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SPECIAL SPHERICALLY SYMMETRIC SOLUTIONS IN $f(R)$ GRAVITY

Background. ***$f(R)$ gravity is a natural generalization of General Relativity, where the Lagrangian in the Einstein-Hilbert action is replaced by a more complicated function. This theory is extensively used in considerations of the early inflation, late inflation and in the dark matter models. Variation of the action in this theory leads to the fourth order equations with respect to the physical metric (Jordan frame). We study the conformally transformed metric (Einstein frame) in a static spherically symmetric (SSS) case.***

Methods. ***We use the well-known transformation, which allows to reduce the problem to usual Einstein equations with an additional scalar field dubbed scalaron. The focus is on the SSS solutions in case of large dimensionless parameter $M\mu\to\infty$, , where M is the configuration mass and μ is the scalaron mass in geometrized units.***

Results. ***A class of scalaron potentials is identified that covers well-known f(R) models and permits a straightforward analytical description of SSS objects in the region where the solutions are significantly different from the Schwarzschild case for $M\mu\gg 1$. A representation of the structure equations is found that enables numerical integration. The results may be useful for testing numerical algorithms in case of sufficiently large $M\mu$.***

Conclusions. ***Approximate analytic relations are found for the SSS solutions of the $f(R)$ gravity in the Einstein frame on a finite interval of radial variable in the limit $M\mu\to\infty$ under special assumption on the initial data. The results show universal properties of solutions for a wide class of scalaron potentials.***

Keywords: ***relativistic objects, naked singularities, modified gravity.***

Background

Among various modifications of the General Relativity caused mainly by cosmological problems, $f(R)$ gravity is probably the most direct and natural generalization (see for reviews Sotiriou & Faraoni, 2010; De Felice, & Tsujikawa, 2010). This theory has been extensively used in considerations of the early inflation, late inflation and in the dark matter models. Natural questions arise about the possible role of the $f(R)$ gravity in relativistic astrophysical objects. In this case we deal with the total mass M of the astrophysical configuration that can be combined with μ to yield a dimensionless (in geometrized units) parameter $M\mu$. In different $f(R)$ theories, the scalaron masses vary, e.g., from $\sim 10^{13}$ Gev to ~ 4 meV. Even this, comparatively small, value of μ corresponds to the length scale $l_\mu=\mu^{-1}\sim 5\cdot 10^{-3}$ cm yielding $M\mu\gg 10^{10}$ for typical masses of astrophysical objects in AGNs and create difficulties with a standard software arising in numerical modeling.

In this paper we consider properties of static spherically symmetric (SSS) solutions in the limit of high $M\mu$ for a class of astrophysically interesting scalaron potentials arising in the Einstein frame of the $f(R)$ gravity. First of all, we mean the well-known potentials with an extended plateau, like the Starobinsky model (1980), and/or table-top and hill-top potentials discussed by Shtanov, Sahni, & Mishra (2023).

Methods

The $f(R)$ gravity deals with the fourth-order system for physical metric $g_{\mu\nu}$ (Jordan frame). Using a well-known transformation

$$\hat{g}_{\mu\nu}=e^{2\xi}g_{\mu\nu},$$

it is possible to find a scalar field ξ dubbed scalaron[1], such that the dynamics and structure of the physical system iss described by usual Einstein equations for $\hat{g}_{\mu\nu}$ with an additional equation for ξ (Einstein frame, see, e.g., Sotiriou, & Faraoni, 2010; De Felice, & Tsujikawa, 2010). The right-hand side of these Einstein equations and the equation for the scalaron contains a self-interaction potential $W(\xi)$ defined parametrically

$$e^{2\xi}=f'(u)\,,\quad W(\xi)=\frac{1}{2}e^{-4\xi}[f(u)-uf'(u)]\,.$$

For a SSS space-time we use the Schwarzschild coordinate system (curvature coordinates) with

$$d\hat{s}^2\equiv\hat{g}_{\mu\nu}dx^\mu dx^\nu=e^{\alpha}dt^2-e^{\beta}dr^2-r^2dO^2, \tag{1}$$

where $r>0$, $\alpha\equiv\alpha(r)$, $\beta\equiv\beta(r)$; $dO^2=d\theta^2+\sin^2\theta d\varphi^2$ stands for the metric element on the unit sphere.

