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Оцінка Беррі-Ессеєна для майже критичних гіллястих процесів з імміграцією

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Berry–Esseen bound for nearly critical branching processes with immigration

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У даній роботі ми розглядаємо майже критичні гіллясті процеси з імміграцією. Було встановлено оцінку швидкості збіжності в центральній граничній теоремі для флуктуацій таких процесів. При виводі оцінки швидкості збіжності до граничного розподілу було використано добре відому нерівність Ессеєна, що дозволяє оцінювати близькість функцій розподілу за близькістю їх характеристичних функцій. Відомо, що теорема Беррі-Ессеєна дає результат, який неможливо в загальному випадку покращити, або, як кажуть - правильну оцінку. Варто відмітити, що в нашій оцінці залежність сталих від розподілу прямих нащадків однієї частинки та потоку імміграції є більш складною ніж в класичній оцінці Беррі-Ессеєна. Отримана оцінка узагальнює та уточнює попередню відому оцінку для флуктуації подібних процесів.

Ключові слова: Гіллясті процеси, імміграція, нерівність Ессеєна.

In this paper, we consider a nearly critical branching process with immigration. We obtain the rate of convergence in central limit theorem for nearly critical branching processes with immigration.

Key Words: Branching process, immigration, Esseen inequality
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1 Introduction

Let for each $n \in \mathbb{N}$ $\left\{\xi_{k,j}^{(n)}, k, j \in \mathbb{N}\right\}$ and $\left\{\varepsilon_k^{(n)}, k \in \mathbb{N}\right\}$ be the two independent families of independent identically distributed random variables with non-negative integer values. The sequence of branching processes with immigration $\{X_k^{(n)}, k \geqslant 0\}$, $n \in \mathbb{N}$ is defined by the following recursion relation

$$X_0^{(n)} = 0, \quad X_k^{(n)} = \sum_{j=1}^{X_{k-1}^{(n)}} \xi_{k,j}^{(n)} + \varepsilon_k^{(n)}, \quad k, n \in \mathbb{N}. \quad (1)$$

For a fixed $n \in \mathbb{N}$ we can interpret $X_k^{(n)}$ as the size of $k-$th generation of a population and $\xi_{k,j}^{(n)}$ is the number of offsprings of the $j-$th individual in the $(k-1)-$st generation and $\varepsilon_k^{(n)}$ is the number of immigrants contributing to the $k-$th generation.

Branching processes models are extensively used in various parts of natural sciences, among others in biology, epidemiology, physics, computer sciences. We refer to monographs by Athreya and Ney [1], Rahimov [17], Haccou et al. [6] where one may find a recent developments and various applications of these process.

Assume that for all $n \in \mathbb{N}$

$$m_n = \mathrm{E}\xi_{1,1}^{(n)}, \quad \sigma_n^2 = \mathrm{var}\,\xi_{1,1}^{(n)}, \quad \lambda_n = \mathrm{E}\varepsilon_1^{(n)}, \quad b_n^2 = \mathrm{var}\,\varepsilon_1^{(n)}$$

exist and finite.

Означення 1.1. Branching processes with immigration defined by (1) is called nearly critical with rate $\alpha \in \mathbb{R}$ if $m_n = 1 + \alpha n^{-1} + o\left(n^{-1}\right)$ as $n \to \infty$.

This notation was introduced for the first time by Chan and Wei [3] in order to study AR(1) processes.

Introduce the random process

$$X_n(t) = X^{(n)}_{[nt]}, \quad t \in \mathbb{R}_+, \quad n \in \mathbb{N}, \qquad (2)$$

where $[a]$ denotes the lower integer part of number a. It is clear that $\{X_n(t), n \in \mathbb{N}\}$ is a sequence of random elements with values in the Skorokhod space $D[0,\infty)$, which is the space of non–negative functions on $[0,\infty)$ that are right continuous and have left limits.

There have been a lot of research works devoted to the asymptotic behavior of the process defined by (2). For instance, in case when $m_n = m = 1$ and with finite moments $\sigma_n^2 = \sigma^2 > 0$, $\lambda_n = \lambda$ and $b_n^2 = b^2 > 0$, Wei and Winnicki [23] proved that the random processes $n^{-1}X_n(t)$ weakly converge in Skorokhod space $D[0,\infty)$ to the solution of some stochastic differential equation. Later, the case $m_n = 1 + \alpha n^{-1} + o\left(n^{-1}\right)$ ($\alpha \in \mathbb{R}$), $\sigma_n^2 \to \sigma^2 > 0$ as $n \to \infty$ was studied by Sriram[22], who proved the weak convergence of random processes $n^{-1}X_n(t)$ in Skorokhod space $D[0,\infty)$ to the solution of stochastic differential equation. Ispány et al. [8] showed that Sriram's [22] result still holds if condition $\sigma_n^2 \to \sigma^2$ replaced by $\sigma_n^2 \to 0$ as $n \to \infty$. They also proved fluctuation limit theorem in case where $\sigma_n^2 \to 0$, namely, the sequence $n^{-\frac{1}{2}}(X_n(t) - \mathrm{E}X_n(t))$ has a limit process $X(t)$. The process $\{X(t), t \in \mathbb{R}\}$ turns out to be an Ornstein–Uhlenbeck type process driven by a time changed Wiener process. Further systematic investigations are due to Ispány [7], Györfi et al.[5], Li [14], [15] Rahimov [18],[19], Khusanbaev [10],[11],[12],[13] and see also references therein. From these papers, it follows that the necessary norming factor and the possible limit process strongly depend on the various conditions that are supposed to hold for the offspring and immigration processes.

