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BOUNDARY-VALUE PROBLEM FOR A NONLOCAL IN TIME AND SPACE FRACTIONAL-DIFFERENTIAL ANALOGUE OF THE BIPARABOLIC EVOLUTIONARY EQUATION

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КРАЙОВА ЗАДАЧА ДЛЯ НЕЛОКАЛЬНОГО ЗА ЧАСОМ І ПРОСТОРОМ ДРОБОВО-ДИФЕРЕНЦІЙНОГО АНАЛОГУ БІПАРАБОЛІЧНОГО ЕВОЛЮЦІЙНОГО РІВНЯННЯ

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Abstract. In this work, the formulation and solution of a nonstationary boundary value problem for a fractional-differential analog (with nonlocality in the temporal and spatial variables) of the well-known biparabolic evolutionary partial differential equation of the fourth order are presented. Additionally, the conditions for the existence of a regular solution to the specified problem are provided.

Keywords: boundary value problems for a finite interval, biparabolic evolution equation, fractional-differential analogues, non-classical models, Caputo fractional derivatives, regular solutions.

Анотація. В роботі виконана постановка і одержано розв'язок нестаціонарної крайової задачі для дробово-диференційного аналогу (з нелокальністю за часовою і просторовою змінними) щодо відомого біпараболічного еволюційного диференційного рівняння з частинними похідними четвертого порядку та наведено умови існування регулярного розв'язку зазначеної задачі.

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Ключові слова: крайові задачі для скінченного інтервалу, біпараболічне еволюційне рівняння, дробово-диференціальні аналоги, некласичні моделі, дробові похідні Капуто, регулярні розв'язки.

1. Introduction

The problems of mathematical and computer modeling of the dynamics of complex transport processes are among the pressing areas of development in applied mathematics and cybernetics. A significant focus is placed on studying the fundamental principles of the dynamics and control of heat and mass transfer processes [1–6]. Classical mathematical models of transport process dynamics are based on linear parabolic equations and assume infinite propagation speed of disturbances, leading to several well-known paradoxes.

In [1], a generalization of the classical Fourier heat conduction equation was proposed in the form:

$$Lu \equiv \mu_1 L_1 u + \mu_2 L_2 u = f, \tag{1}$$

where $L_2 = L_1 L_1$, L_1 is the heat conduction (diffusion) operator: $L_1 = \frac{\partial}{\partial t} - \kappa\Delta$, Δ is the Laplace operator with respect to the spatial variables, $u(x,t)$ is the temperature, f is the source function, μ_1, μ_2 are real parameters, and $\kappa = \text{const} > 0$.

Equation (1) was derived in [1] from the law of energy conservation when the energy e and flux $\vec{q}$ are defined by the relations:

$$e = e_0 + c_\nu(u - u_0) + \tau_r c_v L_1 u,$$
$$\vec{q} = -\lambda\nabla u - \tau_r c_v \kappa \nabla\left(L_1 u\right),$$

where $u = u(x,t)$ is the temperature, c_ν is the heat capacity, λ is the thermal conductivity coefficient, $\kappa = \lambda/c_\nu$, τ_r is a real parameter, and ∇ is the Hamiltonian operator. It has been proven that the biparabolic equation (1) is invariant with respect to the Galilean group and can therefore be used to describe thermal and diffusion processes that are independent of the inertial systems in which they are observed [1]. Compared to the classical linear parabolic equation, equation (1) more accurately describes evolutionary processes and allows for the investigation of special regimes, including those with a finite propagation speed of disturbances [1].

The biparabolic equation (1) has been frequently used to model the dynamics of various evolutionary processes [5, 6]. Recent studies have focused on the features of anomalous transport processes based on the ideas of fractional-order integro-differentiation [7, 8]. For example, in works [9–11], a fractional-differential analog of the biparabolic evolutionary equation was introduced, and several model boundary value problems accounting for the phenomenon of temporal nonlocality were solved, including processes of geofiltration in fractured-porous media and filtration-consolidation processes in water-saturated soils.

