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OPTIMIZATION OF THE MODEL OF φ-SUB-GAUSSIAN GENERALIZED FRACTIONAL BROWNIAN MOTION

Simulation methods for stochastic processes are used in many scientific areas, and special attention is given to the fractional Brownian motion. There are many results on simulation of Gaussian fractional Brownian motion. In this paper, we consider simulation of processes from more general class, namely, simulation of a strictly φ-sub-Gaussian generalized fractional Brownian motion defined on the finite domain $[0,1]$. In the beginning of the article, we presented preliminary information which is necessary for understanding the results of our study. In the main part of our paper, for the strictly φ-sub-Gaussian generalized fractional Brownian motion with some given function φ optimal values of the parameters necessary for simulation with given reliability and accuracy in the space $C([0;1])$ are found. This gives us a significant reduction in the number of terms needed to ensure the reliability and accuracy of our model. Also, in the paper some results of calculations and simulation are given. As expected, in our case the obtained values of terms in the model are significantly smaller than the corresponding numbers for the models with the values of parameters that are not optimal.

K e y w o r d s: ***φ-sub-Gaussian generalized fractional Brownian motion, optimization of a model, reliability and accuracy of a model, simulation, stochastic process.***

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Introduction

Simulation methods for stochastic processes are used in many scientific areas, and special attention is given to the fractional Brownian motion, see, for example, papers (Banna et al., 2019; Pashko et al., 2019), as well as the works (Cai, Huan, & Zhu, 2022; Kozachenko & Rozora, 2019; Linda & Allen, 2010; Mode, 2011; Molz & Liu, 1997; Pashko & Rozora, 2018; Pashko, Rozora, & Synyavska, 2021).

In the article (Kozachenko, Sottinen, & Vasylyk, 2005), there was proposed an algorithm for simulation of a strictly φ-sub-Gaussian generalized fractional Brownian motion (φ-GFBM) with given reliability and accuracy in the space $C([0;1])$. The simulation was based on a series expansion of the fractional Brownian motion. The authors proved an estimate of the accuracy of the simulation in the space $C([0;1])$ of continuous functions equipped with the usual sup-norm.

In the article (Kozachenko & Vasylyk, 2006), this study was continued and there was considered the case of simulation of sub-Gaussian generalized fractional Brownian motion, that is, the case when $\varphi(x) = \frac{x^2}{2}, x \in \mathbb{R}$.

In the paper (Vasylyk & Lovytska, 2021), there were derived conditions, under which the model based on a series representation approximates a strictly φ-sub-Gaussian generalized fractional Brownian motion with given reliability and accuracy in the space $C([0;1])$ in the case, when $\varphi(x) = \frac{|x|^p}{p}$, $|x| \geqslant 1$, $p > 1$. Also, necessary simulation parameters were calculated and models of sample paths of corresponding processes were constructed for various values of the Hurst parameter H and for given reliability and accuracy. Then in (Vasylyk et al., 2021), the authors derived conditions, under which the model based on a series representation approximates a strictly φ-sub-Gaussian generalized fractional Brownian motion with given reliability and accuracy in the space $C([0;1])$ in the case, when $\varphi(x) = \exp|x| - |x| - 1, x \in \mathbb{R}$.

But in all the above-mentioned articles the parameters necessary for construction of the model were chosen from acceptable intervals for reasons of computational convenience, and the problem of optimization of the model was not considered.

In this paper, we consider simulation of the strictly φ-sub-Gaussian generalized fractional Brownian motion on the finite domain $[0,1]$ with covariance function

$$R_H(t,s) = \mathsf{E}Z(s)Z(t) = \frac{1}{2}(t^{2H} + s^{2H} - |s-t|^{2H})$$

and the Hurst index H, which takes values in the interval $(0,1)$. For the strictly φ-sub-Gaussian generalized fractional Brownian motion with the function $\varphi(x) = \frac{x^2}{\omega}\mathbb{I}_{\{|x|<1\}} + \frac{|x|^\omega}{\omega}\mathbb{I}_{\{|x|\geqslant 1\}}$, $\omega \geqslant 2$, optimal values of the parameters necessary for simulation in the space $C([0;1])$ are found. This gives us a significant gain in the number of terms needed to ensure the reliability and accuracy of our model.

The plan of the paper is as follows.

In Sections 1 and 2 we present preliminary information which is necessary for undestanding the results of our study. Section 3 contains our main results. And Section 4 presents some results of calculations and simulation, as well as comparison of our results with the results obtained in the paper (Vasylyk & Lovytska, 2021).

