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THE AVERAGING METHOD FOR THE OPTIMAL CONTROL PROBLEM OF A PARABOLIC INCLUSION WITH FAST-OSCILLATING COEFFICIENTS ON A FINITE TIME INTERVAL

In this paper we investigate the optimal control problem for a parabolic differential inclusion with rapidly oscillating variables in the finite interval. There are many approaches intended for the investigation of control problems for differential equations and inclusions. In particular, the asymptotic methods are used fairly extensively. Among these methods, we can especially mention the averaging method, which was mathematically rigorously substantiated by M. M. Krylov and M. M. Bogolyubov. The well-known Krasnoselski – Krein theorem and its multi-valued analogue play an essential role for the investigation of the above-mentioned problems. The averaging method was substantiated, in particular, for ordinary differential inclusions, inclusions with partial derivatives, and inclusions with the Hukuhara derivative. When dealing with multi-valued mappings one faces specific problems, such as closedness, convexity of the family of solutions, existence of limit solutions, selection of solutions with given properties, etc. However, the well-developed apparatus of mathematical analysis applied to the study of multi-valued functions makes it possible to apply the averaging method to the optimal control problem described above. Thus, using the averaging method, the convergence of optimal controls and optimal trajectories of solutions of the exact problem to optimal control and the trajectory of the averaged problem is proved in the paper.

Keywords: *optimal control problem, parabolic differential inclusion, averaging method, rapidly oscillating variables.*

AMS 2020 classification: 49J21, 35K5

Introduction

The intensive development of science and technology regularly stimulates the search for effective methods of control various natural, economic, social, and technical processes. Mathematical models of such situations are optimal control problems for different classes of evolutionary systems. The existence and properties of solutions of evolutionary systems were studied in the works of (Zadoyanchuk, & Kas'yanov, 2007; Zadoianchuk, & Kas'yanov, 2012; Kasimova, Kupenko, & Tsyganivska, 2023). Interesting results that establish the conditions for the practical stability of evolutionary systems were obtained in (Pichkur, 2019; Pichkur, & Linder, 2021; Pichkur, Linder, & Tairova, 2021). Among the studies dedicated to the tasks of optimal control of evolutionary systems, the following works should be noted (Kapustyan, & Nakonechnyi, 1999; Kapustyan, O. V., Kapustyan, O. A., & Sukretna, 2009; Koval'chuk, Lavrova, & Mohyl'ova, 2021; Hermosilla, & Palladino, 2022; Hermosilla, Palladino, & Vilches, 2024). But considerable attention is paid to mathematical models of processes in the form of differential equations with a small parameter. For their solution asymptotic methods are widely used, in particular, the averaging method, the strict mathematical justification of which was proposed by M. M. Krylov and M. M. Bogolyubov. Since the differential inclusion is a natural generalization of the differential equation, the next step in the development of asymptotic methods was the justification of the averaging method for differential inclusions. The main idea of this approach is that a non-autonomous differential equation (or inclusion) is matched by an autonomous differential equation using the averaging method (Koval'chuk, Mohyl'ova, & Shovkoplyas, 2020; Kapustian et al., 2022). This makes it possible to apply effective numerical methods for solving the averaged control problem. The optimal control problem for systems of differential inclusions with fast-oscillating parameters is considered in (Zhuk, Kasimova, & Ryzhov, 2022; Kichmarenko et al., 2023; Kasimova, Zhuk, & Tsyganivska, 2023).

The object of research in the present paper is the optimal control problem for a parabolic differential inclusion with fast-oscillating parameters. The aim and objectives of the research in the present paper are the application of the averaging method to the study of the mentioned problem.

In the present paper, we use the averaging method for the investigation of the optimal control problem for a parabolic differential inclusion with rapidly oscillating variables in the finite interval. In particular, by using the direct method of variational calculus, the solvability of the original and averaged problems is proved. Moreover, the convergence of optimal controls and optimal trajectories of the exact problem to the optimal control and optimal trajectory of the averaged problem is established.

1. Methods

The methods of the research – the averaging method and the direct method of the calculus of variations.

2. Statement of the problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. In cylinder $Q_T = (0, T) \times \Omega$ we consider an initial boundary-value problem for a parabolic inclusion:

$$\begin{cases} \frac{\partial y}{\partial t} \in Ay + f\left(\frac{t}{\varepsilon}, y(t,x)\right) + g(y)u, \ (t,x) \in Q_T, \\ y|_{\partial\Omega} = 0, \\ y|_{t=0} = y_0(x). \end{cases} \tag{1}$$

Here $\varepsilon > 0$ is a small parameter, f is a given multivalued mapping, g is a given real-valued mapping, A is an elliptic operator, y is an unknown state function, u is an unknown control function, which are determined by requirements

$$u \in \mathcal{U} \subseteq L^2(Q_T), \tag{2}$$

$$\mathcal{J}(y,u) = \int_\Omega q\big(x, y(T,x)\big)dx + \int_{Q_T} u^2(t,x)dtdx \ \to inf, \tag{3}$$

where q is a given function.

