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Резонансні усталені хлюпання у вертикальних баках, що рухаються періодично у тривимірному просторі

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Resonant steady-state sloshing in upright tanks performing a three-dimensional periodic motion

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*Розвиваються аналітичніпідходи до гідростатичнихкапілярних (меніск) проблем у нескінченому горизонтальному каналі та осесиметричномуконтейнері. Для цих геометричних випадків знаходження капілярнихменісків зводиться до задачіізневідомою границею ізспеціальною системою звичайних диференціальнихрівнянь. Їх розв'язки описують капілярнікриві, які виникають в результаті перетину капілярнихменісків та чи перечногоперерізу, чи меридіональної площини (в залежностівід форми контейнера). Подальшідослідженнякапілярних хвиль вимагають знати аналітичнінаближенння цих капілярних кривих у метриціCn, n ≥ 3. Метою може бути побудова аналітичних наближених розв'язківвідповідних систем звичайних диференціальнихрівнянь. Ця стаття присвячена дослідженню границь застосувань аналітичних наближень Тейлора та Паде, які було запропоновано для цього класу капілярних задач у 1984 роціБарняком та Тимохою.**Ключові слова: хлюпання рідини, демпфування, усталені хвилі.*

Analytical approaches to hydrostatic capillary (meniscus) problem in infinite horizontal channel and axisymmetric container are developed. For these geometric cases, finding the capillary menisci reduces to freeboundary problems for special systems of ordinary differential equations. Their solutions describe capillary curves, which appear as intersections of the capillary menisci and (depending on the container type) either crosssection or meridional plane. Further studies on capillary waves require to know analytical approximations of these capillary curves in the Cn, n ≥ 3 metrics. An objective may consists of constructing analytical approximate solutions of the corresponding systems of ordinary differential equations. The present paper focuses on limits of applicability of the Taylorpolynomial and Pad´e approximations, which were proposed for this class of capillary problems in 1984 by Barnyak&Timokha.
Keywords: sloshing, damping, steady-state waves.

Статтю представив д.т.н., проф. Лимарченко О.С.

The present paper considers two capillary surface problems for partlyfilled infinite channels and axisymmetric reservoirs. Finding the capillary surface (meniscus) reduces to boundary value problems for systems of ordinary diffrential equations. The ODEs describe capillary lines, which are an intersection of either cross or meridional plane, respectively. We show that the capillary lines are solutions of one-parameter families of the Cauchy problem for the ODEs. Following Barnyak & Timokha [1], we construct the Taylor and Pad´e approximations of these solutions. Their radii of convergence are estimated. Whereas the capillary lines for channels may be effectively approximated by using both Taylor and Pad´e approximations, the approximations are less accurate for axisymmetric reservoirs.

Capillary surface in infinite channels

Consider the Oz-symmetric and, generally speaking, closed infinite channel (horizontal tube) whose rigid walls are defined by the function $y=\pm f(z)$ as in fig. 1. The tube is partly filled with a liquid whose hydrostatic shape, which is affected by gravity force (parallel to Oz) and surface tension, is bounded with the capillary surface and the wetted tank surface $S_0 = x, y, z: -\infty < x < \infty,\ y, z \in O$. In the cross-section, Σ_0 is fully determined by capillary curve l_0 but S_0 is defined by l_1

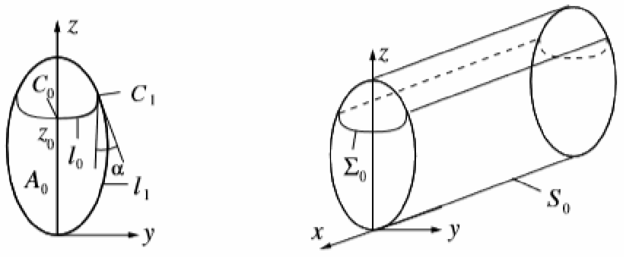

Figure 1. Capillary surface Σ_0 in an infinite closed channel (horizontal tube). Three-dimensional and cross-section views. Capillary curve l_0 is resulted from intersection of Σ_0 and the *Oyz* plane. Curve l_1 implies the intersection with the wetted tank surface. The present study assumes that the tank surface is defined as the single-valued presentation $y = \pm f(z)$ The gravity acceleration is parallel to the Oz axis.

Mathematical formulation

The problem on the capillary curve l_0 is furthermore considered in nondimensional statement, which appears after introducing the characteristic length r_0 of the two-dimensional cross-sectional area A_0 where $s = 0$ implies the starting point, $C_0 = (0, z_0)$, on the Oz-axis, but s_1 is the actual length of l_0 and implies the contact point

$c_1 = (y(s_1) = f(z(s_1)), z(s_1))$ of l_0 and l_1

$$y'' - -z'(B_0\, z + c),\ z'' - y'(B_0\, z + c), \qquad (1)$$

where B_0 is the Bond number ($B_0 = \dfrac{\rho g r_0^2}{T_s}$).