In absence of non-gravitational fields, the nontrivial equations for the static metric (1) in the Einstein frame are (see, e.g., Zhdanov, Stashko, & Shtanov, 2024).

$$\frac{d}{dr}(\alpha+\beta)=6r\left(\frac{d\xi}{dr}\right)^2, \tag{2}$$

[1] In fact, the canonical scalaron is obtained by rescaling of ξ; however, we prefer to use the dimensionless ξ here. This explains the multipliers in front of $w(\xi)$ and $w'(\xi)$ below.

$$\frac{d}{dr}(\alpha-\beta)=-\frac{2}{r}+\frac{2e^{\beta}}{r}\left[1-r^2W(\xi)\right]. \quad (3)$$

Where $\xi\equiv\xi(r)$. Equations (2), (3) must be supplemented by an equation for the scalaron :

$$\frac{d}{dr}\left[r^2e^{\frac{\alpha-\beta}{2}}\frac{d\xi}{dr}\right]=\frac{r^2}{6}e^{\frac{\alpha-\beta}{2}}W'(\xi). \quad (4)$$

Scalaron potentials and initial conditions. It was repeatedly pointed out (see, e.g., Shtanov, Sahni, & Mishra, 2023) that the scalaron potentials, which have a plateau-like form (as in the case of the Starobinsky model) and/or the table-top (flattened hill-top) potentials are preferable for physical reasons. In what follows we restrict ourselves to positive ξ; however, our method can easily be extended to negative ξ. We consider the scalaron potentials such that

$$W(\xi)=\mu^2w(\xi), \quad (5)$$

where

$$w(\xi)=3\xi^2\big(1+O(\xi)\big),\ w'(\xi)=6\xi\big(1+O(\xi)\big) \quad (6)$$

for $|\xi| << 1$. We also assume

$$|w(\xi)|\le w_0,\ |w'(\xi)|\le w_1, \quad (7)$$

and

$$\gamma(\xi)=1-\frac{w(\xi)}{3\xi^2}>0,\quad \xi>0, \quad (8)$$

where w_0, w_1 and $\gamma(\xi)$ do not depend on μ. These properties are fulfilled in case of the quadratic $f(R)$ gravity, however, the scope of application of these conditions is much wider.

Approximate solutions for $M\mu\gg 1$. We consider some "scalarization region" $(0,r_0)$, where significant deviations from the Schwarzschild solution are expected, assuming $r_0\sim(100\div1000)r_g$. However, at r_0 we assume that the metric takes the Schwarzschild values

$$\exp[\alpha(r_0)]=1-r_g/r_0,\quad \exp[\beta(r_0)]=\left(1-r_g/r_0\right)^{-1},$$

and $\xi(r_0)=\xi_0$, $|\xi_0|\ll1$. It is essential that we assume that r_0,ξ_0 are fixed when $M\mu\to\infty$. In fact, these conditions can be modified to arbitrary but moderate values of $\alpha(r_0)$ and $\beta(r_0)$.

It may be difficult to perform direct numerical integration of the basic equations in the form (2), (3), (4) for $M\mu\gg1$ with a standard software because of exponentially large numbers involved. In this view, we introduce new independent variable p by means of the relations

$$r\mu=X_0+\frac{p}{X_0}\equiv X_0U(p),\ U(p)=1+\frac{p}{X_0^2},$$

where $X_0=\mu r_0\gg1$ and the interval $r\in(0,r_0]$ corresponds to negative $p\in(-X_0^2,0]$. We shall move from $p=0$ to negative values.

Denote

$$\chi=\frac{\alpha+\beta}{2},\quad Y=U\exp\left(\frac{\alpha-\beta}{2}\right). \quad (9)$$

Equation (2) yields

$$\frac{d\chi}{dp}=3U\left(\frac{d\xi}{dp}\right)^2, \quad (10)$$

Equation (3) multiplied by $\exp[(\alpha-\beta)/2]$ can be transformed to

$$\frac{dY}{dp}=e^{\chi}\left[\frac{1}{X_0^2}-U^2w(\xi)\right]. \quad (11)$$

From equation (4) we get

$$\frac{d}{dp}\left[UY\frac{d\xi}{dp}\right]=\frac{U^2}{6X_0^2}e^{\chi}w'(\xi). \quad (12)$$