It is also worth noting that an important area of research concerning the considered biparabolic mathematical model of heat and mass transfer processes involves the study of various inverse problems within this framework. These problems are often ill-posed in the sense of Hadamard, for which various regularization methods are developed. For instance, in [12], a modified quasi-boundary method was applied to solve an inverse problem of determining an unknown source function. In [13], a general inverse initial-value problem for the biparabolic differential equation was studied, while in [14], final-value problems for the biparabolic equation with statistical discrete data were investigated. Closed-form solutions for certain nonstationary one-dimensional direct and inverse boundary value problems concerning anomalous filtration dynamics in a layered geoporous medium of finite thickness, posed as problems for the fractional-differential generalization of the biparabolic evolutionary equation with fourth-order partial derivatives with respect to the spatial variable, were obtained in [15].

This work also presented the formulation and solution of direct and inverse model boundary value problems in geofiltration dynamics based on a special combined mathematical filtration model with coupling conditions and established the conditions for the existence of regular solutions to the considered problems.

2. Boundary value problems

Taking into account temporal nonlocality, the operator L_1 is expressed as:

$$L_1 = {}^C D_{0t}^{\alpha} - \kappa \Delta$$

(here ${}^C D_{0t}^{\alpha}$ is the Caputo fractional derivative operator [7] of order α, $0 < \alpha \leq 1$ with respect to t), and the corresponding fractional-differential analog of the biparabolic equation (1) is written as follows [9]:

$$\mu_1 L_1 u(x,t) + \mu_2 L_1 L_2 u(x,t) = f, \tag{2}$$

where

$$L_1 := {}^C D_{0t}^{\alpha_1} - \kappa_1 \Delta, \quad L_2 := {}^C D_{0t}^{\alpha_2} - \kappa_2 \Delta, \tag{3}$$

${}^C D_{0t}^{\alpha_1}, {}^C D_{0t}^{\alpha_2}$ are the Caputo fractional derivative operators with respect to t of orders α_1, α_2, respectively ($0 < \alpha_1, \alpha_2 \leq 1$), and μ_1, μ_2 are known constants.

Unlike the results of works [9–11,15], this study examines a nonlocal fractional-differential analog of the classical biparabolic equation (1), both in time and space. Extending equations (2) and (3) to account not only for memory effects but also for the anomalous dynamics of the process with respect to the spatial variable x, we derive the following model equation:

$$L_1 u(x,t) + \tau_r L_1 L_2 u(x,t) = f, \tag{4}$$

where

$$L_1 = {}^C D_{0t}^{\alpha_1} - \kappa_1 \, {}^C D_{0x}^{\gamma}, \; L_2 = {}^C D_{0t}^{\alpha_2} - \kappa_2 \, {}^C D_{0x}^{\gamma} \; (0 < \alpha_1, \alpha_2 \leq 1, 1 < \gamma \leq 2). \tag{5}$$

${}^C D_{0t}^{\alpha}, {}^C D_{0x}^{\gamma}$ are Caputo fractional derivative operators of orders α and γ with respect to variables t and x, respectively ($\kappa_1, \kappa_2, \tau_r = \text{const} > 0$).

Equation (4) (under the conditions of (5)) is a nonlocal model equation in both time and spatial variables, which generalizes the classical biparabolic equation (1) and significantly differs from previous fractional-differential analogs of the biparabolic evolution equation by accounting not only for memory effects but also for spatial correlations.