1. φ-sub-Gaussian random variables and processes

Let us recall some facts about φ-sub-Gaussian random variables and processes.

Definition 1. (Buldygin & Kozachenko, 2000) A continuous even convex function φ is called an *Orlicz N-function*, if $\varphi(0) = 0, \varphi(x) > 0$ as $x \neq 0$ and the following conditions hold: $\lim_{x\to 0} \frac{\varphi(x)}{x} = 0,$ $\lim_{x\to\infty} \frac{\varphi(x)}{x} = \infty$.

The function φ^* defined by $\varphi^*(x) = \sup_{y\in\mathbb{R}}(xy - \varphi(y))$ is called *the Young-Fenchel transform (or convex conjugate)* of the function φ.

We say that φ satisfies the *Condition Q*, if φ is such an Orlicz N-function that $\liminf\limits_{x\to 0} \frac{\varphi(x)}{x^2} = c > 0$, where the case $c = \infty$ is possible (Giuliano, Kozachenko, & Nikitina, 2003).

In what follows we will always deal with Orlicz N-functions for which condition Q holds.

Examples of Orlicz N-functions, for which the condition Q is satisfied:

$$\varphi(x) = \frac{|x|^\alpha}{\alpha}, \quad 1 < \alpha \leqslant 2,$$
$$\varphi(x) = \begin{cases} \frac{|x|^\alpha}{\alpha}, & |x| \geqslant 1; \\ \frac{|x|^2}{\alpha}, & |x| < 1, \end{cases} \quad \alpha > 2. \tag{1}$$

Definition 2. (Buldygin & Kozachenko, 2000) Let $\{\Omega, \mathcal{F}, \mathbf{P}\}$ be a standard probability space. The random variable ζ is said to be φ-sub-Gaussian (or belongs to the space $\mathrm{Sub}_\varphi(\Omega)$), if $\mathsf{E}\zeta = 0$, $\mathsf{E}\exp\{\lambda\zeta\}$ exists for all $\lambda \in \mathbb{R}$ and there exists a constant $a > 0$ such that for all $\lambda \in \mathbb{R}$ the following inequality holds: $\mathsf{E}\exp\{\lambda\zeta\} \leqslant \exp\{\varphi(\lambda a)\}$.

The space $\mathrm{Sub}_\varphi(\Omega)$ is a Banach space with respect to the norm (Buldygin & Kozachenko, 2000)

$$\tau_\varphi(\zeta) = \inf\{a > 0 : \mathsf{E}\exp\{\lambda\zeta\} \leqslant \exp\{\varphi(a\lambda)\},$$

which is called the φ-sub-Gaussian standard of the random variable ζ.

Centered Gaussian random variables $\zeta \sim N(0, \sigma^2)$ are φ-sub-Gaussian with $\varphi(x) = \frac{x^2}{2}$ and $\tau_\varphi^2(\zeta) = \mathsf{E}\zeta^2 = \sigma^2$. In the case $\varphi(x) = \frac{x^2}{2}$, φ-sub-Gaussian random variables are simply sub-Gaussian.

The important property of φ-sub-Gaussian random variables is the exponential estimate for their tail probabilities. If ζ is a φ-sub-Gaussian random variable, then for all $u > 0$

$$P\{|\zeta| > u\} \leqslant 2\exp\left\{-\varphi^*\left(\frac{u}{\tau_\varphi(\zeta)}\right)\right\}. \tag{2}$$

Definition 3. A family Δ of random variables $\zeta \in \mathrm{Sub}_\varphi(\Omega)$ is called strictly φ-sub-Gaussian (see (Buldygin & Kozachenko, 2000)), if there exists a constant C_Δ such that for all countable sets I of random variables $\zeta_i \in \Delta$, $i \in I$, the following inequality holds:

$$\tau_\varphi\left(\sum_{i\in I}\lambda_i\zeta_i\right) \leqslant C_\Delta\left(\mathsf{E}\left(\sum_{i\in I}\lambda_i\zeta_i\right)^2\right)^{1/2}. \tag{3}$$

The constant C_Δ is called the *determining* constant of the family Δ.

The linear closure of a strictly φ-sub-Gaussian family Δ in the space $L_2(\Omega)$ is the strictly φ-sub-Gaussian with the same determining constant.

Definition 4. The random process $X = \{X(t), t \in T\}$ is called (strictly) φ-sub-Gaussian if the family of random variables $\{X(t), t \in T\}$ is (strictly) φ-sub-Gaussian.