The problem of estimating the rate of convergence in the central limit theorem (CLT) is one of the traditional problems in probability theory. It attracted the attention of many researchers during the last six decades. However, most of the attention was paid to clarifying the estimate of the Berry–Esseen theorem, which considers the uniform distance between the distribution functions of the normalized sums of independent random variables and the standard normal law.

However, few scientific papers are devoted to studying of the rate of weak convergence in limit theorems for branching processes. Apparently, such a problem was first studied by Nagaev and Mukhamedkhanova in [16]. They obtained the rate of convergence in the well-known Yaglom's limit theorem on the weak convergence of the critical Galton–Watson branching process to the exponential distribution. We note that the constants in this estimate have a more complicated relationship to the distribution of offsprings as compared to the classical Berry-Esseen estimate. Later, Borovkov [2] established the rate of weak convergence for the Galton–Watson branching processes (without immigration) to the Feller process. We also refer to Rahimov and Sirazhdinov [20] where one can find several types of estimations for fluctuation of (1). Recently, Khusanbaev [9] obtained the rate of convergence in CLT for fluctuation of process (1). Motivated by that paper we continue studying the rate of convergence in one-dimensional functional limit theorem 2.9 of Ispány [7].

More precisely, Ispány [7] proved the following a fluctuation limit theorem for the process defined in (2).

Theorem 1.1. *Assume that $b_n^2 > 0$ for all $n \in \mathbb{N}$ such that $nb_n^2 \to \infty$ as $n \to \infty$ and*

1) $m_n = 1 + \alpha n^{-1} + o\left(n^{-1}\right)$ as $n \to \infty$ for some $\alpha \in \mathbb{R}$;

2) $\sigma_n^2 = \sigma^2 n^{-1} + o\left(n^{-1}\right)$ as $n \to \infty$ for some $\sigma^2 \geqslant 0$;

3) $\mathrm{E}\left(\left(\xi_{1,1}^{(n)} - m_n\right)^2 I(U_n)\right) = o\left(n^{-1}\right)$ as $n \to \infty$ for all $\theta > 0$, where $U_n = \left\{\left|\xi_{1,1}^{(n)} - m_n\right| > \theta\sqrt{nb_n^2}\right\}$ and $I\{A\}$ denotes an indicator of event A;

4) $\lambda_n = \lambda b_n^2 + o\left(b_n^2\right)$ as $n \to \infty$ for some $\lambda \geqslant 0$;

5) $\mathrm{E}\left(\left(\varepsilon_{1,1}^{(n)} - \lambda_n\right)^2 I(U_n')\right) = o\left(b_n^2\right)$ as $n \to \infty$ for all $\theta > 0$, where $U_n' = \left\{\left|\varepsilon_1^{(n)} - \lambda_n\right| > \theta\sqrt{nb_n^2}\right\}$.

Then

$$\left(nb_n^2\right)^{-\frac{1}{2}}\left(X_n(t) - \mathrm{E}X_n(t)\right) \Rightarrow \chi(t), \quad t \in \mathbb{R}_+ \quad \text{as} \quad n \to \infty$$

where convergence holds in Skorokhod space $D[0,\infty)$, and $\{\chi(t), t \in \mathbb{R}_+\}$ is an Ornstein–Uhlenbeck type process defined by the following *stochastic differential equation*

$$d\chi(t) = \alpha\chi(t)\,dt + \sqrt{\sigma^2\mu(t)+1}dW(t), \quad \chi(0)=0,$$

where $W(t)$ *is a standard Wiener process and* μ *is defined by* $\mu(t)=\lambda\int_0^t e^{\alpha s}ds$.

Note that the limiting process $\chi(t)$ may be represented as

$$\chi(t) = \int_0^t e^{\alpha(t-s)}dM(s), \quad t\in\mathbb{R}_+,$$

where $\{M(t), t\in\mathbb{R}_+\}$ is a Wiener process $M(t)=W(T(t)), t\in\mathbb{R}_+$ with $T(t)=\int_0^t\rho(s)ds$ and

$$\rho(t) = \lambda\sigma^2\int_0^t e^{\alpha s}ds + 1.$$

From the properties of the stochastic integral [4] we can obtain

$$\mathrm{E}\chi(t)=0, \quad \mathrm{var}(\chi(t)) = \int_0^t e^{2\alpha(t-s)}\rho(s)ds.$$

Consequently,

$$\beta^2 := \mathrm{var}(\chi(1)) = \begin{cases} \frac{\sigma^2\lambda}{2}+1, & \alpha=0, \\ \frac{\sigma^2\lambda}{2\alpha^2}(e^\alpha-1)^2 + \frac{e^{2\alpha}-1}{2\alpha}, & \alpha\neq 0 \end{cases}$$

Hence, $\chi(1)$ has normal distribution with zero mean and variance β^2.