We consider the problem of finding the solution in the domain

$$\Omega = \{(x,t) : 0 < x < 1, 0 < t < T < \infty\}$$

for the equation

$$\left({}^C D_{0t}^{\alpha_1} - \kappa_1 \, {}^C D_{0x}^{\gamma}\right) u(x,t) + \\ +\tau_r \left({}^C D_{0t}^{\alpha_1} - \kappa_1 \, {}^C D_{0x}^{\gamma}\right) \left({}^C D_{0t}^{\alpha_2} - \kappa_2 \, {}^C D_{0x}^{\gamma}\right) u(x,t) = f(x), \tag{6}$$

under the following boundary conditions:

$$u(0,t) = u(1,t) = 0, \; {}^C D_{0x}^{\gamma} u(x,t)\big|_{x\in\{0,1\}} = 0 \quad (t \geq 0)\,, \tag{7}$$

$$u(x,0) = \varphi(x), \quad {}^C D_{0t}^{\alpha_2} u(x,t)\big|_{t=0} = 0 \quad (0 \leq x \leq 1)\,, \tag{8}$$

where $\varphi(x)$ is the initial distribution function, $f(x)$ is the source function, $\tau_r = \text{const} > 0$, and $u(x,t)$ is the sought function.

We introduce a new unknown function

$$\upsilon(x,t) = \tau_r L_2 u(x,t) + u(x,t), \tag{9}$$

where L_2 is defined by (5). Then the problem (6)–(8) reduces to solving the following pair of boundary value problems for the unknown functions $\upsilon(x,t)$ and $u(x,t)$:

$$\left({}^C D_{0t}^{\alpha_1} - \kappa_1 \, {}^C D_{0x}^{\gamma}\right) \upsilon(x,t) = f(x), \tag{10}$$

$$\upsilon(0,t) = \upsilon(1,t) = 0 \quad (t \geq 0)\,, \tag{11}$$

$$\upsilon(x,0) = \psi(x) \quad (0 \leq x \leq 1)\,, \tag{12}$$

$$\tau_r \left({}^C D_{0t}^{\alpha_2} - \kappa_2 \, {}^C D_{0x}^{\gamma}\right) u(x,t) + u(x,t) = \upsilon(x,t), \tag{13}$$

$$u(0,t) = u(1,t) = 0 \quad (t \geq 0)\,, \tag{14}$$

$$u(x,0) = \varphi(x) \quad (0 \leq x \leq 1)\,, \tag{15}$$

where $\psi(x) = \varphi(x) - \tau_r \kappa_2 \, {}^C D_{0x}^{\gamma} \varphi(x)$ is a known function.

3. Regular solution

The solution of the boundary value problem (10)–(12) is sought in the form of a series

$$\upsilon(x,t) = \sum_{n=1}^{\infty} \upsilon_n(t) X_n(x) \quad (x,t) \in \Omega, \tag{16}$$

where $X_n(x) = x^{\gamma-1} E_{\gamma,\gamma}(\lambda_n x^{\gamma})$ are the eigenfunctions of the Sturm-Liouville problem of the form

$$ {}^C D_x^{\gamma} X(x) = \lambda X(x), \; X(0) = X(1) = 0, \; x \in (0,1), \tag{17}$$

and $E_{\gamma,\gamma}(z)$ is the two-parameter Mittag-Leffler function [16].

The problem (17) was studied in particular in works [17–21], which showed that the eigenfunctions $X_n(x)$ exist only for eigenvalues λ_n, which are zeros of the two-parameter Mittag-Leffler function $E_{\gamma,\gamma}(\lambda)$. These eigenfunctions take the form $\{X_n(x)\}_{n=1}^{\infty} = \{x^{\gamma-1}E_{\gamma,\gamma}(\lambda_n(1-x)^{\gamma})\}_{n=1}^{\infty}$. This system forms a non-orthogonal basis in $L^2(0,1)$ [18–21].