Example. (Kozachenko & Koval'chuk, 1998) Let a family of random variables $\{\xi_k,\ k = \overline{1,\infty}\}$ be a strictly φ-sub-Gaussian with determining constant C_ξ, and let $X(t) = \sum\limits_{k=1}^{\infty}\xi_k f_k(t)$, where $\{f_k(t), t \in T, k \geqslant 1\}$ is a sequence of nonrandom functions and the series converges in mean square. Then the random process $X = \{X(t), t \in T\}$ is strictly φ-sub-Gaussian with determining constant C_ξ.

2. Generalized fractional Brownian motion

Let $\{B(t), t \in [0,1]\}$ be a fractional Brownian motion with Hurst parameter $H \in (0,1)$. This means that $B(t)$ is a centered Gaussian process with covarince function of the form

$$\mathsf{E}B(s)B(t) = \frac{1}{2}(t^{2H} + s^{2H} - |s-t|^{2H}).$$

In the paper (Dzhaparidze & Zanten, 2002), it is shown that $B(t)$ can be represented in the following way:

$$B(t) = \sum_{n=1}^{\infty}\frac{\sin x_n t}{x_n}X_n + \sum_{n=1}^{\infty}\frac{1-\cos y_n t}{y_n}Y_n, \tag{4}$$

where the series in (4) are mean square convergent, $\{X_n, n=1,2,\ldots\}$ and $\{Y_n, n=1,2,\ldots\}$ are independent Gaussian variables such that $\mathsf{E}X_n = \mathsf{E}Y_n = 0$,

$$\mathsf{Var}X_n = 2C_H^2 x_n^{-2H} J_{1-H}^{-2}(x_n), \quad \mathsf{Var}Y_n = 2C_H^2 y_n^{-2H} J_{-H}^{-2}(y_n), \quad C_H^2 = \pi^{-1}\Gamma(1+2H)\sin(\pi H),$$

$x_1 < x_2 < \ldots,$ are positive real zeroes of the Bessel function J_{-H}, and $y_1 < y_2 < \ldots,$ are positive real zeroes of the Bessel function J_{1-H}.

The representation (4) can be rewritten in the following form:

$$B(t) = \sum_{n=1}^{\infty} \tilde{X}_n c_n \sin x_n t + \sum_{n=1}^{\infty} \tilde{Y}_n d_n (1 - \cos y_n t),$$

where $\{\tilde{X}_n, n=1,2,\ldots\}$ and $\{\tilde{Y}_n, n=1,2,\ldots\}$ are independent Gaussian centered random variables such that $\mathsf{E}\tilde{X}_n^2 = \mathsf{E}\tilde{Y}_n^2 = 1$,

$$c_n = \frac{\pi^H \sqrt{2c}}{x_n^{H+1} J_{1-H}(x_n)}, \qquad n = 1,2,\ldots, \tag{5}$$

$$d_n = \frac{\pi^H \sqrt{2c}}{y_n^{H+1} J_{-H}(y_n)}, \qquad n = 1,2,\ldots, \tag{6}$$

$$c = \frac{\Gamma(2H+1)\sin(\pi H)}{\pi^{2H+1}}. \tag{7}$$

Definition 5. A stochastic process $Z = \{Z(t), t \in [0,1]\}$ is called generalized fractional Brownian motion (GFBM) with a Hurst parameter $H \in (0,1)$, if $\mathsf{E}Z(t) = 0$ and

$$R_H(t,s) = \mathsf{E}Z(s)Z(t) = \frac{1}{2}(t^{2H} + s^{2H} - |s-t|^{2H}).$$

If, in addition, the process $Z = \{Z(t), t \in [0,1]\}$ belongs to a space $\mathrm{Sub}_\varphi(\Omega)$, then it is said to be φ-sub-Gaussian generalized fractional Brownian motion (φ-GFBM).

Let

$$Z(t) = \sum_{n=1}^{\infty} \xi_n c_n \sin x_n t + \sum_{n=1}^{\infty} \eta_n d_n (1 - \cos y_n t), \tag{8}$$

where $t \in [0,1]$, c_n and d_n are given in (5) and (6), ξ_n, η_n are independent centered random variables such that $\mathsf{E}\xi_n^2 = \mathsf{E}\eta_n^2 = 1, n = 1,2,\ldots$. Then the following proposition holds.