System (1) should be equipped with the initial conditions $y(0) = z'(0) = 0$, $y'(0) = 1$ as well as one can suggest $z(0) = z_0 > 0$.

The Taylor approximation

M. Barnyak was most probably the first one who proposed to adopt the Taylor approximation for solving the capillary meniscus problem. Postulating this approximation

$$Z = \sum_{k=1}^{N} b_k S^{2k-2},\ \ Y = \sum_{k=1}^{N} a_k S^{2k-1}, N \to \infty \qquad (2)$$

$a_1 = 1, b_1 = \epsilon,$

$$b_{j+1} = \frac{b}{2j(2j-1)} \sum_{k=1}^{j} b_k a_{j-k+1}(2(j-k)+1), \qquad (3)$$

$$a_{j+1} = \frac{b}{j(2j+1)} \sum_{k=1}^{j} b_k a_{j-k+2}(j-k+1), j \geq 1.$$

When N is finite, the Taylor approximation (2) is applicable for $|S| \leq S_T(N, \varepsilon, \epsilon)$ $|S| \leq S_T(N, \varepsilon, \epsilon) < R_T$, where ϵ is a given accuracy. An estimate of the radius $S_T(N, \varepsilon, \epsilon)$ follows from the condition $S_T = \max S_0$ such that

$$\left| Z'^2(S) + Y'^2(S) - 1 \right| \leq \epsilon,\ 0 < S < S_0(N, \xi, \varepsilon)$$

This means that using the Taylor solution (2) deduces the Pag´e approximant

$$Y = S\frac{1 + \Sigma_{i=1}^{L} p_i^{(Y)} S^2}{1 + \Sigma_{i=1}^{M} q_i^{(Y)} S^2};$$

$$Y = \frac{\xi + \Sigma_{i=1}^{L} p_i^{(Z)} S^2}{1 + \Sigma_{i=1}^{M} q_i^{(Z)} S^2};\ \ M + L = N \qquad (4)$$

For given N, M (the number of simple poles accounted for ξ and the accuracy ε, one can define radius of convergence of (4) as $S_p = \max S$. Because the Pad´e approximant should account for the nearest simple poles, we expect to improve accuracy and increase radius of convergence with respect to the Taylor polynomials, i.e. $S_T = S_p$. Axisymmetric capillary surface.

Mathematical formulations

Cavities of revolution may provide either axisymmetric or exotic (nonsymmetric) capillary surface. The exotic surface was theoretically predicted and, later on, validated in the Space experiments [2]. In the present paper, we exclusively focus on studying the axisymmetric capillary meniscus whose mathematical formulation reduces, as in the previous section, to the system of ODEs

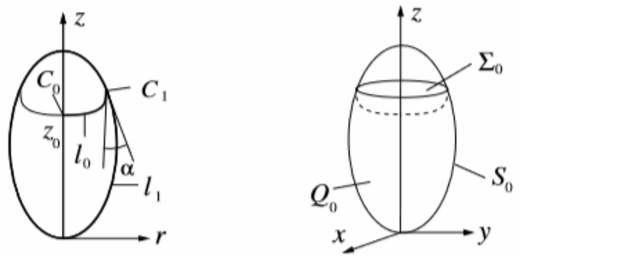

Figure 1.The same as in fig.1 but for containers of revolution and threedimensional axisymmetric capillary surfaces.

$$r'' - -z'\left(B_0\, z - \frac{z'}{r} + c\right),$$

$$z'' = r'\left(B_0\, z - \frac{z'}{r} + c\right), \qquad (5)$$

the Taylor and Pad´e approximations of L_0^*.

The solution R, Z can also be suggested as analytical functions of $S \in C$ at $S=0$. In the contrast to the previous section, we cannot prove that R and Z are meromorphic functions but only show, following [1], that R and Z may have the simple pole singularity. This means that using the Taylor and Pad´e approximations of the Cauchy problem has no rigorous mathematical argumentation but could be considered as a numerical experiment. Adopting the Taylor polynomials

$$R = \sum_{k=1}^{N} a_k S^{2k-1}, \qquad Z = \sum_{k=1}^{N} b_k S^{2k-2}, N \to \infty \quad , \qquad (6)$$

leads to the recurrence formulas

$$a_1 = 1, \qquad b_1 = \varepsilon,$$

$$b_{j+1} = \frac{1}{(2j)^2\left[b\ \sum_{m=1}^{j} a_{j-m+1}[2(j-m)+1]\sum_{l=1}^{m} a_l b_{m-l+1} - 4j\sum_{m=1}^{j-1}[m a_{j-m+l} b_{m+1}\right],]} \qquad (7)$$

$$a_{j-1} = -\frac{1}{j(2j+1)\left[\sum_{m=1}^{j-1} a_{m+1}a_{j-m+1}m(2m+1) + \sum_{m=1}^{j} b_{j-m+2}(j-m+1)\left(b\sum_{i=1}^{m}[b_l a_{m-l+1} - 2mb_{m+1})\right], j > 1.]}$$