The main purpose of this paper is to obtain the rate of convergence of the random variables $(nb_n^2)^{-\frac{1}{2}}(X_n(1)-\mathrm{E}X_n(1))$ to $\chi(1)$. The approach we use in the proofs comes from Khusanbaev [9]. However, we have to pay more attention to the asymptotics of mean and variances of immigration.

The remainder of the paper is organized as follows. Section 2 contains some notations and assumptions. The main result and examples are given in Section 3. Section 4 is devoted to the proof of the main theorem.

2 Some notations and assumptions

Before giving our main result, we need to some notations and conditions.

For each $n\in\mathbb{N}$, let $S_k^{(n)}, k\geqslant 0$ be the Galton–Watson branching process defined by

$$S_0^{(n)}=1, \quad S_k^{(n)} = \sum_{i=1}^{S_{k-1}^{(n)}} \xi_{k,j}^{(n)}.$$

Then it is well–known [1]

$$\mathrm{E}S_k^{(n)} = m_n^k, \quad \mathrm{E}\left[S_k^{(n)}\right]^2 = \frac{m_n^{k-1}\left(m_n^{k-1}-1\right)}{m_n-1}b_n^2 + m_n^{2k},$$

$$\mathrm{E}\left[S_k^{(n)}\right]^3 = \frac{m_n^{k-1}\left(m_n^{2k}-1\right)}{m_n^2-1}\gamma_n +$$

$$+\frac{3m_n^{k-1}\left(m_n^k-1\right)\left(m_n^{k-1}-1\right)}{(m_n-1)\left(m_n^2-1\right)}\left(b_n^2+m_n^2-m_n\right)^2 +$$

$$+\frac{3m_n^{k-1}\left(m_n^k-1\right)}{m_n-1}\left(b_n^2+m_n^2-m_n\right) + m_n^k,$$

where $\gamma_n = \mathrm{E}\xi_{1,1}^{(n)}\left(\xi_{1,1}^{(n)}-1\right)\left(\xi_{1,1}^{(n)}-2\right)$.

Let $f_n(s)$ be a generating function of $\xi_{1,1}$. Then, plainly, $f_n^{(k)}(s)$ is the generating function of $S_k^{(n)}$.

Further, we put

$$h_n(s) = \mathrm{E}s^{\varepsilon_1^{(n)}}, \quad \varphi_n(s) = \mathrm{E}s^{X_n^{(n)}}, \quad |s|\leqslant 1.$$

Suppose that for all $n\in\mathbb{N}$ the variables

$$\beta_n^2 := \frac{\mathrm{var}(X_n(1))}{n} =$$

$$\begin{cases} \frac{\lambda_n\sigma_n^2(n-1)}{2} + b_n^2, \ m_n=1, \\ \frac{\lambda_n\sigma_n^2\left(m_n^{n-1}-1\right)\left(m_n^n-1\right)}{n(m_n-1)\left(m_n^2-1\right)} + \frac{b_n^2\left(m_n^{2n}-1\right)}{n\left(m_n^2-1\right)}, \ m_n\neq 1. \end{cases}$$

$$\chi_n =$$

$$\begin{cases} n\gamma_n + \frac{3}{2}\sigma_n^4 n(n-1) + 3n\sigma_n^2 + 1, \ m_n=1, \\ \frac{\gamma_n}{m_n\left|m_n^2-1\right|} + \frac{3\left(\sigma_n^2+m_n^2-m_n\right)^2}{m_n|m_n-1|\left|m_n^2-1\right|} + \frac{3\left|\sigma_n^2+m_n^2-m_n\right|}{m_n|m_n-1|}, \ m_n\neq 1. \end{cases}$$

$$\tau_n = \mathrm{E}\varepsilon_1^{(n)}\left(\varepsilon_1^{(n)}-1\right)\left(\varepsilon_1^{(n)}-2\right)$$

exist and finite.

Set

$$L_n = A_nB_n,$$

where

$$A_n = \max\left\{(1+\lambda_n)^3;\ b_n^2+\lambda_n^2;\ \tau_n\right\},$$

$$B_n = A_n\chi_n\max\left\{1;\ m_n^n\right\}.$$

As usual, denote by $\Phi_\beta(x)$ normal distribution function with zero mean and variance β^2 and $\rho(F,G)$ stands for uniform distance (Kolmogorov's metric) between distribution functions $F(x)$ and $G(x)$. The symbol C denotes a positive number which may vary from place to place.

The proof of the main theorem is based on the following inequality (see [21] Chap.II, p.376), which allows us to estimate the proximity of distribution functions by the proximity of their characteristic functions.