We consider the problem of finding an approximate solution of (1)–(3) by transition to averaged coefficients. For this purpose we assume that uniformly w.r.t. $y \in \mathbb{R}$ there exists

$$\bar{f}(y) := \lim_{T\to\infty} \frac{1}{T}\int_0^T f(s,y)ds. \tag{4}$$

We consider the following optimal control problem

$$\begin{cases} \frac{\partial y}{\partial t} \in Ay + \bar{f}(y) + g(y)u, \ (t,x) \in Q_T, \\ y|_{\partial\Omega} = 0, \\ y|_{t=0} = y_0(x). \end{cases} \tag{5}$$

$$u \in \mathcal{U} \subseteq L^2(Q_T), \tag{6}$$

$$\mathcal{J}(y,u) = \int_\Omega q\big(x, y(T,x)\big)dx + \int_{Q_T} u^2(t,x)dtdx \quad inf. \tag{7}$$

Under the natural assumptions on f, g, u, q we prove, that the optimal control problem (1)–(3) has a solution $\overline{y}^\varepsilon, \overline{u}^\varepsilon$, i.e. for every $u \in \mathcal{U}$ and for any solution y^ε of (1) with control u we have

$$\mathcal{J}\left(\overline{y}^\varepsilon, \overline{u}^\varepsilon\right) \le \mathcal{J}(y^\varepsilon, u).$$

Note that we can apply similar arguments to problem (5)–(7).

Assume that $\overline{y}, \overline{u}$ is a solution of (5)–(7). The main goal of the paper is to prove the convergence

$$\mathcal{J}\left(\overline{y}^\varepsilon, \overline{u}^\varepsilon\right) \to \mathcal{J}(\overline{y}, \overline{u}), \ \varepsilon \to 0. \tag{8}$$

3. Preliminaries and Notations

We suppose that the following assumptions for the parameters of the problem are fulfilled.

(**f**₁) *Multi-valued function* $f: \mathbb{R}_+ \times \mathbb{R} \to conv\ (\mathbb{R})$ *is continuous and there exists* $C_1 > 0$ *such that*

$$\forall t \ge 0 \ \forall y \in \mathbb{R}$$
$$\|f(t,y)\|_+ := \sup_{\xi \in f(t,y)} \|\xi\|_{\mathbb{R}} \le C_1(1 + \|y\|_{\mathbb{R}}), \tag{9}$$

where $\|\xi\|_{\mathbb{R}}$ *denotes the Euclidian norm of* $\xi \in \mathbb{R}$;

(**g**₁) *function* $g: \mathbb{R} \to \mathbb{R}$ *is continuous and there exists* $C_2 > 0$ *such that*

$$\forall y \in \mathbb{R} \qquad \|g(y)\|_{\mathbb{R}} \le C_2; \tag{10}$$

(**q**₁) *function* $q: \Omega \times \mathbb{R} \to \mathbb{R}$ *is Carateodori function and there exists* $C_3 > 0$ *and functions* $\mathcal{K}_1 \in L^2(\Omega)$, $\mathcal{K}_2 \in L^1(\Omega)$ *such that*

$$\|q(x,\xi)\|_{\mathbb{R}} \le C_3\|\xi\|_{\mathbb{R}} + \mathcal{K}_1(x), \ \ q(x,\xi) \ge \mathcal{K}_2(x) \tag{11}$$

for all $\xi \in \mathbb{R}$ *and a.a.* $x \in \Omega$;

(**u**₁) $\mathcal{U} \subseteq L^2(Q)$ *is closed and convex,* $0 \in \mathcal{U}$;

(**A**₁) *uniformly w.r.t.* $y \in \mathbb{R}$ *there exists the limit* (4).

Let us consider a triplet of Hilbert spaces $V \subset H \subset V'$ with compact dense embeddings, and the scalar product on $V \times H$ we denote by $(\cdot,\cdot)$. We denote by $\|\cdot\|$ and $|\cdot|$ the norm in V and H respectively. We will use the following spaces

$$V = H_0^1(\Omega), \ H = L^2(\Omega), \ V' = H^{-1}(\Omega). \tag{12}$$

In the sequel we shall fix time interval $[0,T]$; in addition to the triplet V, H, V' we also consider spaces $L^2(0,T;V)$, $L^2(0,T;H)$, $L^2(0,T;V')$ which are the spaces of the square summarable functions defined on the interval $[0,T]$ and with values, respectively, in V, H, V'.

Let us consider the space

$$W = \left\{ y \in L^2(0,T;V) : \ y' = \frac{\partial y}{\partial t} \in L^2(0,T;V') \right\}$$

with norm

$$\|\cdot\|_W = \|\cdot\|_{L^2(0,T;V)} + \|\cdot\|_{L^2(0,T;V')}.$$

Remark 1. The space W is continuously embedded into $\mathrm{C}(0,\mathrm{T};\mathrm{H})$. (Zgurovsky, & Mel'nik, 2004)

Remark 2. The space W is compactly embedded into $\mathrm{L}^2(0,\mathrm{T};\mathrm{H})$. (Zgurovsky, & Mel'nik, 2004)

Let us consider a differential inclusion from (1) in the following form:

$$y' - Ay \in F(t, y),\ 0 \le t \le T, \tag{13}$$

$$y(0) = y_0, \tag{14}$$

where $F\colon \mathbb{R}_+ \times \mathbb{R} \to conv\,(\mathbb{R})$,

$$F(t, y) = f\left(\frac{t}{\varepsilon}, y(t, x)\right) + g(y)u(t).$$

Definition 1. We say that the problem (13) – (14) has a weak solution if there exists $y \in W$ and $f \in L^2(0, T; H)$, $f(t) \in F\big(t, y(t)\big)$, a. e. in $[0, T]$ such that

$$y' - Ay(t) = f(t),$$

$$y(0) = y_0.$$

In the sequel we denote by Ξ_ε (or Ξ) a set of all pairs $\{y, u\}$, where y is a solution of (1) (or (5)) with control u.

Let us consider the result about solvability of the problem (1)–(3) (resp. (5)–(7)).