Proposition 1. (Kozachenko, Sottinen, & Vasylyk, 2005) *The series in (8) converge in mean square and the covariance function of the process Z is:*

$$EZ(s)Z(t) = \frac{1}{2}(t^{2H} + s^{2H} - |s-t|^{2H}).$$

According to Definition 5, the process Z, defined by (8), is a generalized fractional Brownian motion.

Theorem 1. (Kozachenko, Sottinen, & Vasylyk, 2005) *The series in (8) converge uniformly with probability one to the process $Z(t)$, and the process $Z(t)$ is uniformly continuous on $[0,1]$ with probability one.*

Let $\xi_n, \eta_n, n = 1,2,\ldots,$ be independent identically distributed strictly φ-sub-Gaussian random variables, such that $\mathsf{E}\xi_n^2 = \mathsf{E}\eta_n^2 = 1, n = 1,2,\ldots$.

Then, according to Example 1, Proposition 1 and Definition 5, the process $Z = \{Z(t), t \in [0,1]\}$ defined by the representation

$$Z(t) = \sum_{n=1}^{\infty} \xi_n c_n \sin x_n t + \sum_{n=1}^{\infty} \eta_n d_n (1 - \cos y_n t), \tag{9}$$

where c_n and d_n are given in (5) and (6), is a strictly φ-sub-Gaussian generalized fractional Brownian motion.

In what follows, we assume that the function $\varphi(\sqrt{\cdot})$ is convex.

Definition 6. The model $\tilde{Z}$ approximates the process Z with given *reliability* $1-\nu$, $0 < \nu < 1$, and *accuracy* $\delta > 0$ in $C([0,1])$, if

$$\mathbf{P}\left(\sup_{t\in[0,1]} |Z(t) - \tilde{Z}(t)| > \delta\right) \leqslant \nu.$$

A natural model for Z, defined by the expansion (9), would be the truncated series

$$\sum_{n=1}^{N} \left(c_n \sin(x_n t)\, \xi_n \;+\; d_n \big(1 - \cos(y_n t)\big)\, \eta_n\right).$$

As in (Kozachenko, Sottinen, & Vasylyk, 2005), we assume that the constants c_n and d_n and the zeros x_n, y_n are only calculated approximately, since there are fast-to-compute asymptotic formulas for the zeros x_n and y_n (Watson, 1944, p. 506).

Let $\tilde{c}_n$ and $\tilde{d}_n$ be the approximated values of the c_n and d_n, respectively. Let

$$|\tilde{c}_n - c_n| \leqslant \gamma_n^c, \quad |\tilde{d}_n - d_n| \leqslant \gamma_n^d,$$

$n = 1, \ldots, N$. The errors γ_n^c and γ_n^d are assumed to be known. Let $\tilde{x}_n$ and $\tilde{y}_n$ be approximations of the corresponding zeros x_n and y_n with error bounds

$$|\tilde{x}_n - x_n| \leqslant \gamma_n^x, \quad |\tilde{y}_n - y_n| \leqslant \gamma_n^y.$$

The error bounds γ_n^x and γ_n^y are also assumed to be known.

Then, the model of the process Z is

$$\tilde{Z}(t) = \sum_{n=1}^{N} \Big(\tilde{c}_n \sin(\tilde{x}_n t)\, \xi_n \;+\; \tilde{d}_n \big(1 - \cos(\tilde{y}_n t)\big)\, \eta_n \Big), \tag{10}$$

and simulation error has the form

$$\begin{aligned} \Delta_t := Z(t) - \tilde{Z}(t) \quad &= \sum_{n=1}^{N} \Big\{ \Big(c_n \sin(x_n t) - \tilde{c}_n \sin(\tilde{x}_n t) \Big)\, \xi_n + \Big(d_n \big(1 - \cos(y_n t)\big) - \tilde{d}_n \big(1 - \cos(\tilde{y}_n t)\big) \Big)\, \eta_n \Big\} + \\ &+ \sum_{n=N+1}^{\infty} \Big\{ c_n \sin(x_n t)\, \xi_n \;+\; d_n \big(1 - \cos(y_n t)\big) \eta_n \Big\}. \end{aligned}$$

In order to estimate Δ in $C([0,1])$, it was necessary to estimate $\tau_\varphi(\Delta_t)$ and $\tau_\varphi(\Delta_t - \Delta_s)$ for all $s, t \in [0,1]$, and it was done in (Kozachenko, Sottinen, & Vasylyk, 2005). Then on the basis of the obtained estimates for the simulation error and using the corresponding estimates of the distributions of supremums of φ-sub-Gaussian random processes, the following theorem on the simulation of φ-sub-Gaussian generalized fractional Brownian motion with given reliability and accuracy in the space $C([0;1])$ was proved in (Kozachenko, Sottinen, & Vasylyk, 2005, Theorem 4.3).