Lemma 1 (Esseen's inequality)**.** *Let $F(x)$ and $G(x)$ be two distribution functions with characteristic functions $f(t)$ and $g(t)$, respectively. Assume $G(x)$ has derivative $G'(x)$ with $\sup_x |G'(x)| \leqslant C$. Then, for arbitrary $x \in \mathbb{R}$ and $T > 0$*

$$\rho(F,G) \leqslant \frac{2}{\pi}\int_0^T \left|\frac{f(t)-g(t)}{t}\right| dt + \frac{24}{\pi T}\sup_x \left|G'(x)\right|.$$

3 Main result

Now we state our main result.

Theorem 3.1. *Let $b_n^2 > 0$, $n \in \mathbb{N}$, $nb_n^2 \to \infty$ and $L_n = o\left(\sqrt{nb_n^2}\right)$ as $n \to \infty$. Assume that $\min\left(\frac{\beta_n^2}{b_n^2}, \beta^2\right) > 0$ and the following conditions hold:*

1) $m_n = 1 + \alpha n^{-1} + o\left(n^{-1}\right)$ as $n \to \infty$ for some $\alpha \in \mathbb{R}$;

2) $\sigma_n^2 = \sigma^2 n^{-1} + o\left(n^{-1}\right)$ as $n \to \infty$ for some $\sigma^2 \geqslant 0$;

3) $\lambda_n = \lambda b_n^2 + o\left(b_n^2\right)$ as $n \to \infty$ for some $\lambda \geqslant 0$;

4) $\tau = \limsup_{n\to\infty} \tau_n \leqslant \infty$;

5) $\gamma_n = \gamma n^{-1} + o\left(n^{-1}\right)$ as $n \to \infty$ for some $\gamma > 0$.

Then distribution functions of normalized variables

$$F_n(x) = \mathrm{P}\left(\left(nb_n^2\right)^{-1/2}\left(X_n(1) - \mathrm{E}\left(X_n(1)\right)\right) < x\right), \quad -\infty < x < \infty,$$

such that for all $n \geqslant 2$,

$$\rho(F_n, \Phi_\beta) \leqslant \frac{CL_n}{\sqrt{nb_n^2}} + \frac{\left|\frac{\beta_n^2}{b_n^2} - \beta^2\right|}{\min\left(\frac{\beta_n^2}{b_n^2}, \beta^2\right)}, \qquad (3)$$

where C depends on variables α, σ^2, λ, b^2, β^2, τ, γ.

Remark 1. *Theorem 2 is a generalization of the main result of paper [9] for the cases $b_n^2 \to 0$ and $b_n^2 \to \infty$ as $n \to \infty$ (see condition 3). Assumption $nb_n^2 \to \infty$ and the condition 3) imply that the variable b_n^2 may converge to a finite number (zero or non-zero) or infinity. Khusanbaev [9] obtained the rate of convergence in one-dimensional Theorem 2.2 of Ispány et al. [8].*

Remark 2. *Note that both terms are important on the right-hand side of (3) since each of first or second terms on the right hand side of (3) may be dominating in different situations.*

As an example, we give the following corollary.

Наслідок 1. *1) Let $\xi_{1,1}^{(n)}$ has a Bernoulli distribution with mean $1-\alpha n^{-1}$, $\alpha \geqslant 0$ and the random variable $\varepsilon_1^{(n)}$ is distributed as follows:*

$$\mathrm{P}(\varepsilon_1^{(n)} = 0) = 1 - \frac{1}{n}, \ \ \mathrm{P}(\varepsilon_1^{(n)} = [\ln n]) = \frac{1}{n}, \ \ n \in \mathbb{N}.$$

Then it is easy to see that the conditions 1), 2) and 5) are satisfied. Moreover, we have $\lambda_n = \frac{\ln n}{n}$, $b_n^2 \sim \frac{\ln^2 n}{n}$, $n \to \infty$ and thus the condition 3) holds. The condition 4) fulfills since $\tau_n \sim \frac{\ln^3 n}{n}$ and $\tau < \infty$. Consequently, all conditions of Theorem 2 hold in the case when immigration variance tends to zero.

2) Let us consider the case when the immigration variance tends to infinity such that $nb_n^2 \to \infty$ as $n \to \infty$. Let $\xi_{1,1}^{(n)}$ is distributed as below and take for $n \in \mathbb{N}$

$$\mathrm{P}(\varepsilon_1^{(n)} = 0) = 1 - \frac{1}{\ln n}, \quad \mathrm{P}(\varepsilon_1^{(n)} = [\ln n]) = \frac{1}{\ln n}.$$

In this case, we have $\lambda_n = 1$, $b_n^2 = \ln n - 1$ and $\tau_n \sim \ln n$ and all conditions of Theorem 2 hold.