Theorem 1. *Under assumptions* (f₁) – (A₁) *problem* (1)–(3) (*resp. problem* (5)–(7)) *has a solution* $\overline{y}_\varepsilon, \overline{u}_\varepsilon$ (*resp.* $\{\overline{y}, \overline{u}\}$).

Proof. In what follows, assume that $\varepsilon > 0$ is fixed. First of all, note that by Theorem 1 (Denkowski, & Mortola, 1993) the set of admissible pairs Ξ_ε is not empty. For further investigations let us consider some a priory estimates for solutions. Taking into account the definition of weak solution for parabolic inclusion, suppose that $\forall \varphi \in H_0^1(\Omega)$

$$\frac{d}{dt}(y, \varphi) + (\nabla y, \nabla \varphi) = (f_1(t), \varphi) + (g(y)u, \varphi)$$

$$\text{for a.a. } t \in [0, T],\ f_1(t) \in f\big(t, y(t)\big). \tag{15}$$

From (15) we get the following equality

$$\int_0^s \big(y'(t), y(t)\big)\, dt + \int_0^s (\nabla y, \nabla y)\, dt = \int_0^s \big(f_1(t), y(t)\big)\, dt + \int_0^s \big(g(y)u, y(t)\big)\, dt. \tag{16}$$

Using integrating by parts, Young's inequality and Assumption (g₁), we get

$$|y(s)|^2 + 2C \int_0^s \|y(t)\|^2 dt \le |y(0)|^2 + \int_0^s (|f_1(t)|^2 + |y(t)|^2) dt + C_2\Big(\int_0^s |u(t)|^2 dt + \int_0^s |y(t)|^2 dt\Big), \tag{17}$$

where C is the constant from the inequality $|\nabla y|^2 \ge C\|y\|^2$ for an arbitrary $y \in H_0^1(\Omega)$.

Using (9) we have

$$|f_1(t)|^2 \le 2(C_1^2|\Omega| + C_1^2|y(t)|^2),$$

where $|\Omega|$ is Lebesgue's measure of bounded set Ω.

Then from (17) we have:

$$|y(s)|^2 + 2C \int_0^s \|y(s)\|^2 dt \le |y(0)|^2 + \int_0^s 2C_1^2|\Omega| dt + \int_0^s (2C_1^2 + C_2 + 1)|y(t)|^2 dt + C_2 \int_0^s |u(t)|^2 dt.$$

Then using Gronwall inequality we obtain

$$|y(t)|^2 \le \big(|y(0)|^2 + 2C_1^2 T|\Omega| + C_2\|u\|_{L^2(0,T;H)}^2\big) \cdot e^{(2C_1^2 + C_2 + 1)T} := M_1. \tag{18}$$

From (17) and (18) we conclude that $\exists M > 0$:

$$\int_0^T \|y(t)\|^2 dt \le M. \tag{19}$$

From (18) we have:

$$|f_1(t)|^2 \le 2C_1^2(|\Omega| + M_1),$$

and as a consequence

$$|f_1(t)| \le \sqrt{2}C_1(|\Omega| + M_1)^{1/2} \text{ a. e. in } [0, T]. \tag{20}$$

Due to (15), (1), (10), (19), (20) we conclude that there exists $L > 0$

$$\int_0^T \|y'(t)\|_{V'}^2 dt \le L, \tag{21}$$

and in consequence, there exists

$$M_2 := \sqrt{2}C_1 T^{1/2}(|\Omega| + M_1)^{\frac{1}{2}}$$

such that

$$\|f_1\|_{L^2(0,T;H)} \le M_2. \tag{22}$$

Taking into account (18), and Ass. (q₁) we get that

$$\mathcal{J}(y, u) \le C_3|\Omega|^{\frac{1}{2}}\sqrt{M_1} + |\Omega|^{\frac{1}{2}}\|\mathcal{K}_1(x)\|_{L^2(\Omega)} + \|u\|_{L^2(0,T;H)}^2 < \infty. \tag{23}$$

Thus the cost functional in (7) has sense.

Now let $\{y_n, u_n\}_{n\ge1}$ be a minimizing sequence, that is,

$$\lim_{n\to\infty} \mathcal{J}(y_n, u_n) = \inf_{y,u\in\Xi_\varepsilon} \mathcal{J}(y, u) =: \overline{\mathcal{J}}^{\varepsilon}, \tag{24}$$

Due to Ass. (q₁) for an arbitrary $y, u \in \Xi_\varepsilon$

$$\mathcal{J}(y, u) \ge -\|\mathcal{K}_2\|_{L^1(\Omega)},$$

therefore

$$\overline{\mathcal{J}}^{\varepsilon} \geq -\|\mathcal{K}_2\|_{L^1(\Omega)} \succ \infty.$$

From (24) for sufficiently large u

$$\mathcal{J}(y_n, u_n) \leq \overline{\mathcal{J}}^{\varepsilon} + 1. \tag{25}$$

On the other hand

$$\mathcal{J}(y_n, u_n) \geq -\|\mathcal{K}_2\|_{L^1(\Omega)} + \|u_n\|^2_{L^2(0,T;H)}. \tag{26}$$

Inequalities (25), (26) imply that $\{u_n\}_{n\geq 1}$ is bounded in $L^2(0,T;H)$, therefore up to a subsequence

$$u_n \to u \ \text{ weakly in } \ L^2(0,T;H). \tag{27}$$

Due to convexity of $\mathcal{U}$ we have that $u \in \mathcal{U}$. From (18), (19), (21), (22) we get that $\{y_n\}_{n\geq 1}$ is bounded in $L^2(0,T;V) \cap L^\infty(0,T;H)$, $\frac{\partial y_n}{\partial t}_{n\geq 1}$ is bounded in $L^2(0,T;V')$.

