2N_{i_1}^2N_{i_2}^2 + \\ +2N_{i_1}^2N_{i_3}^2 - 2N_{i_1}^2 + N_{i_2}^2N_{i_3}^2 - N_{i_2}^2 - N_{i_3}^2 + 1\big).$$

By the Isserlis formula,

$$EN_{i_1}^2N_{i_2}^2N_{i_3}^2 = EN_{i_1}^2EN_{i_2}^2EN_{i_3}^2 + \\ +2EN_{i_1}^2(EN_{i_2}N_{i_3})^2 + 2EN_{i_2}^2(EN_{i_1}N_{i_3})^2 + \\ +2EN_{i_3}^2(EN_{i_1}N_{i_2})^2 + \\ +8EN_{i_1}N_{i_2}EN_{i_1}N_{i_3}EN_{i_2}N_{i_3};$$

$$EN_{i_1}^4N_{i_2}^2N_{i_3}^2 = 3(EN_{i_1}^2)^2EN_{i_2}^2EN_{i_3}^2 + \\ +12EN_{i_1}^2EN_{i_2}^2(EN_{i_1}N_{i_3})^2 + \\ +12EN_{i_1}^2EN_{i_3}^2(EN_{i_1}N_{i_2})^2 + \\ +6(EN_{i_1}^2)^2(EN_{i_2}N_{i_3})^2 + \\ +48EN_{i_1}^2EN_{i_1}N_{i_2}EN_{i_2}N_{i_3}EN_{i_1}N_{i_3} + \\ +24(EN_{i_1}N_{i_2})^2(EN_{i_1}N_{i_3})^2.$$

Therefore,

$$EZ_{i_1}^2Z_{i_2}Z_{i_3} = \sigma^8E\big(N_{i_1}^4N_{i_2}^2N_{i_3}^2 - N_{i_1}^4N_{i_2}^2 - \\ -N_{i_1}^4N_{i_3}^2 + N_{i_1}^4 - 2N_{i_1}^2N_{i_2}^2N_{i_3}^2 + \\ +2N_{i_1}^2N_{i_2}^2 + 2N_{i_1}^2N_{i_3}^2 - 2N_{i_1}^2 + \\ +N_{i_2}^2N_{i_3}^2 - N_{i_2}^2 - N_{i_3}^2 + 1\big) = \\ = 3\sigma^8 + 12\sigma^4\rho(i_1-i_3)^2 + \\ +12\sigma^4\rho(i_1-i_2)^2 + 6\sigma^4\rho(i_2-i_3)^2 + \\ +48\sigma^2\rho(i_1-i_2)\rho(i_2-i_3)\rho(i_1-i_3) + \\ +24\rho(i_1-i_2)^2\rho(i_1-i_3)^2 - 3\sigma^8 - \\ -12\sigma^4\rho(i_1-i_2)^2 - 3\sigma^8 - 12\sigma^4\rho(i_1-i_3)^2 + \\ +3\sigma^8 - 2(\sigma^8 + 2\sigma^4\rho(i_2-i_3)^2 + \\ +2\sigma^4\rho(i_1-i_3)^2 + 2\sigma^4\rho(i_1-i_2)^2 + \\ +8\sigma^2\rho(i_1-i_2)\rho(i_1-i_3)\rho(i_2-i_3)) + \\ +2\sigma^8 + 4\sigma^4\rho(i_1-i_2)^2 + 2\sigma^8 + \\ +4\sigma^4\rho(i_1-i_3)^2 - 2\sigma^8 + 2\sigma^4\rho(i_2-i_3)^2 = \\ = 32\sigma^2\rho(i_1-i_2)\rho(i_1-i_3)\rho(i_2-i_3) + \\ +24\rho(i_1-i_2)^2\rho(i_1-i_3)^2 + 4\sigma^4\rho(i_2-i_3)^2.$$

Finally,

$$EZ_{i_1}Z_{i_2}Z_{i_3}Z_{i_4} = \sigma^8E(N_{i_1}^2-1)(N_{i_2}^2-1)\times \\ \times(N_{i_3}^2-1)(N_{i_4}^2-1) = \sigma^8E\big(N_{i_1}^2N_{i_2}^2N_{i_3}^2N_{i_4}^2 - \\ -N_{i_1}^2N_{i_2}^2N_{i_3}^2 - N_{i_1}^2N_{i_2}^2N_{i_4}^2 + N_{i_1}^2N_{i_2}^2 - \\ -N_{i_1}^2N_{i_3}^2N_{i_4}^2 + N_{i_1}^2N_{i_3}^2 + N_{i_1}^2N_{i_4}^2 - \\ -N_{i_1}^2 - N_{i_2}^2N_{i_3}^2N_{i_4}^2 + N_{i_2}^2N_{i_3}^2 + \\ +N_{i_2}^2N_{i_4}^2 - N_{i_2}^2 + N_{i_3}^2N_{i_4}^2 - N_{i_3}^2 - N_{i_4}^2 + 1\big).$$