4 Proof of the main result

Доведення. We consider only the case $m_n > 1$. The statement of Theorem for the cases $m_n = 1$ and $m_n < 1$ can be proved by analogous considerations. By Esseen inequality

$$\sup_x \left|F_n(x) - \Phi_\beta(x)\right| \leqslant \frac{2}{\pi}\int_0^T \left|\frac{g_n(t) - g(t)}{t}\right| dt + \frac{24}{\pi T}\sup_x \left|\Phi'_\beta(x)\right|, \quad (4)$$

where

$$g_n(t) = e^{-it\mathrm{E}X_n^{(n)}\big/\sqrt{nb_n^2}}\varphi_n\left(e^{it\big/\sqrt{nb_n^2}}\right),$$

$$g(t) = e^{-\frac{t^2}{2}\beta^2}, \quad \sup_x \left|\Phi'_\beta(x)\right| = \frac{1}{\sqrt{2\pi}\beta}.$$

In relation (4), a positive number $T = T_n$ may be chosen arbitrary.

We set

$$T_n = \frac{\sqrt{nb_n^2}}{24L_n}.$$

Now we have to estimate the integral (4). For this aim, we represent it as the sum of three integrals:

$$J_n^{(1)}+J_n^{(2)}+J_n^{(3)}=\int_0^T\frac{\Delta_n^{(1)}(t)}{t}dt+ +\int_0^T\frac{\Delta_n^{(2)}(t)}{t}dt+\int_0^T\frac{\Delta_n^{(3)}(t)}{t}dt, \quad (5)$$

where

$$\Delta_n^{(1)}(t)=\left|e^{-it\mathrm{E}X_n^{(n)}\Big/\sqrt{nb_n^2}}\varphi_n\left(e^{it\Big/\sqrt{nb_n^2}}\right)- -e^{-it\frac{\lambda_n}{\sqrt{nb_n^2}}\frac{m_n^n-1}{m_n-1}+\sum\limits_{k=0}^{n-1}\left(z_{nk}-\frac{z_{nk}^2}{2}\right)}\right|,$$

$$\Delta_n^{(2)}(t)=\left|e^{-it\frac{\lambda_n}{\sqrt{nb_n^2}}\frac{m_n^n-1}{m_n-1}+\sum\limits_{k=0}^{n-1}\left(z_{nk}-\frac{z_{nk}^2}{2}\right)}-e^{-\frac{\beta_n^2}{b_n^2}\frac{t^2}{2}}\right|,$$

$$\Delta_n^{(3)}(t)=\left|e^{-\frac{\beta_n^2}{b_n^2}\frac{t^2}{2}}-e^{-\beta^2\frac{t^2}{2}}\right|,$$

where $z_{nk}=h_n\left(f_n^{(k)}\left(e^{it\Big/\sqrt{nb_n^2}}\right)\right)-1$ and we have to bound each $\Delta_n^{(i)}(t)$, $i=1,2,3$ and evaluate integrals $J_n^{(i)}$, $i=1,2,3$, separately.

Consider $\Delta_n^{(1)}(t)$. It is well-known that the generating function of $X_n^{(n)}$ is defined by

$$\varphi_n(s)=\prod_{k=0}^{n-1}h_n\left(f_n^{(k)}(s)\right).$$

Since $\mathrm{E}\left(\xi_{1,1}^{(n)}\right)^3<\infty$, then we may use Taylor expansion with Lagrange form of remainder

$$f_n^{(k)}\left(e^{it\Big/\sqrt{nb_n^2}}\right)= =1+it\frac{\mathrm{E}S_k^{(n)}}{\sqrt{nb_n^2}}-\frac{t^2}{2}\frac{\mathrm{E}\left(S_k^{(n)}\right)^2}{nb_n^2}+\theta_1\frac{|t|^3\,\mathrm{E}\left(S_k^{(n)}\right)^3}{nb_n^2\sqrt{nb_n^2}}, \quad |\theta_1|\leqslant 1. \quad (6)$$

Similarly, by the assumption $\tau_n<\infty$ we may decompose the generating function $h_n(s)$ in Taylor series with Lagrange form of remainder

$$h_n(s)= =1+(s-1)\,\mathrm{E}\varepsilon_1^{(n)}+\frac{(s-1)^2}{2}\mathrm{E}\varepsilon_1^{(n)}\left(\varepsilon_1^{(n)}-1\right)+ +\theta_2(s-1)^3\tau_n, |\theta_2|\leqslant 1. \quad (7)$$

Substituting (6) in (7), we obtain

$$h_n\left(f_n^{(k)}\left(e^{it\Big/\sqrt{nb_n^2}}\right)\right)=1+\frac{it}{\sqrt{nb_n^2}}\lambda_n\mathrm{E}S_k^{(n)}- -\frac{t^2}{2nb_n^2}\lambda_n\mathrm{E}\left(S_k^{(n)}\right)^2- -\frac{t^2}{2nb_n^2}\lambda_n\left(\mathrm{E}S_k^{(n)}\right)^2\left(b_n^2+\lambda_n^2-\lambda_n\right)+Q_n^{(k)}(t)= =1+\frac{itm_n^k}{\sqrt{nb_n^2}}\lambda_n-\frac{t^2\lambda_n\sigma_n^2}{2nb_n^2}\frac{m_n^{k-1}\left(m_n^k-1\right)}{m_n-1}- -\frac{t^2m_n^{2k}}{2nb_n^2}\left(b_n^2+\lambda_n^2\right)+Q_n^{(k)}(t),$$