Using Compactness Lemma (Lions, 1969) we conclude that up to a subsequence

$$\begin{gathered} y_n \to y \text{ weakly in } L^2(0,T;V), \\ y_n \to y \text{ in } L^2(0,T;H), \\ \forall t \in [0,T] y_n(t) \to y(t) \text{ weakly in } H, \\ y_n(t,x) \to y(t,x) \text{ a.e. in } \mathcal{Q}_T, \\ y_n' \to y' \text{ weakly in } L^2(0,T;V'). \end{gathered} \tag{28}$$

Let us show that $y, u \in \Xi_\varepsilon$.We have that y_n is a weak solution of the following problem,

$$\begin{cases} \frac{\partial y_n}{\partial t} = Ay_n + f_{1n} + g(y_n)u_n, \\ y_n|_{\partial\Omega} = 0, \\ y_n|_{t=0} = y_0(x), \end{cases} \tag{29}$$

where $f_{1n}(t) \in f\big(t, y_n(t)\big)$.

From (9) and boundness of y_n in $L^2(0,T;H)$ we deduce that

$$f_{1n} \to f_1 \text{ in } L^2(0,T;H).$$

From (28) we get

$$y_n \to y \text{ weakly in } W. \tag{30}$$

From Lemma 3.2 (Lions, 1969) we have that $y_n \to y$ in $C(0,T;H)$, and $f_1 \in f\big(t, y(t)\big)$, having that from the Dominated Convergence Theorem and (28) we see that

$$g(y_n) \to g(y) \text{ in } L^2(0,T;H), \ \ n \to \infty.$$

Due to pointwise convergence

$$q\big(x, y_n(T,x)\big) \to q\big(x, y(T,x)\big) \text{ a. e. in } \mathcal{Q}_T,$$

Fatou's lemma and weak convergence (27) we have

$$\overline{\mathcal{J}}^{\varepsilon} = \lim_{n\to\infty} \mathcal{J}(y_n, u_n) \geq \underline{\lim}_{n\to\infty} \int_{\Omega} q\big(x, y_n(T,x)\big)dx + \underline{\lim}_{n\to\infty} \int_{\mathcal{Q}_T} u^2(t,x)dtdx \geq \mathcal{J}(y,u).$$

Therefore y, u is a solution of (1)–(3).

4. Main result

We assume that $\forall \eta > 0$, $\exists \delta > 0$, $\forall t \geq 0$, $\forall y, z \in \mathbb{R}$

$$\|y - z\|_{\mathbb{R}} < \delta \Longrightarrow \mathrm{dist}_{\mathrm{H}}\big(f(t,y), f(t,z)\big) < \eta, \tag{31}$$

where $\mathrm{dist}_{\mathrm{H}}(A, B)$ is Hausdorff metric between sets A and B.

Theorem 2. *Suppose that assumptions* (f1), (g1), (q1), (u1), (A1), (4) and (31) *hold and, moreover, the problem* (5) *has the unique solution for every* $u \in \mathcal{U}$. *Let* $\{\overline{y}^{\varepsilon}, \overline{u}^{\varepsilon}\}$ *be a solution of* (1)–(3). *Then*

$$\mathcal{J}(\overline{y}^{\varepsilon}, \overline{u}^{\varepsilon}) \to \mathcal{J}(\overline{y}, \overline{u}), \ \varepsilon \to 0 \tag{32}$$

and up to subsequence

$$\overline{y}^{\varepsilon} \to \overline{y} \ \ \textit{in} \ \ L^2(0,T;H), \tag{33}$$

$$\overline{u}^{\varepsilon} \to \overline{u} \ \ \textit{weakly in} \ \ L^2(0,T;H), \tag{34}$$

where $\{\overline{y}, \overline{u}\}$ *is a solution of* (5) – (7).

Proof. Let $\varepsilon_n \to 0$, $\overline{y}^n, \overline{u}^n$ be a solution of (1)–(3) for $\varepsilon = \varepsilon_n$. Due to the optimality of $\overline{y}^n, \overline{u}^n$ we have

$$\mathcal{J}(\overline{y}^n, \overline{u}^n) \leq \mathcal{J}(y_n, 0)$$

where y_n is a solution of (1) with $\varepsilon = \varepsilon_n$ and $u \equiv 0$.

Then from (23) and (26)

$$-\|\mathcal{K}_2\|_{L^1(\Omega)} + \left\|\overline{u}^n\right\|^2_{L^2(0,T;H)} \leq C_3|\Omega|^{\frac{1}{2}}\sqrt{M_1} + |\Omega|^{\frac{1}{2}}\|\mathcal{K}_1\|_{L^2(\Omega)}. \tag{35}$$

Repeating arguments used in the proof of Theorem 1 we conclude that on subsequence for some $\hat{y}$, $\hat{u}$:

$$\overline{u}^n \to \hat{u} \text{ weakly in } L^2(0,T;H),\ n \to \infty; \tag{36}$$

$$\overline{y}^n \to \hat{y} \text{ in the sense of (28)},\ n \to \infty. \tag{37}$$

Let us prove that $\hat{y}, \hat{u} \in \Xi$, i. e. $\hat{y}$ is a solution of the averaged problem (5) with control $\hat{u}$. For this purpose it is sufficient to pass to the limit in the equality

$$\left(\overline{y}^n(T),\varphi\right)_H - (y_0,\varphi)_H + \int_0^T\left(\nabla\overline{y}^n,\nabla\varphi\right)_H dt = \int_0^T\left(f_1^{\varepsilon_n}(t),\varphi\right)_H dt + \int_0^T\left(g\left(\overline{y}^n\right)\overline{u}^n,\varphi\right)_H dt \tag{38}$$

where

$$f_1^{\varepsilon_n}(t) \in f\left(\frac{t}{\varepsilon_n},\overline{y}^n\right),$$

for an arbitrary $\varphi \in V$.