By the Isserlis formula,

$$EN_{i_1}^2N_{i_2}^2N_{i_3}^2N_{i_4}^2 = EN_{i_1}^2EN_{i_2}^2EN_{i_3}^2EN_{i_4}^2 + \\ +2\big(EN_{i_1}^2EN_{i_2}^2(EN_{i_3}N_{i_4})^2 + EN_{i_1}^2EN_{i_3}^2(EN_{i_2}N_{i_4})^2 + \\ +EN_{i_1}^2EN_{i_4}^2(EN_{i_2}N_{i_3})^2 + EN_{i_2}^2EN_{i_3}^2(EN_{i_1}N_{i_4})^2 + \\ +EN_{i_2}^2EN_{i_4}^2(EN_{i_1}N_{i_3})^2 + EN_{i_3}^2EN_{i_4}^2(EN_{i_1}N_{i_2})^2\big) + \\ +8\big(EN_{i_1}^2EN_{i_2}N_{i_3}EN_{i_3}N_{i_4}EN_{i_2}N_{i_4} + \\ +EN_{i_2}^2EN_{i_1}N_{i_3}EN_{i_3}N_{i_4}EN_{i_1}N_{i_4} + \\ +EN_{i_3}^2EN_{i_1}N_{i_4}EN_{i_2}N_{i_4}EN_{i_1}N_{i_2} +$$

$$+EN_{i_4}^2 EN_{i_1}N_{i_2}EN_{i_2}N_{i_3}EN_{i_1}N_{i_3}\big)+$$
$$+4\big((EN_{i_1}N_{i_2})^2(EN_{i_3}N_{i_4})^2+$$
$$+(EN_{i_1}N_{i_3})^2(EN_{i_2}N_{i_4})^2+$$
$$+(EN_{i_1}N_{i_4})^2(EN_{i_2}N_{i_3})^2\big)+$$
$$+16\big(EN_{i_1}N_{i_2}EN_{i_2}N_{i_3}EN_{i_3}N_{i_4}EN_{i_1}N_{i_4}+$$
$$+EN_{i_1}N_{i_3}EN_{i_2}N_{i_3}EN_{i_1}N_{i_4}EN_{i_2}N_{i_4}+$$
$$+EN_{i_1}N_{i_2}EN_{i_1}N_{i_3}EN_{i_2}N_{i_4}EN_{i_3}N_{i_4}\big).$$

Thus, we have

$$EZ_{i_1}Z_{i_2}Z_{i_3}Z_{i_4} = 4\big(\rho(i_1-i_2)^2\rho(i_3-i_4)^2+$$
$$+\rho(i_1-i_3)^2\rho(i_2-i_4)^2+\rho(i_1-i_4)^2\rho(i_2-i_3)^2\big)+$$
$$+16\big(\rho(i_1-i_2)\rho(i_2-i_3)\rho(i_3-i_4)\rho(i_1-i_4)+$$
$$+\rho(i_1-i_3)\rho(i_2-i_3)\rho(i_1-i_4)\rho(i_2-i_4)+$$
$$+\rho(i_1-i_2)\rho(i_1-i_3)\rho(i_2-i_4)\rho(i_3-i_4)\big).$$

Collecting everything,

$$E(V_N-EV_N)^4 = 70N\sigma^8+240\sigma^4\sum_{i_1\neq i_2}\rho(i_1-i_2)^2+$$
$$+3\sum_{i_1\neq i_2}\big(24\rho(i_1-i_2)^4+32\sigma^4\rho(i_1-i_2)^2+4\sigma^8\big)+$$
$$+6\sum_{i_1\neq i_2\neq i_3}(32\sigma^2\rho(i_1-i_2)\rho(i_1-i_3)\rho(i_2-i_3)+$$
$$+24\rho(i_1-i_2)^2\rho(i_1-i_3)^2+4\sigma^4\rho(i_2-i_3)^2)+$$
$$+\sum_{i_1\neq i_2\neq i_3\neq i_4}\big(4\rho(i_1-i_2)^2\rho(i_3-i_4)^2+$$
$$+4\rho(i_1-i_3)^2\rho(i_2-i_4)^2+4\rho(i_1-i_4)^2\rho(i_2-i_3)^2+$$
$$+16\rho(i_1-i_2)\rho(i_2-i_3)\rho(i_3-i_4)\rho(i_1-i_4)+$$
$$+16\rho(i_1-i_3)\rho(i_2-i_3)\rho(i_1-i_4)\rho(i_2-i_4)+$$
$$+16\rho(i_1-i_2)\rho(i_1-i_3)\rho(i_2-i_4)\rho(i_3-i_4)\big)=$$
$$=70N\sigma^8+\sum_{j=1}^{8}A_{j,N},$$