where $Q_n^{(k)}(t)$ admits the representation:

$$Q_n^{(k)}(t)=\theta_1\frac{|t|^3\,\mathrm{E}\left(S_k^{(n)}\right)^3}{nb_n^2\sqrt{nb_n^2}}\lambda_n+ +\frac{it\mathrm{E}\left(S_k^{(n)}\right)}{\sqrt{nb_n^2}}\left[\frac{t^2\mathrm{E}\left(S_k^{(n)}\right)^3}{2nb_n^2}+ +\theta_1\frac{|t|^3\,\mathrm{E}\left(S_k^{(n)}\right)^3}{nb_n^2\sqrt{nb_n^2}}\right]\left(b_n^2+\lambda_n^2-\lambda_n\right)+ +\frac{1}{2}\left[-\frac{t^2\mathrm{E}\left(S_k^{(n)}\right)^2}{2nb_n^2}+\theta_1\frac{|t|^3\,\mathrm{E}\left(S_k^{(n)}\right)^3}{nb_n^2\sqrt{nb_n^2}}\right]^2\times \times\left(b_n^2+\lambda_n^2-\lambda_n\right)+\theta_2\times \times\left[\frac{it\mathrm{E}S_k^{(n)}}{\sqrt{nb_n^2}}-\frac{t^2\mathrm{E}\left(S_k^{(n)}\right)^2}{2nb_n^2}+\theta_1\frac{|t|^3\,\mathrm{E}\left(S_k^{(n)}\right)^3}{nb_n^2\sqrt{nb_n^2}}\right]^3\tau_n.$$

Then, it is easily seen that for $|t|\leqslant T_n$, we have the following estimate

$$1-\left|h_n\left(f_n^{(k)}\left(e^{it\Big/\sqrt{nb_n^2}}\right)\right)\right|\leqslant|z_{nk}|\leqslant \leqslant\frac{|t|}{\sqrt{nb_n^2}}m_n^k\left(1+\lambda_n\right)\leqslant\frac{1}{24}. \quad (8)$$

Therefore, for $|t|\leqslant T_n$, it is valid the representation

$$\prod_{k=0}^{n-1}h_n\left(f_n^{(k)}\left(e^{it\Big/\sqrt{nb_n^2}}\right)\right)=e^{\sum\limits_{k=0}^{n-1}\ln(1+z_{nk})},$$

where $\ln z$ means the main value of the logarithm of a complex number z ($\ln z=\ln|z|+i\arg z$, $-\pi<\arg z<\pi$).

Now by using the following inequalities

$$|e^x-1|\leqslant |x|\,e^{|x|},\quad \left|\ln(1+x)-x+\frac{x^2}{2}\right|\leqslant |x|^3,\quad |x|\leqslant\frac{1}{2},\quad (9)$$

we obtain

$$\Delta_n^{(1)}(t)\leqslant\left|e^{-it\frac{\lambda_n}{\sqrt{nb_n^2}}\frac{m_n^n-1}{m_n-1}}\right|\left|e^{\sum\limits_{k=0}^{n-1}\left(z_{nk}-\frac{z_{nk}^2}{2}\right)}\right|\times$$
$$\times\sum_{k=0}^{n-1}|z_{nk}|^3\,e^{\sum\limits_{k=0}^{n-1}|z_{nk}|^3}.$$

First we will estimate $|z_{nk}|^3$. By (8), for $|t|\leqslant T_n$, we have

$$|z_{nk}|^3\leqslant\frac{|t|^3}{nb_n^2\sqrt{nb_n^2}}m_n^{3k}(1+\lambda_n)^3\leqslant\frac{t^2}{24nb_n^2}m_n^{2k},$$

and thus

$$\sum_{k=0}^{n-1}|z_{nk}|^3\leqslant\frac{t^2}{24nb_n^2}\sum_{k=0}^{n-1}m_n^{2k}=\frac{t^2}{24nb_n^2}\frac{m_n^{2n}-1}{m_n-1}.\quad (10)$$

On the other hand,

$$\sum_{k=0}^{n-1}|z_{nk}|^3\leqslant\frac{|t|^3}{nb_n^2\sqrt{nb_n^2}}(1+\lambda_n)^3\sum_{k=0}^{n-1}m_n^{3k}\leqslant$$
$$\leqslant\frac{|t|^3}{nb_n^2\sqrt{nb_n^2}}(1+\lambda_n)^3\frac{m_n^{3n}-1}{m_n-1}.\quad (11)$$