The possibility of passing to the limit in the left hand side of (38) is a direct consequence of (37). From the Dominated Convergence Theorem we see that

$$g\left(\overline{y}^n\right) \to g(\hat{y}) \ \text{ in } L^2(0,T;H),\ n \to \infty.$$

Then (37) implies convergence in the last term of (38). Let us prove that $\forall T > 0$, $\forall y \in V$

$$\int_{Q_T} f_1^{\varepsilon_n}(t)\,\varphi dtdx \to \int_{Q_T} f_1^{\varepsilon_n}(t)\varphi dtdx,\ n \to \infty. \tag{39}$$

Let us consider the following inequalities:

$$\int_\Omega\int_0^T f_1^{\varepsilon_n}(t)\varphi dtdx - \int_\Omega\int_0^T \overline{f}(\hat{y})\varphi dtdx =$$

$$= \mathrm{dist_H}\left(\int_\Omega\int_0^T f_1^{\varepsilon_n}(t)\varphi dtdx, \int_\Omega\int_0^T \overline{f}(\hat{y})\varphi dtdx\right) \le \mathrm{dist_H}\left(\int_\Omega\int_0^T f\left(\frac{t}{\varepsilon_n},\overline{y}^n(t,x)\right)\varphi dtdx, \int_\Omega\int_0^T \overline{f}(\hat{y})\varphi dtdx\right).$$

We will show that

$$\mathrm{dist_H}\left(\int_\Omega\int_0^T f\left(\frac{t}{\varepsilon_n},\overline{y}^n(t,x)\right)\varphi dtdx, \int_\Omega\int_0^T \overline{f}(\hat{y})\varphi dtdx\right) \to 0. \tag{40}$$

First of all let us note that due to (4) and (Hermes, 1968) $\forall 0 < a < b$ $\forall\psi \in H$ we have

$$\mathrm{dist_H}\left(\int_\Omega\int_a^b f\left(\frac{t}{\varepsilon_n},\psi(x)\right)\varphi dtdx, \int_\Omega\int_a^b \overline{f}\left(\psi(x)\right)\varphi dtdx\right) 0. \tag{41}$$

Due to Egorov's theorem $\forall\delta > 0$ $\exists Q_1^\delta \subset Q_T$ such that $\mu\left(Q_1^\delta\right) < \delta$ and

$$\overline{y}^n \to \hat{y} \text{ uniformly on } Q_T \setminus Q_1^\delta, \text{ as } n \to \infty. \tag{42}$$

Here μ is Lebesgue's measure on

$$(0,T) \times \Omega \subset \mathbb{R}^{n+1}.$$

On the other hand, there exists a sequence of step functions

$$y^m(t,x) = \sum_{k=1}^m y_k^m(x)\chi_{A_k^m}(t), \qquad \{y_k^m\} \subset H,$$

$\{A_k^m = (a_k^m, b_k^m)\}$ is a covering of $(0,T)$ such that

$$y^m \to \hat{y} \ \text{ in } \ L^2(0,T;H) \ \text{ and a.e. in } \ Q_T. \tag{43}$$

Moreover, $\forall\delta > 0$ $\exists Q_2^\delta \subset Q_T$ such that

$$\mu\left(Q_2^\delta\right) < \delta$$

and

$$y^m \to \hat{y} \ \text{ uniformly on } \ Q_T \setminus Q_2^\delta, \text{ as } m \to \infty. \tag{44}$$

Further we have

$$\mathrm{dist_H}\left(\int_{Q_T} f\left(\frac{t}{\varepsilon_n},\overline{y}^n(t,x)\right)\varphi dtdx, \int_{Q_T} \overline{f}\left(\hat{y}(t,x)\right)\varphi dtdx\right) \le \mathrm{dist_H}\left(\int_{Q_T} f\left(\frac{t}{\varepsilon_n},\overline{y}^n(t,x)\right)\varphi dtdx, \int_{Q_T} f\left(\frac{t}{\varepsilon_n},\hat{y}(t,x)\right)\varphi dtdx\right) +$$

$$+\mathrm{dist_H}\left(\int_{Q_T} f\left(\frac{t}{\varepsilon_n},\hat{y}(t,x)\right)\varphi dtdx, \int_{Q_T} \overline{f}\left(\hat{y}(t,x)\right)\varphi dtdx\right) =: I_1^{(n)} + I_2^{(n)}.$$