By denoting

$$Z=-X_0UY\frac{d\xi}{dp}, \quad (13)$$

we obtain two first-order equations

$$\frac{d\xi}{dp}=-\frac{Z}{X_0UY}, \quad (14)$$

$$\frac{dZ}{dp}=-\frac{U^2}{6X_0}e^{\chi}w'(\xi). \quad (15)$$

Substitution of (14) into (2) yields

$$\frac{d\chi}{dp}=\frac{3Z^2}{UY^2}. \quad (16)$$

Now we have a closed system of four equations (11), (14), (15), (16) in a normal form, which is ready for the backwards numerical integration, starting from $p=0$. Correspondingly, we set the initial data at $p=0$, which corresponds to $r=r_0>r_g$:

$$\xi(0)=\xi_0,\ Y_0\equiv Y(0)=1-\frac{r_g}{r_0},\ \chi(0)=0,\quad Z_0=-X_0Y_0\left(\frac{d\xi}{dp}\right)_0=\xi_0\sqrt{1-\frac{r_g}{r_0}}.$$

Numerical investigation show that there is a transition region T close r_0 to with a jump-like change of $Y(r)$, but where

$$\xi(p)\approx\xi_0,\quad Z(p)\approx Z_0,\quad U(p)\approx1.$$

We will see that to satisfy these relations it is sufficient to restrict T as follows:

$$T=\{p\in(p_1,0)\},\ p_1\simeq-\sqrt{X_0}.$$

For sufficiently large $M\mu$ the X_0^{-2} term in the right-hand side of (11) can be neglected and we have

$$\frac{d\chi}{dp}=\frac{3Z^2}{Y^2},\quad \frac{dY}{dp}=-e^{\chi}w(\xi).$$

Combining these equations, we get

$$\frac{1}{Y^2}\frac{dY}{dp}=-\frac{w(\xi_0)}{3Z_0^2}e^{\chi}\frac{d\chi}{dp},$$

whence

$$\frac{1}{Y(p)}=\frac{1}{Y_0}+\frac{w(\xi_0)}{3Z_0^2}\left[e^{\chi(p)}-1\right]. \tag{17}$$

From initial conditions $Y_0/Z_0^2=1/\xi_0^2$ from (8) follows $1-Y_0w(\xi_0)/(3Z_0^2)=\gamma(\xi_0)>0$. Then

$$\frac{Y_0}{Y(p)}\approx\frac{1}{Y_*}+e^{\chi(p)},\quad Y_*=\frac{1}{\gamma(\xi_0)}. \tag{18}$$

Using (18) and $Z\approx Z_0$ from initial conditions, equation (16) can be integrated to obtain

$$\ln\left(\frac{\gamma(\xi_0)+e^{\chi}}{\gamma(\xi_0)+1}\right)-\chi-\frac{\gamma(\xi_0)(1-e^{\chi})}{[\gamma(\xi_0)+e^{\chi}][\gamma(\xi_0)+1]}=-\frac{3\gamma^2(\xi_0)}{Y_0}\xi_0^2X_0^2\ln\left(\frac{r}{r_0}\right)$$

where we take account of $\chi(0)=0$. Then for $-\sqrt{X_0}\le p\le 0$ we have inequality

$$\chi(p)\lesssim-3\frac{\gamma^2(\xi_0)}{Y_0}\xi_0^2\,|p|. \tag{19}$$

Now we can justify the approximation made for the transition region *T*. Using (19) and (7), we get from (14)

$$|Z(z)-Z_0|\le\frac{C_1(\xi_0)}{X_0},\quad |p|\le\sqrt{X_0},$$

and using (18), from (14)

$$|\xi(p)-\xi_0|\approx\frac{Z_0}{X_0}(\gamma(\xi_0)+e^{\chi})\le\frac{C_2(\xi_0)}{\sqrt{X_0}},$$

where $-\sqrt{X_0}\le p\le 0$ and constants $C_1(\xi_0),C_2(\xi_0)$ do not depend on X_0. These are rough estimates, but they are sufficient for further consideration in case of $X_0\to\infty$ and fixed ξ_0 to justify approximations in the region *T*.