The system of eigenfunctions for the problem adjoint to (17) is defined as [18, 19, 21]:

$$\{Y_n(x)\}_{n=1}^{\infty} = \{(1-x)^{\gamma-1}E_{\gamma,\gamma}(\lambda_n(1-x)^{\gamma})\}_{n=1}^{\infty}.$$

The systems $\{X_n(x)\}_{n=1}^{\infty}$ and $\{Y_n(x)\}_{n=1}^{\infty}$ form a biorthogonal system of functions [18, 19]. By expanding the initial condition function $\psi(x)$ and source function $f(x)$ in a series with respect to the eigenfunctions of (17) as follows:

$$\psi(x) = \sum_{n=1}^{\infty} \psi_n X_n(x), \quad f(x) = \sum_{n=1}^{\infty} f_n X_n(x), \tag{18}$$
$$\left(\psi_n = (\psi(x), Y_n(x))_{L^2(0,1)},\ f_n = (f(x), Y_n(x))_{L^2(0,1)}\right),$$

we obtain from (10)–(12) the following sequence of Cauchy problems to determine the unknown functions $\upsilon_n(t) (n \in \mathbb{N})$ in the series expansion (16):

$$^{C}D_t^{\alpha_1}\upsilon_n(t) - \kappa_1\lambda_n\upsilon_n(t) = f_n \quad (n \in \mathbb{N}), \tag{19}$$
$$\upsilon_n(0) = \psi_n \quad (n \in \mathbb{N}), \tag{20}$$

where

$$\psi_n = \varphi_n(1 - \tau_r\kappa_2\lambda_n), \quad \varphi_n = (\varphi(x), Y_n(x))_{L^2(0,1)} \quad (n \in \mathbb{N}). \tag{21}$$

The solution to problems (19), (20) based on [7] can be written as

$$\nu_n(t) = \left(\psi_n + \frac{f_n}{\kappa_1\lambda_n}\right) E_{\alpha_1}(\kappa_1\lambda_n t^{\alpha_1}) - \frac{f_n}{\kappa_1\lambda_n}\,(n \in \mathbb{N}), \tag{22}$$

where $f_n (n \in \mathbb{N})$ is defined in accordance with (18), and $E_\alpha(z)$ is the one-parameter Mittag-Leffler function [16].

Similarly, by seeking the solution $u(x,t)$ of problem (13)–(15) as an expansion in terms of the eigenfunctions $X_n(x)$ of problem (17), specifically,

$$u(x,t) = \sum\nolimits_{n=1}^{\infty} u_n(t)\, X_n(x) \quad (x,t) \in \Omega, \tag{23}$$

we derive, based on (13)–(15), the following sequence of Cauchy problems:

$$^{C}D_{0t}^{\alpha_2} u_n(t) + \omega_n u_n(t) = \frac{1}{\tau_r}\upsilon_n(t)\ (n \in \mathbb{N}), \tag{24}$$
$$u_n(0) = \varphi_n\,(n \in \mathbb{N}), \tag{25}$$

where $\omega_n = \frac{1}{\tau_r} - \kappa_2\lambda_n\,(n \in \mathbb{N})$, φ_n is given by (21), and the functions $\upsilon_n(t)$ are defined according to (22).

The solution to problems (24), (25), based on [7], is written as

$$u_n(t) = \varphi_n E_{\alpha_2}\left(-\omega_n t^{\alpha_2}\right)$$
$$+\frac{1}{\tau_r}\int_0^t (t-\tau)^{\alpha_2-1} E_{\alpha_2,\alpha_2}\left(-\omega_n (t-\tau)^{\alpha_2}\right)\nu_n(\tau)\,d\tau$$
$$= \left(\varphi_n + \frac{f_n}{\kappa_1\tau_r\omega_n\lambda_n}\right) E_{\alpha_2}\left(-\omega_n t^{\alpha_2}\right)$$
$$-\frac{f_n}{\kappa_1\tau_r\omega_n\lambda_n} + \frac{1}{\tau_r}\left(\psi_n + \frac{f_n}{\kappa_1\lambda_n}\right) J_n(t)\,, \tag{26}$$

where the following notation is introduced:

$$J_n(t) = \int_0^t (t-\tau)^{\alpha_2-1} E_{\alpha_2,\alpha_2}\left(-\omega_n (t-\tau)^{\alpha_2}\right) E_{\alpha_1}\left(\kappa_1\lambda_n\tau^{\alpha_1}\right) d\tau \quad (n \in \mathbb{N}). \tag{27}$$