where

$$A_{1,N}=240\sigma^4\sum_{i_1\neq i_2}\sigma^4\rho(i_1-i_2)^2,$$
$$A_{2,N}=72\sum_{i_1\neq i_2}\rho(i_1-i_2)^4,$$
$$A_{3,N}=3\sum_{i_1\neq i_2}(32\sigma^4\rho(i_1-i_2)^2+4\sigma^8),$$
$$A_{4,N}=144\sum_{i_1\neq i_2\neq i_3}\rho(i_1-i_2)^2\rho(i_1-i_3)^2,$$
$$A_{5,N}=192\sigma^2\sum_{i_1\neq i_2\neq i_3}\rho(i_1-i_2)\rho(i_1-i_3)\rho(i_2-i_3),$$
$$A_{6,N}=24\sigma^4\sum_{i_1\neq i_2\neq i_3}\rho(i_2-i_3)^2,$$
$$A_{7,N}=4\sum_{i_1\neq i_2\neq i_3\neq i_4}\big(\rho(i_1-i_2)^2\rho(i_3-i_4)^2+$$
$$+\rho(i_1-i_3)^2\rho(i_2-i_4)^2+\rho(i_1-i_4)^2\rho(i_2-i_3)^2\big),$$
$$A_{8,N}=16\times$$
$$\times\sum_{i_1\neq i_2\neq i_3\neq i_4}\big(\rho(i_1-i_2)\rho(i_2-i_3)\rho(i_3-i_4)\rho(i_1-i_4)+$$
$$+\rho(i_1-i_3)\rho(i_2-i_3)\rho(i_1-i_4)\rho(i_2-i_4)+$$
$$+\rho(i_1-i_2)\rho(i_1-i_3)\rho(i_2-i_4)\rho(i_3-i_4)\big).$$

Let is consider each sum separately:

$$A_{1,N}=240\sigma^4\sum_{i_1\neq i_2}\rho(i_1-i_2)^2=|i_1-i_2=m|=$$
$$=480\sigma^4\sum_{m=1}^{N-1}(N-m)\rho(m)^2.$$

Therefore,

$$\frac{A_{1,N}}{N^2}\leqslant\frac{480\sigma^4}{N}\sum_{m=1}^{\infty}\rho(m)^2\to 0, N\to\infty.$$

Further,

$$A_{2,N}=72\sum_{i_1\neq i_2}24\rho(i_1-i_2)^4=|i_1-i_2=m|=$$
$$=144\sum_{k=1}^{N-1}(N-m)\rho(m)^4.$$

Similarly to $A_{1,N}$, $\frac{A_{2,N}}{N^2}\to 0, N\to\infty$. Further,

$$A_{3,N}=3\sum_{i_1\neq i_2}(32\sigma^4\rho(i_1-i_2)^2+4\sigma^8)=$$
$$=192\sigma^4\sum_{m=1}^{N-1}(N-m)\rho(m)^2+12\sigma^8(N^2-N).$$

Hence,

$$\frac{A_{3,N}}{N^2}\to 12\sigma^8, N\to\infty.$$

Denoting $i_1-i_2=m, i_1-i_3=n$, we can estimate

$$A_{4,N}=144\sum_{i_1\neq i_2\neq i_3}\rho(i_1-i_2)^2\rho(i_1-i_3)^2\leqslant$$
$$\leqslant 144N\sum_{m,n=-N}^{N}\rho(m)^2\rho(n)^2\leqslant$$
$$\leqslant 144N\sum_{m,n=-\infty}^{\infty}\rho(m)^2\rho(n)^2.$$