Further,

$$\left|e^{\sum\limits_{k=0}^{n-1}\left(z_{nk}-\frac{z_{nk}^2}{2}\right)}\right|\leqslant e^{-\frac{t^2}{2nb_n^2}\sum\limits_{k=0}^{n-1}\left(\frac{\lambda_n\sigma_n^2m_n^k\left(m_n^k-1\right)}{m_n(m_n-1)}+m_n^{2k}b_n^2\right)}\times$$
$$\times e^{\sum\limits_{k=0}^{n-1}\left(\delta_{nk}^{(1)}(t)+\delta_{nk}^{(2)}(t)\right)}\left|e^{it\frac{\lambda_n}{\sqrt{nb_n^2}}\frac{m_n^n-1}{m_n-1}}\right|\quad (12)$$

where

$$\delta_{nk}^{(1)}(t)=\frac{|t|}{\sqrt{nb_n^2}}m_n^k\lambda_n\left|Q_n^{(k)}(t)\right|+$$
$$+\frac{1}{2}\left(\frac{t^2}{2nb_n^2}\lambda_n\mathrm{E}\left(S_k^{(n)}\right)^2+\frac{t^2}{2nb_n^2}m_n^{2k}\left(b_n^2+\lambda_n^2-\lambda_n\right)+\right.$$
$$\left.+\left|Q_n^{(k)}(t)\right|\right)^2,$$

$$\delta_{nk}^{(2)}(t)=\theta_1\frac{|t|^3\,\mathrm{E}\left(S_k^{(n)}\right)^3}{nb_n^2\sqrt{nb_n^2}}\lambda_n+$$
$$+\frac{itm_n^k}{\sqrt{nb_n^2}}\theta_1\frac{|t|^3\,\mathrm{E}\left(S_k^{(n)}\right)^3}{nb_n^2\sqrt{nb_n^2}}\left(b_n^2+\lambda_n^2-\lambda_n\right)+$$
$$+\frac{1}{2}\left[-\frac{t^2\mathrm{E}\left(S_k^{(n)}\right)^2}{2nb_n^2}+\theta_1\frac{|t|^3\,\mathrm{E}\left(S_k^{(n)}\right)^3}{nb_n^2\sqrt{nb_n^2}}\right]^2\times$$
$$\times\left(b_n^2+\lambda_n^2-\lambda_n\right)+$$
$$+\theta_2\left[\frac{it\mathrm{E}S_k^{(n)}}{\sqrt{nb_n^2}}-\frac{t^2\mathrm{E}\left(S_k^{(n)}\right)^2}{2nb_n^2}+\theta_1\frac{|t|^3\,\mathrm{E}\left(S_k^{(n)}\right)^3}{nb_n^2\sqrt{nb_n^2}}\right]^3\tau_n.$$

We may notice that for $|t|\leqslant T_n$

$$\left|Q_n^{(k)}(t)\right|\leqslant\frac{1}{5}\frac{t^2}{nb_n^2}m_n^{2k},\quad\left|\delta_{nk}^{(1)}(t)\right|\leqslant\frac{1}{100}\frac{t^2}{nb_n^2}m_n^{2k},$$

$$\left|\delta_{nk}^{(2)}(t)\right|\leqslant\frac{4}{25}\frac{t^2}{nb_n^2}m_n^{2k}.\quad (13)$$

From (12) and taking estimations (13) into account, we may write

$$\left|e^{\sum\limits_{k=0}^{n-1}\left(z_{nk}-\frac{z_{nk}^2}{2}\right)}\right|\leqslant$$
$$\leqslant e^{-\frac{t^2}{2nb_n^2}\frac{\lambda_n\sigma_n^2\left(m_n^{n-1}-1\right)\left(m_n^n-1\right)}{\left(m_n^2-1\right)(m_n-1)}-\frac{t^2}{2n}\frac{m_n^{2n}-1}{m_n^2-1}}\times$$
$$\times e^{\frac{17}{100}\frac{t^2}{nb_n^2}\frac{m_n^{2n}-1}{m_n^2-1}}.\quad (14)$$

Recalling relations (10), (11) and (14) we have the bound

$$\Delta_n^{(1)}(t)\leqslant\frac{|t|^3}{nb_n^2\sqrt{nb_n^2}}(1+\lambda_n)^3\frac{m_n^{3n}-1}{m_n-1}\times$$
$$\times e^{-\frac{t^2}{2nb_n^2}\left[\frac{\lambda_n\sigma_n^2\left(m_n^{n-1}-1\right)\left(m_n^n-1\right)}{\left(m_n^2-1\right)(m_n-1)}+\frac{11}{25}\frac{m_n^{2n}-1}{m_n-1}\right]}e^{-\frac{t^2}{2n}\frac{m_n^{2n}-1}{m_n^2-1}}.\quad (15)$$