Theorem 2. (Kozachenko, Sottinen, & Vasylyk, 2005) *Let b and α be such that $0 < b < \alpha < H$. The model $\tilde{Z}$, defined by (10), approximates the process Z, defined by (9), with given reliability $1 - \nu$, $0 < \nu < 1$, and accuracy $\delta > 0$ in $C([0,1])$, if the following three inequalities are satisfied:*

$$\gamma_0 < \delta, \quad \frac{\beta\gamma_0}{\gamma_\alpha} < \frac{\delta}{2^\alpha(\exp\{\varphi(1)\} - 1)^\alpha}, \quad \nu \geqslant 2\exp\left\{-\varphi^*\left(\frac{\delta}{\gamma_0} - 1\right)\right\} \left(\frac{1}{2^b(1 - \frac{b}{\alpha})} \left(\frac{\gamma_\alpha \delta}{\beta\gamma_0}\right)^{\frac{b}{\alpha}} l^{(-1)}\left(\frac{\delta}{\gamma_0} - 1\right) + 1 \right)^{\frac{2}{b}},$$

where the parameters $\gamma_0 = \gamma_0(N)$, $\gamma_\alpha = \gamma_\alpha(N)$ and $\beta = \min\{\gamma_0, \frac{\gamma_\alpha}{2^\alpha}\}$ are expressed through the estimates derived for the simulation error, and $l^{(-1)}$ is the generalised inverse function of the density function l of the function φ.

Remark 1. The density function l of the Orlicz N-function φ is nondecreasing right continuous function such that $l(u) > 0$ as $u > 0$, $l(0) = 0$, $l(u) \to \infty$ as $u \to \infty$ and $\varphi(x) = \int_0^{|x|} l(u)\mathrm{d}u, \quad x \in \mathbb{R}$. The function $l^{(-1)}$ is the generalised inverse function of l, i.e. $l^{(-1)}(x) := \sup\{v \geqslant 0 : l(v) \leqslant x\}$.

Also we need the following Corollary (Kozachenko, Sottinen, & Vasylyk, 2005, Corollary 4.7). Assume that the constants c_n and d_n, as well as the zeros x_n and y_n are calculated exactly.

Denote $a_\varphi := \tau_\varphi(\xi_n) = \tau_\varphi(\eta_n)$.

Corollary 1. (Kozachenko, Sottinen, & Vasylyk, 2005) *Suppose that there is no approximation error, i.e. $\gamma_n^c = \gamma_n^d = \gamma_n^x = \gamma_n^y = 0$. Then the conditions of the Theorem 2 are satisfied if*

$$N \geqslant \max\left\{ \left(\frac{A_0}{\delta}\right)^{\frac{1}{H}} + 1;\ \left(\frac{A_0(\exp\{\varphi(1)\} - 1)^\alpha}{\delta}\right)^{\frac{1}{H}} + 1;\ 2\left(\frac{A_0}{A_\alpha}\right)^{\frac{1}{\alpha}} \right\} \tag{11}$$

and

$$\nu \geqslant 2\exp\left\{-\varphi^*\left(\frac{\delta N^H}{A_0} - 1\right)\right\} \times \left(\frac{(\delta A_\alpha)^{\frac{b}{\alpha}} (N+1)^{2Hb/\alpha}}{2^b\left(1 - \frac{b}{\alpha}\right) A_0^{\frac{2b}{\alpha}} N^{\frac{(H-\alpha)b}{\alpha}}} l^{-1}\left(\frac{\delta(N+1)^H}{A_0} - 1\right) + 1 \right)^{\frac{2}{b}}, \tag{12}$$

where

$$A_0 = a_\varphi\sqrt{\frac{5c}{2H}}, \quad A_\alpha = 2^{1-\alpha} a_\varphi \pi^\alpha \sqrt{\frac{c}{H - \alpha}}.$$

3. Main results

Based on the above results, we derive the conditions under which the model of the form (10) constructed for the optimal values of the parameters b and α approximates the process of a strictly φ-sub-Gaussian generalized fractional Brownian motion with given reliability and accuracy in the space $C([0;1])$ in the case when

$$\varphi(x)=\begin{cases}\frac{|x|^{\omega}}{\omega}, & |x|\geqslant 1;\\ \frac{x^2}{\omega}, & |x|<1,\end{cases}\qquad \omega\geqslant 2.$$

The next theorem presents the main results of our study.