Hence, we get

$$\frac{A_{4,N}}{N^2} \to 0, N \to \infty.$$

Making similar index changes,

$$|A_{5,N}| \leqslant 192\sigma^2 N \sum_{m,n=-N}^{N} |\rho(m)\rho(n)\rho(n-m)| \leqslant$$

$$\leqslant 192\sigma^2 N \sum_{m,n=-\infty}^{\infty} \rho(m)\rho(n)\rho(n-m);$$

$$|A_{8,N}| \leqslant 48N \sum_{m,n,l=-N}^{N} \rho(m)\rho(n)\rho(l)\rho(m+n+l) \leqslant$$

$$\leqslant 48N \sum_{m,n,l=-\infty}^{\infty} \rho(m)\rho(n)\rho(l)\rho(m+n+l),$$

therefore,

$$\frac{A_{5,N}}{N^2} \to 0, \quad \frac{A_{8,N}}{N^2} \to 0, \quad N \to \infty.$$

Further,

$$A_{6,N} = 24\sigma^4 \sum_{i_1 \neq i_2 \neq i_3} \rho(i_2 - i_3)^2 =$$

$$= 24\sigma^4 (N-2) \sum_{i_2 \neq i_3} \rho(i_2 - i_3)^2 =$$

$$= 48\sigma^4 (N-2) \sum_{m=1}^{N-1} (N-m)\rho(m)^2.$$

Therefore,

$$\frac{A_{6,N}}{N^2} = 96\sigma^4 \Big(1 - \frac{2}{N}\Big) \sum_{m=1}^{N-1} \Big(1 - \frac{m}{N}\Big)\rho(m)^2 \to$$

$$\to 48\sigma^4 \sum_{m=1}^{\infty} \rho(m)^2, N \to \infty.$$

Finally, let us deal with $A_{7,N}$:

$$A_{7,N} = 12 \sum_{i_1 \neq i_2 \neq i_3 \neq i_4} \rho(i_1 - i_2)^2 \rho(i_3 - i_4)^2 =$$

$$= 12 \sum_{i_1 \neq i_2, i_3 \neq i_4 \{i_1, i_2\} \neq \{i_3, i_4\}} \rho(i_1 - i_2)^2 \rho(i_3 - i_4)^2 -$$

$$-12 \cdot 4 \sum_{i_1 \neq i_2 \neq i_3} \rho(i_1 - i_2)^2 \rho(i_3 - i_4)^2.$$

Now

$$\sum_{\substack{i_1 \neq i_2, i_3 \neq i_4 \\ \{i_1, i_2\} \neq \{i_3, i_4\}}} \rho(i_1 - i_2)^2 \rho(i_3 - i_4)^2 =$$

$$= \Big(\sum_{i_1 \neq i_2} \rho(i_1 - i_2)^2\Big)^2 - \sum_{i_1 \neq i_2} \rho(i_1 - i_2)^4 =$$

$$= \Big(2 \sum_{m=1}^{N-1} (N-m)\rho(m)^2\Big)^2 - \frac{A_{2,N}}{72}.$$

Therefore,

$$A_{7,N} = 48 \Big(\sum_{m=1}^{N-1} (N-m)\rho(m)^2\Big)^2 - \frac{A_{2,N}}{6} - \frac{A_{4,N}}{3},$$

hence

$$\lim_{N\to\infty} \frac{A_{7,N}}{N^2} = 48 \lim_{N\to\infty} \left(\sum_{m=1}^{N-1} \Big(1 - \frac{m}{N}\Big) \rho(m)^2\right)^2 =$$

$$= 48 \left(\sum_{m=1}^{\infty} \rho(m)^2\right)^2.$$

Collecting everything,

$$\frac{E(V_N - EV_N)^4}{N^2} \to 12\sigma^8 + 48\sigma^4 \sum_{m=1}^{\infty} \rho(m)^2 +$$

$$+48 \Big(\sum_{m=1}^{\infty} \rho(m)^2\Big)^2 =$$

$$= 12 \Big(\sum_{m=-\infty}^{\infty} \rho(m)^2\Big)^2, N \to \infty. \quad \square$$

Theorem 3.2. *$\hat{\sigma}^2$ is an asymptotically normal estimator of σ^2 with asymptotic variance $2\sum_{i=-\infty}^{\infty} \rho(i)^2$.*

Proof. From Lemmas 1, 2 it follows that

$$\lim_{N\to\infty} \frac{E(V_N - EV_N)^4}{D(V_N)^2} =$$

$$= \lim_{N\to\infty} \frac{\frac{E(V_N - EV_N)^4}{N^2}}{\left(\frac{D(V_N)}{N}\right)^2} =$$

$$\frac{12\left(\sum_{i=-\infty}^{\infty} \rho(i)^2\right)^2}{(2\sum_{i=-\infty}^{\infty} \rho(i)^2)^2} = 3.$$

So we get the statement as a conclusion of the Fourth moment theorem.

$\square$

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