Thus, the formal solution to the considered problem (6)–(8) is defined by relations (23), (26), (27). It can be shown that under certain additional conditions, this solution is regular. To this end, assume that the initial condition function $\varphi(x)$ and the source function $f(x)$ satisfy the following constraints: $\varphi(x), f(x) \in C^2[0,1]$, $\varphi(0) = \varphi(1) = \varphi'(0) = 0$, $f(0) = f(1) = f'(0) = 0$. Then, as is known [22, 23],

$$|\varphi_n| \le \frac{C_1}{|\lambda_n|^2}, \quad |f_n| \le \frac{C_2}{|\lambda_n|^2} \quad (C_1, C_2 > 0, n \in \mathbb{N})\,. \tag{28}$$

Considering the known property of the two-parameter Mittag-Leffler function [16]:

$$|E_{\eta,\xi}(z)| \le \frac{M_1}{1+|z|} \quad (z \in \mathbb{C})\,, \tag{29}$$

$$\left(0 < \eta < 2, \mu \le |\arg(z)| \le \pi,\ \mu \in \left(\frac{\pi\eta}{2},\ \min\{\pi, \pi\eta\}\right),\ \xi \in \mathbb{R},\ M_1 = \text{const} > 0\right),$$

and taking into account (28), (29), along with the following inequality for the eigenfunctions $X_n(x)$ [18, 20, 21]:

$$|X_n(x)| \le \frac{C}{|\lambda_n|\,x} \quad (C > 0, x > 0, n \in \mathbb{N})\,,$$

we obtain the following estimate for the series terms in (23):

$$|u_n(t)\,X_n(x)| \le \frac{L}{|\lambda_n|^3\,x} \quad (L = \text{const} > 0, t > 0, x > 0, n \in \mathbb{N})\,. \tag{30}$$

Since the eigenvalues λ_n of problem (17) with $\mathrm{Im}(\lambda_n) > 0$ satisfy the properties [18–22]:

a) $|\lambda_k| < |\lambda_{k+1}|$ for $k \ge 1$,
b) for sufficiently large n and $\arg(\lambda_n) > \frac{\alpha\pi}{2}$, we have

$$\left|e^{\lambda_n t}\right| < 1, \quad |\lambda_n| \sim \mathrm{O}(n^\gamma) \quad (1 \le \gamma \le 2),$$

then, considering (30), it follows that the majorant for the series (23) is the convergent generalized harmonic series

$$\sum_{n=1}^{\infty} n^{-3\sigma} \quad (1 < \sigma < 2)\,.$$

By the Weierstrass test, the series (23) is uniformly convergent on the set $\overline{\Omega}$, and its sum is a continuous function on this set: $u\,(x,t) \in C\left(\overline{\Omega}\right)$. The convergence of all other series with corresponding derivatives of partial solutions is proven similarly.

Thus, there exists a regular solution to the considered problem, and (taking into account the completeness of the system of functions $\{Y_n(x)\}$, $n \in \mathbb{N}$) it is easy to establish its uniqueness.

4. Special cases

Let us briefly discuss some special cases of the considered problem.