Owing to (19), for $p_1=-\sqrt{X_0}$ and $M\mu$ large enough

$$\exp[\chi(p_1)]<\exp\left(-3\gamma^2(\xi_0)Y_0\xi_0^2\sqrt{X_0}\right)\ll\gamma(\xi_0);$$

this means that $Y(p)$ practically reaches maximal value Y_* on a left end of interval *T*.

Now we can consider $p<p_1$ and we can deal with the exact equations (11) and (15). In consequence of (16) function $\chi(p)$ is monotonous, therefore $\exp\chi(p)<\exp\chi(p_1)$ is bounded by very small constant

$$e^{\chi(p)}<\exp\left(-3\frac{\gamma^2(\xi_0)}{Y_0}\xi_0^2\sqrt{X_0}\right).$$

This value enters into right-hand sides of (11) and (15) as an exponentially small factor (for $X_0\to\infty$). Though we have a very large interval of $p\in(-X_0^2,p_1)$, this factor suppresses these right-hand sides and leads to practically constant values of

$$Y(p)=Y_f\equiv\lim_{r\to 0}Y(r),\quad Z(p)=Z_f\equiv\lim_{r\to 0}Z(r), \tag{20}$$

and

$$Z_f\approx Z_0,\quad Y_f/Y_0\approx Y_*, \tag{21}$$

for $M\mu\gg 1$. This means

$$r\exp\left(\frac{\alpha-\beta}{2}\right)\approx\text{const},\quad r\frac{d\xi}{dr}\approx\text{const}$$

with very good accuracy for large $M\mu$. Owing to the above estimates we get from (14) in this region (including $r\to 0$)

$$\xi(r)=-\frac{\xi_0}{\sqrt{Y_0}}\gamma(\xi_0)X_0\ln\left(\frac{r}{r_0}\right)+\xi_0, \tag{22}$$

and from (16)

$$\chi(r)=3\frac{\xi_0^2}{Y_0}\gamma^2(\xi_0)X_0^2\ln\left(\frac{r}{r_0}\right). \tag{23}$$

The metric coefficients in the Einstein frame for $p<p_1$ are:

$$e^{\alpha}=\frac{Y_0}{\gamma(\xi_0)}\left(\frac{r}{r_0}\right)^{H-1},\quad e^{\beta}=\frac{\gamma(\xi_0)}{Y}\left(\frac{r}{r_0}\right)^{H+1}, \tag{24}$$

where $H\equiv 3\xi_0^2\gamma^2(\xi_0)X_0^2/Y_0\gg 1$ for $M\mu\gg 1$ and fixed ξ_0.

Results

The scalaron potential of the quadratic Starobinsky model is

$$W(\xi)=\frac{3}{4}\mu^2\left(1-e^{-2\xi}\right)^2; \tag{25}$$

it satisfies conditions (5)–(8) for $\xi>0$. In the region of positive ξ it agrees very well with current cosmological data (see, e.g., Shtanov, Sahni, & Mishra, 2023). As ξ increases, $w'(\xi)\to 0$.

In numerical investigations we typically worked with $\xi_0\sim 10^{-3}\div 10^{-5}$, $r_0\sim(10\div 1000)r_g$, $M\mu\sim 10^5\div 10^{30}$. Numerical simulations confirm that approximate formulas (20), (21) hold with a good accuracy for $M\mu\gtrsim 10^{11}$. Examples of dependence of limiting parameter Y_f upon $M\mu$ are shown in Fig. 1 for different sizes of the strong scalaron region corresponding to different values of the scalar charge. In fig. 1 we assumed $\xi_0=0.001$, but the other values of ξ_0 from the interval $10^{-3}\div 10^{-5}$ show the same behavior.

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Our formulas are also were tested using the hilltop and tabletop potentials described by Shtanov, Sahni, & Mishra, 2023. The results are essentially the same, though the numerical values of $\gamma(\xi)$ are of course different for different potentials. The reason lies in the similar behavior of the potentials (5), (6), (8) near minimum. Some qualitative difference in case of the hilltop potentials is due to sign of the right-hand side of (15) because in this case the potential goes to zero after the maximum. In case of (25) $w(\xi) \to 3/4$ as $\xi > 0$ increases and there is only a small region near the center ($r \sim 1/\mu$, $U(p) \to 0$), where the right-hand side of (11) changes its sign. However, these differences are almost invisible because of strong suppression of this right-hand side by $\exp(\chi)$.