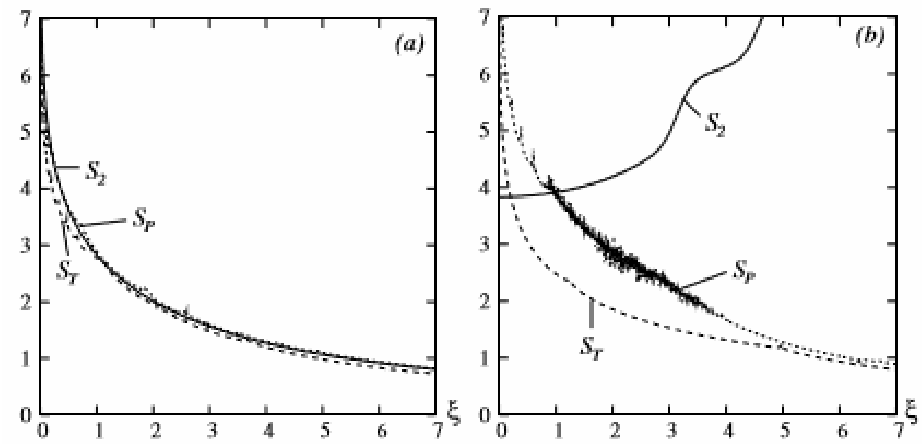

Figure 2. $T = 40, c = 10^7$ and M = 6 . The case (*a*) $B_0 > 0$ and (*b*) $B_0 < 0$.

The constructed approximations provide an accurate approximation when $S_T \geq S_2$ (for the Taylor polynomials) and/or $S_p \geq S_2$ (for the Pad´e approximants).

Fig. 2 represents results of numerical experiments on the radii of convergence $S_T(\xi), S_P(\xi)$ as well as $S_2(\xi)$ as functions of ξ for $N = 40$, $M = 6$ and $\in = 10^{-7}$. When $B_0 > 0$ (the panel a), the Pad´e approximant slightly improves the accuracy so that $S_P \geq S_T$. However, this improvement is not as strong as for channels. Most likely, there are either other types of singularities in the complex plane (not only simple poles) or many of the simple poles are located relatively close to *S=0*. Practically, usage of the Pad´e approximant guarantees rather accurate solution for the positive Bond number except, perhaps, for small ξ . Numerical estimates of $S_T(\xi), S_P(\xi)$ and S_2 (ξ) for negative Bond numbers ($B_0 < 0$) are presented in fig. 2 (*b*). Here, we see that switching from Taylor to Pad´e approximation may significantly increase the radius of convergence as $\varepsilon \leq 5$. However, this does not help. Neither Taylor nor Pad´e approximations are practically applicable for solving the capillary problem with the negative Bond number. The reason is that $S_2 \to \infty$ with increasing ξ, namely, the solution becomes non-periodic in the limiting case.

Conclusion

The present paper tests Taylor and Pad´e approximations of the capillary meniscus problem in infinite channels and axisymmetric containers. This continues the study by Barnyak&Timokha [1] who suggested that rational approximation may significantly improve the numerical accuracy. We showed that, for the positive Bond number ($B_0 > 0$), using the Pad´e approximants may indeed provide an accurate solution of the capillary meniscus problem, except, perhaps, for large Bond numbers, when the capillary curve rapidly changes its behaviour at the contact line. In the contrast, neither Taylor polynomials nor rational approximations can guarantee getting an accurate analytical approximate solution for negative values of B_0. Our approach reduces the problem to an one-parameter set of the Cauchy problems and, as long as $B_0 < 0$, there are critical values of this parameter when one must find the solution on large interval that is impossible by using our two analytical approximate methods.

Список використаних джерел

1. Barnyak M.Ya. On finding approximate analytical solutions of planar and axisymmetric single-connected capillary surfaces in the form of rational functions / M. Ya. Barnyak, A. N. Timokha // In Book: "Numerical-Analytical Methods for Investigation of Dynamics and Stability of Multidimensional Systems" / Lukovsky, I. (Ed.).– 1984.– P. 38–47.– Institute of Mathematics. Academy of Sciences of the UkrSSR, (inRussian).
2. Concus P., Finn, R., Weislogel M. Capillary surfaces in an exotic container: results from Space experiments / P. Concus, R. Finn, M. Weislogel // Journal of Fluid Mechanics.– 1999.– 394.– P. 119-135.

References

1. BARNYAK, M. Ya., TIMOKHA, A. N. (1984) On finding approximate analytical solutions of planar and axisymmetric single-connected capillary surfaces in the form of rational functions. In Book: *"Numerical-Analytical Methods for Investigation of Dynamics and Stability of Multidimensional Systems"*. Kiev: Institute of Mathematics. Academy of Sciences of the UkrSSR.
2. CONCUS, P., FINN, R., WEISLOGEL, M. (1999) Capillary surfaces in an exotic container: results from Space experiments. *Journal of Fluid Mechanics*. 394. pp.119-135.

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