In the case $\alpha_1 = \alpha_2 = \alpha$, the obtained solution to problem (6)–(8) simplifies significantly and takes the form (23), where $u_n(t)$ is defined by the following relations that do not involve quadratures:

$$u_n\,(t) = \left(\varphi_n + \frac{f_n}{\kappa_1 \tau_r \omega_n \lambda_n}\right) E_\alpha\left(-\omega_n t^\alpha\right) - \frac{f_n}{\kappa_1 \tau_r \omega_n \lambda_n}$$

$$+ \frac{1}{\tau_r(\omega_n + \kappa_1\lambda_n)} \left(\psi_n + \frac{f_n}{\kappa_1 \lambda_n}\right) \left[E_\alpha(\kappa_1 \lambda_n t^\alpha) - E_\alpha(-\omega_n t^\alpha)\right] \quad (n \in \mathbb{N}). \quad (31)$$

Moreover, under the condition $\kappa_1 = \kappa_2 = \kappa$, in the case of nonlocality only with respect to the time variable (corresponding to $\gamma \to 2$) and in the absence of sources ($f = 0$), we obtain from (31) the following relations [9]:

$$u_n\,(t) = \varphi_n \left(\left(1 - \tau_r \kappa \lambda_n\right) E_\alpha\left(\kappa \lambda_n t^\alpha\right)\right.$$

$$\left. + \tau_r \kappa \lambda_n E_\alpha\left(-\left(\frac{1}{\tau_r} - \kappa\lambda_n\right) t^\alpha\right)\right) \quad (n \in \mathbb{N}), \quad (32)$$

where $\lambda_n = -(n\pi)^2$. In this case, the solution to the problem is defined according to (23), (32).

If, in particular, we set $\alpha \to 1$ and $\gamma \to 2$ in relations (32), we obtain the solution to the corresponding boundary-value problem (6)–(8) within the framework of the standard (differential) biparabolic model in the form [5, 6]:

$$u\,(x,t) = 2 \sum_{n=1}^{\infty} \varphi_n \left(1 + \tau_r \kappa \tilde{\lambda}_n^2 \left(1 - e^{-\frac{t}{\tau_r}}\right)\right) e^{-\kappa \tilde{\lambda}_n^2 t} \sin\left(\tilde{\lambda}_n x\right),$$

where $\varphi_n = \int_0^1 \varphi\,(x) \sin\left(\tilde{\lambda}_n x\right) dx$, and $\tilde{\lambda}_n = n\pi$ $(n \in \mathbb{N})$.

In the special case when $\varphi(x) = u_0 = \text{const}$, this solution takes the form [6]:

$$u\,(x,t) = u_c\,(x,t) + 4\tau_r\kappa \left(1 - e^{-\frac{t}{\tau_r}}\right) \sum_{n=1,3,5,\ldots}^{\infty} \tilde{\lambda}_n^2 e^{-\kappa\tilde{\lambda}_n^2 t} \sin\left(\tilde{\lambda}_n x\right),$$

where $u_c(x,t)$ is the following solution to the corresponding boundary-value problem for the classical linear diffusion equation:

$$u_c(x,t) = 4\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{\tilde{\lambda}_n}e^{-\kappa\tilde{\lambda}_n t}\sin\left(\tilde{\lambda}_n x\right).$$

It is worth noting that one of the well-known interpretations [5, 6] of the biparabolic differential equation defines it as a model equation in the theory of filtration consolidation of water-saturated soil environments (where $u(x,t)$ represents the function of excess pressures).

Numerical experiments have shown [5,6,23] that the biparabolic mathematical model of the filtration-consolidation process is, in some sense, intermediate between two well-known models: the classical model by K. Terzaghi [6] and the model considering the creep of the soil skeleton by V.A. Florin [24,25]. Specifically, the biparabolic model predicts lower pressure values than Florin's model but higher values than the classical mathematical model by Terzaghi.

5. Conclusion

A nonlocal fractional-differential mathematical model for transport processes in both time and space has been introduced, generalizing the well-known biparabolic model [1] for describing heat and diffusion processes with finite propagation speed. The formulation of the problem has been performed, and a closed-form solution to the transient fractional-differential boundary-value problem for a finite interval along the spatial variable has been obtained. The conditions for the existence of a regular solution to the problem have been presented, and some specific cases have been discussed.

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