Hence, the integral $J_n^{(1)}$ does not exceed

$$C\frac{(1+\lambda_n)^3}{nb_n^2\sqrt{nb_n^2}}\frac{m_n^{3n}-1}{m_n-1}\times$$

$$\times\int_0^\infty t^2 e^{-\frac{t^2}{2nb_n^2}\left[\frac{\lambda_n\sigma_n^2\left(m_n^{n-1}-1\right)\left(m_n^n-1\right)}{\left(m_n^2-1\right)\left(m_n-1\right)}+\frac{11}{25}\frac{m_n^{2n}-1}{m_n-1}\right]}\times$$

$$\times e^{-\frac{t^2}{2n}\frac{m_n^{2n}-1}{m_n^2-1}}dt$$

$$\leqslant C\frac{(1+\lambda_n)^3}{nb_n^2\sqrt{nb_n^2}}\frac{m_n^{3n}-1}{m_n-1}\times$$

$$\times\frac{nb_n^2}{\left[\frac{\lambda_n\sigma_n^2\left(m_n^{n-1}-1\right)\left(m_n^n-1\right)}{\left(m_n^2-1\right)\left(m_n-1\right)}+\frac{11}{25}\frac{m_n^{2n}-1}{m_n-1}+b_n^2\frac{m_n^{2n}-1}{m_n^2-1}\right]}\leqslant$$

$$\leqslant C\frac{L_n}{\sqrt{nb_n^2}}. \tag{16}$$

Now we consider the relation $\Delta_n^{(2)}(t)$. From the inequalities (9), it entails

$$\Delta_n^{(2)}(t)\leqslant e^{-\frac{t^2}{2}\frac{\beta_n^2}{b_n^2}}\chi_n(t)\,e^{\chi_n(t)}, \tag{17}$$

where

$$\chi_n(t)=\sum_{k=0}^{n-1}\Bigg\{\left|Q_n^{(k)}(t)\right|+\frac{|t|\,m_n^k\lambda_n}{\sqrt{nb_n^2}}\times$$

$$\times\left(\frac{t^2\lambda_n}{2nb_n^2}\mathrm{E}\left(S_k^{(n)}\right)^2+\frac{t^2}{2nb_n^2}\left(\mathrm{E}S_k^{(n)}\right)^2\times\right.$$

$$\left.\times\left(b_n^2+\lambda_n^2-\lambda_n\right)\right)+$$

$$+\frac{1}{2}\left(\frac{t^2\mathrm{E}\left(S_k^{(n)}\right)^2}{2nb_n^2}+\frac{t^2}{2nb_n^2}\left(\mathrm{E}S_k^{(n)}\right)^2\times\right.$$

$$\left.\times\left(b_n^2+\lambda_n^2-\lambda_n\right)+\left|Q_n^{(k)}(t)\right|\right)^2\Bigg\}.$$

For $|t|\leqslant T_n$, we have

$$\left|Q_n^{(k)}(t)\right|\leqslant C\frac{|t|^3}{nb_n^2\sqrt{nb_n^2}}m_n^{4k}B_n,$$

$$\chi_n(t)\leqslant C\frac{|t|^3}{nb_n^2\sqrt{nb_n^2}}\frac{m_n^{4n}-1}{m_n^4-1}A_nB_n,$$

$$\chi_n(t)\leqslant\frac{1}{4}\frac{|t|^2}{nb_n^2}\frac{m_n^{4n}-1}{m_n^4-1}. \tag{18}$$

Substituting the bound (18) in (17), we get

$$\Delta_n^{(2)}(t)\leqslant C\frac{|t|^3}{nb_n^2\sqrt{nb_n^2}}\frac{m_n^{4n}-1}{m_n^4-1}\times$$

$$\times L_n e^{-\frac{t^2}{2}\frac{\beta_n^2}{b_n^2}}e^{\frac{1}{4}\frac{t^2}{nb_n^2}\frac{m_n^2-1}{m_n^2-1}}\leqslant$$

$$\leqslant C\frac{|t|^3}{nb_n^2\sqrt{nb_n^2}}\frac{m_n^{4n}-1}{m_n^4-1}\times$$

$$\times L_n e^{-\frac{t^2}{2nb_n^2}\left(\frac{\lambda_n\sigma_n^2\left(m_n^{n-1}-1\right)\left(m_n^n-1\right)}{\left(m_n-1\right)\left(m_n^2-1\right)}+b_n^2\frac{m_n^{2n}-1}{m_n^2-1}-\frac{1}{2}\frac{m_n^{2n}-1}{m_n^2-1}\right)}.$$

Consequently, for the integral $J_n^{(2)}$ we may deduce

$$J_n^{(2)}\leqslant C\frac{L_n}{\sqrt{nb_n^2}}. \tag{19}$$

Finally, it remains to estimate the relation $\Delta_n^{(3)}(t)$. Now, the application of the inequality

$$\left|e^{-x}-e^{-y}\right|\leqslant e^{-\min(x,y)}\left|x-y\right|,\quad x,y\geqslant 0$$

gives us the estimate

$$\Delta_n^{(3)}(t)\leqslant\frac{t^2}{2}\left|\frac{\beta_n^2}{b_n^2}-\beta^2\right|e^{-\frac{t^2}{2}\min\left(\frac{\beta_n^2}{b_n^2},\beta^2\right)}. \tag{20}$$

The estimation of the integral $J_n^{(3)}$ is not difficult. Collecting (16), (19), (20), and (5), we get the desired result. □

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