Then due to (42), Hölder's inequality, (18) and (9) we have

$$I_1^{(n)} \le \int_{Q_T} \mathrm{dist_H}\left[f\left(\frac{t}{\varepsilon_n},\hat{y}^n(t,x)\right), f\left(\frac{t}{\varepsilon_n},\hat{y}(t,x)\right)\right]\varphi dtdx \le \int_{Q_T\setminus Q_1^\delta} \mathrm{dist_H}\left[f\left(\frac{t}{\varepsilon_n},\overline{y}^n(t,x)\right), f\left(\frac{t}{\varepsilon_n},\hat{y}(t,x)\right)\right]\varphi dtdx +$$

$$+\int_{Q_1^\delta} \mathrm{dist_H}\left[f\left(\frac{t}{\varepsilon_n},\overline{y}^n(t,x)\right), f\left(\frac{t}{\varepsilon_n},\hat{y}(t,x)\right)\right]\varphi dtdx \le$$

$$\leq \int\limits_{\mathcal{Q}_T \setminus \mathcal{Q}_1^\delta} \mathrm{dist_H}\left[f\left(\frac{t}{\varepsilon_n}, \overline{y}^n(t,x)\right), f\left(\frac{t}{\varepsilon_n}, \hat{y}(t,x)\right)\right] \|\varphi(x)\|_{\mathbb{R}} dtdx \; + 2\int\limits_{\mathcal{Q}_1^\delta} C_1(1+\|y\|_{\mathbb{R}})\, \varphi dtdx \leq$$

$$\leq \int_{\mathcal{Q}_T \setminus \mathcal{Q}_1^\delta} \mathrm{dist_H}\left[f\left(\frac{t}{\varepsilon_n}, \overline{y}^n(t,x)\right), f\left(\frac{t}{\varepsilon_n}, \hat{y}(t,x)\right)\right] \|\varphi(x)\|_{\mathbb{R}} dtdx + 2C_1\|\varphi\|_H \cdot \delta^{\frac{1}{2}} \cdot T + 2C_1 M_1^{\frac{1}{2}} \|\varphi\|_H \cdot \delta^{\frac{1}{2}} \cdot T. \tag{45}$$

Due to (31) for a given $\delta > 0 \; \exists\lambda \; \forall n \geq 1 \; \forall t \geq 0$

$$\|y - z\|_{\mathbb{R}} < \lambda \Rightarrow \mathrm{dist_H}\left(f\left(\frac{t}{\varepsilon_n}, y\right), f\left(\frac{t}{\varepsilon_n}, z\right)\right) \leq \delta^{\frac{1}{2}}.$$

Therefore, choosing n_1 such that $\forall n \geq n_1$

$$\sup_{(t,x) \in \mathcal{Q}_T \setminus \mathcal{Q}_1^\delta} \left\|\overline{y}^n(t,x) - \hat{y}(t,x)\right\| < \lambda$$

we get from (45) that $\forall n \geq n_1$

$$I_1^{(n)} \leq \delta^{\frac{1}{2}} \mu^{\frac{1}{2}}(\mathcal{Q}_T)\|\varphi\|_H \cdot T + 2C_1\|\varphi\|_H \cdot \delta^{\frac{1}{2}} \cdot T + 2C_1 M_1^{\frac{1}{2}}\|\varphi\|_H \cdot \delta^{\frac{1}{2}} \cdot T \leq \tilde{C}(T)\delta^{\frac{1}{2}}. \tag{46}$$

On the other hand, for every step function $y^m(t,x)$ we have due to (41): $\forall m \geq 1$,

$$\mathrm{dist_H}\left(\int\limits_{\mathcal{Q}_T} f\left(\frac{t}{\varepsilon_n}, y^m(t,x)\right)\varphi dtdx\, ,\; \int\limits_{\mathcal{Q}_T} \overline{f}\big(y^m(t,x)\big)\varphi dtdx\right) =$$

$$= \mathrm{dist_H}\left(\sum_{k=1}^{m} \int\limits_{A_k^m} \int\limits_{\Omega} f\left(\frac{t}{\varepsilon_n}, y_k^m(x)\right)\varphi dtdx\, ,\; \sum_{k=1}^{m} \int\limits_{\Omega} \int\limits_{A_k^m} \overline{f}\big(y^m(t,x)\big)\varphi dtdx\right) \leq$$

$$\leq \sum_{k=1}^{m} \mathrm{dist_H}\left(\int_\Omega \int_{A_k^m} f\left(\frac{t}{\varepsilon_n}, y_k^m(x)\right)\varphi dtdx\, , \int_\Omega \int_{A_k^m} \overline{f}\big(y^m(t,x)\big)\varphi dtdx\right) \to 0, \; n \to \infty. \tag{47}$$

So $\forall m \geq 1 \; \exists n_n = n_2(m) \; \forall n \geq n_2$

$$\mathrm{dist_H}\left(\int_{\mathcal{Q}_T} f\left(\frac{t}{\varepsilon_n}, y^m(t,x)\right)\varphi dtdx\, , \int_{\mathcal{Q}_T} \overline{f}\big(y_k^m(t,x)\big)\varphi dtdx\right) \leq \delta. \tag{48}$$

Furthermore, $\exists m_0 \; \forall m \geq m_0 \; \forall n \geq 1$

$$\mathrm{dist_H}\left(\int\limits_{\mathcal{Q}_T \setminus \mathcal{Q}_2^\delta} f\left(\frac{t}{\varepsilon_n}, \hat{y}(t,x)\right)\varphi dtdx\, ,\; \int\limits_{\mathcal{Q}_T \setminus \mathcal{Q}_2^\delta} f\left(\frac{t}{\varepsilon_n}, y^m(t,x)\right)\varphi dtdx\right) \leq$$

$$\leq \int_{\mathcal{Q}_T \setminus \mathcal{Q}_2^\delta} \mathrm{dist_H}\left(f\left(\frac{t}{\varepsilon_n}, \hat{y}(t,x)\right), f\left(\frac{t}{\varepsilon_n}, y^m(t,x)\right)\right) \|\varphi\|_{\mathbb{R}} dtdx \leq \delta^{\frac{1}{2}} \mu^{\frac{1}{2}}(\mathcal{Q}_T)\|\varphi\|_H \cdot T^{\frac{1}{2}}, \tag{49}$$