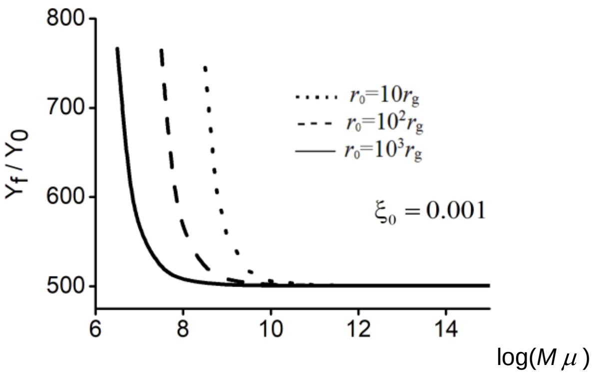

Fig. 1. The case of potential (25): dependence of limiting value Y_f/Y_0 upon $M\mu$ for three sizes r_0 with $\xi(r_0) = 0.001$

Discussion and conclusions

The SSS solutions of the $f(R)$ gravity in the Einstein frame are found for potentials satisfying conditions (5)–(8) that are typical for a number of known $f(R)$ gravity models. We give a representation of the basic equations describing the SSS configuration, which allowed us to derive approximate solutions and to perform numerical calculations for rather high values of $M\mu = 10^6 \div 10^{30}$. The numerical results are in complete correspondence with analytical relations obtained. For large enough $M\mu$, the SSS solutions exhibit simple behavior, corresponding to practically constant values of $rd\xi/dr$ and $r\exp\left((\alpha-\beta)/2\right)$ right up to the naked singularity at the origin, but except very small interval *T* ($\Delta r \sim l_\mu$) of a transition region with a strong jump-like variation of essential parameters. For $r < r_0$ the the solution is significantly different from the Schwarzschild case. It satisfactory describes static spherically symmetric configuration of the $f(R)$ gravity; the larger $M\mu$, the better is the accuracy of the approximation. This makes the results useful to test numerical algorithms in various $f(R)$ models.

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СПЕЦІАЛЬНІ СФЕРИЧНО-СИМЕТРИЧНІ РОЗВ'ЯЗКИ У *f*(*R*) ГРАВІТАЦІЇ

В с т у п . ***f*(*R*) гравітація є природним узагальненням загальної теорії відносності, де лагранжіан у дії Айнштайна – Гільберта замінюється складнішою функцією. Цю теорію широко використовують у розгляді ранньої інфляції, пізньої інфляції та в моделях темної матерії. Варіація дії в цій теорії приводить до рівнянь четвертого порядку відносно фізичної метрики (система Йордана). Ми вивчаємо конформно перетворену метрику (система Айнштайна) у статичному сферично-симетричному (SSS) випадку.**

М е т о д и . ***Ми використовуємо добре відоме перетворення, яке дозволяє звести задачу до звичайних рівнянь Айнштайна з додатковим скалярним полем, яке називають скаляроном. Основну увагу приділено розв'язкам SSS у випадку великого безрозмірного параметра $M\mu \uparrow \infty$, де M – конфігураційна маса, а μ – маса скалярона в геометризованих одиницях.***

Р е з у л ь т а т и . ***Визначено клас потенціалів скалярона, який охоплює відомі моделі f(R) і допускає простий аналітичний опис SSS об'єктів в області, де розв'язки суттєво відрізняються від випадку Шварцшильда, для $M\mu \gg 1$. Знайдено представлення рівнянь структури, яке дозволяє числове інтегрування у випадку достатньо великого $M\mu$. Результати можуть бути корисними для тестування числових алгоритмів.***

В и с н о в к и . ***Знайдено наближені аналітичні співвідношення для розв'язків SSS гравітації в системі відліку Айнштейна на скінченному інтервалі радіальної змінної у границі за спеціального припущення щодо початкових даних. Результати показують універсальні властивості розв'язків для широкого класу скаляронних потенціалів.***

К л ю ч о в і с л о в а : ***релятивістські об'єкти, голі сингулярності, модифікована гравітація.***

Автор заявляє про відсутність конфлікту інтересів. Спонсори не брали участі в розробленні дослідження; у зборі, аналізі чи інтерпретації даних; у написанні рукопису; в рішенні про публікацію результатів.

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