$$\mathrm{dist_H}\left(\int\limits_{\mathcal{Q}_T \setminus \mathcal{Q}_2^\delta} \bar{f}\big(\hat{y}(t,x)\big)\varphi dtdx\, ,\; \int\limits_{\mathcal{Q}_T \setminus \mathcal{Q}_2^\delta} \bar{f}\big(y^m(t,x)\big)\varphi dtdx\right) \leq$$

$$\leq \int_{\mathcal{Q}_T \setminus \mathcal{Q}_2^\delta} \mathrm{dist_H}\left(\bar{f}\big(g(t,x)\big)\varphi dtdx\, , \bar{f}\big(y^m(t,x)\big)\right) \|\varphi\|_{\mathbb{R}} dtdx \leq \delta^{\frac{1}{2}} \mu^{\frac{1}{2}}(\mathcal{Q}_T)\|\varphi\|_H \cdot T^{\frac{1}{2}}. \tag{50}$$

Combining (47)–(50) we get $\forall m \geq m_0 \; \forall n \geq n_2(m)$

$$I_2^{(n)} \leq 2\delta^{\frac{1}{2}} \mu^{\frac{1}{2}}(\mathcal{Q}_T)\|\varphi\|_H \cdot T^{\frac{1}{2}} + \delta \; \leq \tilde{\tilde{C}}(T)\delta^{\frac{1}{2}}. \tag{51}$$

Inequalities (46), (51) imply (40). So we can pass to the limit in (38) and obtain that $\hat{y}, \hat{u} \in \Xi$. Now let us prove that $\hat{y}, \hat{u}$ is an optimal process in (5)–(7).

Fatou's lemma implies

$$\underline{\lim}_{n \to \infty} \mathcal{J}\left(\overline{y}^n, \overline{u}^n\right) \geq \mathcal{J}(\hat{y}, \hat{u}). \tag{52}$$

On the other hand, for every $u \in \mathcal{U}$ and any y_n – solution (1) with control u and $\varepsilon = \varepsilon_n$, we get

$$\mathcal{J}\left(\overline{y}^n, \overline{u}^n\right) \leq \mathcal{J}(y_n, u).$$

Using the same arguments as in the proof of the Theorem 1 for $\{y_n\}$ we derive that $y_n \to y$ in the sense of (28), where y is an unique solution of (5) with control u.

We have the following inequality: $\forall y, u \in \Xi$

$$\mathcal{J}(\hat{y}, \hat{u}) \leq \underline{\lim}_{n \to \infty} \mathcal{J}\left(\overline{y}^n, \overline{u}^n\right) \leq \underline{\lim}_{n \to \infty} \mathcal{J}(y_n, u) = \mathcal{J}(y, u). \tag{53}$$

This means that $\hat{y}, \hat{u}$ is a solution of (5) – (7).

Now we substitute $u = \hat{u}$ in the previous arguments. Then $y = \hat{y}$ due to uniqueness. Thus, from (53) we get

$$\mathcal{J}(\hat{y}, \hat{u}) \leq \underline{\lim}_{n \to \infty} \mathcal{J}\left(\overline{y}^n, \overline{u}^n\right) \leq \mathcal{J}(\hat{y}, \hat{u}). \tag{54}$$

These inequalities mean that up to a subsequence

$$\mathcal{J}\left(\overline{y}^n, \overline{u}^n\right) \to \mathcal{J}(\hat{y}, \hat{u}), \; n \to \infty. \tag{55}$$

Since

$$\mathcal{J}(\hat{y}, \hat{u}) = \inf_{y,u\in\Xi} \mathcal{J}(y,u),$$

then convergence in (55) holds for the whole sequence. Therefore, (32) is proved.

□

Discussion and conclusions

We use the averaging method to study the optimal control problem for a parabolic inclusion with fast-oscillating parameters. In the future, we are planning to apply the averaging method to the study of optimal control problems for evolutionary variational inequalities and differential inclusions of the second order.

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References

Denkowski, Z., & Mortola, S. (1993). Asymptotic behavior of optimal solutions to control problems for systems described by differential inclusions corresponding to partial differential equations. *Journal Optim Theory Appl,* 78, 365–391. https://doi.org/10.1007/BF00939675

Hermosilla Cristopher, & Palladino Michele. (2022). Optimal Control of the Sweeping Process with a Nonsmooth Moving Set. *SIAM Journal on Control and Optimization,* 60(5). https://doi.org/10.1137/21M1405472

Hermosilla, C., Palladino, M., & Hamilton–Jacobi–Bellman Vilches, E. (2024). Approach for Optimal Control Problems of Sweeping Processes. *Applied Mathematics and Optimization,* 90(33). https://doi.org/10.1007/s00245-024-10174-x

Hermes H. (1968). Calculus of Set Valued Functions and Control. *Journal of Mathematics and Mechanics*, 18(1), 47–59.

Kapustian, O. A., Kapustyan, O. V., Ryzhov, A., & Sobchuk, V. (2022). Approximate Optimal Control for a Parabolic System with Perturbations in the Coefficients on the Half-Axis. *Axioms*, 11(4), 175. https://doi.org/10.3390/ axioms11040175

Kapustyan, O. V., Kapustyan, O. A., & Sukretna, A. V. (2009). Approximate bounded synthesis for one weakly nonlinear boundary-value problem. *Nonlinear Oscill,* 12, 297–304. https://doi.org/10.1007/s11072-010-0078-0

Kapustyan, O. A., & Nakonechnyi, A. G. (1999). Optimal bounded control synthesis for a parabolic boundary-value problem with fast oscillatory coefficients. *Journal of Automation and Information Sciences*, 31(12), 33–44. https://doi.org/10.20535 /SRIT.2308-8893.2019.2.08

Kasimova, N. V., Kupenko, O. P., & Tsyganivska, I. M. (2023). Optimal Control Problem for Non-Linear Degenerate Parabolic Variation Inequality: Solvability and Attainability Issues. *Journal of Optimization, Differential Equations and their Applications (JODEA)*, 31(1), 1–21. https://doi.org/10.15421/142301

Kasimova, N., Zhuk, T., & Tsyganivska, I. (2023). Approximate solution of the optimal control problem for non-linear differential inclusion on the semi-axes. *Georgian Mathematical Journal*, 30, (6), 883–889. https://doi.org/10.1515/gmj-2023-2054

Kichmarenko, O. D., Kapustian, O. A., Kasimova, N. V., & Zhuk, T. Yu. (2023). Optimal Control Problem for a Differential Inclusion with Rapidly Oscillating Coefficients on the Semiaxis. *Journal of Mathematical Sciences,* 272, 267–277. https://doi.org/10.1007 /s10958-023-06415-z

Koval'chuk, T. V., Lavrova, O. E., & Mohyl'ova, V. V. (2021). Optimal Control for Some Classes of Dynamic Equations on the Infinite Interval of Time Scale. *Journal of Mathematical Sciences,* 254, 229–245. https://doi.org/10.1007/s10958-021-05300-x

Koval'chuk, T. V., Mohyl'ova, V. V., & Shovkoplyas, T. V. (2020). Averaging Method in Problems of Optimal Control over Impulsive Systems. *Journal of Mathematical Sciences,* 247, 314–327 https://doi.org/10.1007/s10958-020-04804-2

Lions, J.-L. (1969). *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod, Gauthier-Villars.

Pichkur, V. V. (2019). Maximum Sets of Initial Conditions in Practical Stability and Stabilization of Differential Inclusions. In V. Sadovnichiy, & M. Zgurovsky (Eds), *Modern Mathematics and Mechanics. Understanding Complex Systems*. Springer, Cham. https://doi.org/10.1007/978-3-319-96755-4_20

Pichkur, V. V., & Linder, Y. M. (2021). Practical Stability of Discrete Systems: Maximum Sets of Initial Conditions Concept. In V. A. Sadovnichiy, & M. Z. Zgurovsky (Eds), *Contemporary Approaches and Methods in Fundamental Mathematics and Mechanics. Understanding Complex Systems*. Springer, Cham. https://doi.org/10.1007/978-3-030-50302-4_17

Pichkur, V. V., Linder, Y. M., & Tairova, M. S. (2021). On the Practical Stability of Discrete Inclusions with Spatial Components. *Journal of Mathematical Sciences,* 254, 280–286. https://doi.org/10.1007/s10958-021-05304-7

Zadoianchuk, N. V., & Kasyanov, P. O. (2012). Dynamics of solutions of a class of second-order autonomous evolution inclusions. *Cybern Syst Anal*, 48, 414–428. https://doi.org /10.1007/s10559-012-9421-z

Zadoyanchuk, N. V., & Kas'yanov, P. O. (2007). Faedo-Galerkin method for nonlinear second-order evolution equations with Volterra operators. *Nonlinear Oscill,* 10, 203–228. https://doi.org/10.1007/s11072-007-0016-y

Zgurovsky, M. Z., & Mel'nik, V. S. (2004). *Nonlinear analysis and control of physical processes and fields*. Springer. https://doi.org/10.1007/978-3-642-18770-4

Zhuk, T., Kasimova, N., & Ryzhov, A. (2022). Application of the Averaging Method to the Optimal Control Problem of Non-Linear Differential Inclusions on the Finite Interval. *Axioms*, 11(11), 653. https://doi.org/10.3390/axioms11110653

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МЕТОД УСЕРЕДНЕННЯ ДЛЯ ЗАДАЧІ ОПТИМАЛЬНОГО КЕРУВАННЯ ПАРАБОЛІЧНИМ ВКЛЮЧЕННЯМ ЗІ ШВИДКОКОЛИВНИМИ КОЕФІЦІЄНТАМИ НА СКІНЧЕННОМУ ЧАСОВОМУ ІНТЕРВАЛІ

Досліджено задачу оптимального керування для параболічного диференціального включення зі швидкоколивними змінними на скінченному інтервалі. Існує багато підходів до дослідження задач керування для диференціальних рівнянь і включень. Досить часто, зокрема, використовують асимптотичні методи. Серед них можна виокремити метод усереднення, що був строго математично обґрунтований у роботах М. М. Крилова та М. М. Боголюбова. Ключову роль у дослідженні відіграє теорема Красносельського – Крейна, а також її багатозначний аналог. Метод усереднення використовують для звичайних диференціальних включень, а також для включень із частинними похідними та з похідною Хукухара. Багатозначність породжує специфічні проблеми, пов'язані, наприклад, із замкненістю і опуклістю сім'ї розв'язків, існування граничних розв'язків, виділення розв'язків із заданими властивостями. Проте добре розвинений апарат математичного аналізу, застосований до вивчення багатозначних функцій, дає змогу використовувати метод усереднення до згаданих задач оптимального керування. Наприклад, у пропонованій роботі за допомогою метода усереднення доведено збіжність оптимальних керувань та оптимальних траєкторій розв'язків вихідної задачі оптимального керування до оптимальних керувань та оптимальних траєкторій усередненої задачі.

Ключові слова: ***задача оптимального керування, параболічне диференціальне включення, метод усереднення, швидкоколивні змінні.***

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