Theorem 3. *Let Z be a strictly φ-sub-Gaussian generalized fractional Brownian motion with Hurst parameter $H\in(0,1)$ and the function $\varphi(x)=\begin{cases}\frac{|x|^{\omega}}{\omega}, & |x|\geqslant 1;\\ \frac{x^2}{\omega}, & |x|<1,\end{cases}$ $\omega\geqslant 2$. The model (10) approximates the process Z with given reliability $1-\nu$, $0<\nu<1$, and accuracy $\delta>0$ in the space $C([0,1])$, if the following conditions hold:*

$$N\geqslant\left(\frac{a_{\varphi}}{\delta}\sqrt{\frac{5c}{2H}}\right)^{\frac{1}{H}}+1 \quad \text{and} \quad \frac{\pi^2N^2}{8}\exp\left[\frac{1}{H(1-x_G)}-\frac{1}{q}\left(\sqrt{G}-1\right)^{q}\right]\leqslant\nu, \tag{13}$$

where

$$G=\frac{N^{2H}\delta^2 2H}{5ca_{\varphi}^2},\quad x_G=R\left(\frac{8G}{5}\left(\frac{G^{\frac{1}{\omega-1}}}{(1-y_G)^2}\right)^{\frac{1}{y_G}}\right),\quad y_G=R\left(G^{\frac{1}{2(\omega-1)}}\right), \tag{14}$$

$c=\frac{\Gamma(2H+1)\sin(\pi H)}{\pi^{2H+1}}$, $q>0:\frac{1}{\omega}+\frac{1}{q}=1$, *and $R(p)$ is a solution of the following equation with respect to z:*

$$\frac{1}{1-z}+ln(1-z)-lnp-1=0,\ \text{for}\ z\in(0,1),\ p>1. \tag{15}$$

In addition, the resulting values of the parameters $\alpha=Hx_G$ and $b=\alpha y_G$ are optimal, that is, they make it possible to determine the minimum required number of terms N in the model (10) comparing with other possible values $0<b<\alpha<H$.

Proof. Consider conditions of Theorem 2 and Corollary 12 in the case when $\varphi(x)=\frac{x^2}{\omega}\mathbb{I}_{\{|x|<1\}}+\frac{|x|^{\omega}}{\omega}\mathbb{I}_{\{|x|\geqslant1\}}$, $\omega\geqslant 2$. Then

$$\varphi^*(x)=\frac{x^q}{q},q>0:\frac{1}{\omega}+\frac{1}{q}=1,\quad \text{and}\quad l(x)=\varphi'(x)=x^{\omega-1},l^{-1}(x)=x^{\frac{1}{\omega-1}},\ x\geqslant 1.$$

Thus,

$$N\geqslant\max\left\{\left(\frac{A_0}{\delta}\right)^{\frac{1}{H}}+1;\quad 2\left(\frac{A_0}{A_{\alpha}}\right)^{\frac{1}{\alpha}}\right\} \tag{16}$$

and

$$\nu\geqslant 2\exp\left\{-\frac{1}{q}\left(\frac{\delta N^H}{A_0}-1\right)^q\right\}\left(\frac{(\delta A_{\alpha})^{\frac{b}{\alpha}}(N+1)^{\frac{2Hb}{\alpha}}}{2^b\left(1-\frac{b}{\alpha}\right)A_0^{\frac{2b}{\alpha}}N^{\frac{(H-\alpha)b}{\alpha}}}\left(\frac{\delta\,(N+1)^H}{A_0}-1\right)^{\frac{1}{\omega-1}}+1\right)^{\frac{2}{b}}. \tag{17}$$

Recall that $A_0=a_{\varphi}\sqrt{\frac{5c}{2H}}$ and $A_{\alpha}=2^{1-\alpha}a_{\varphi}\pi^{\alpha}\sqrt{\frac{c}{H-\alpha}}$.

Let us consider the second inequality in (16)

$$2\left(\frac{A_0}{A_{\alpha}}\right)^{\frac{1}{\alpha}}=2\left(\frac{a_{\varphi}\sqrt{\frac{5c}{2H}}}{2^{(1-\alpha)}a_{\varphi}\pi^{\alpha}\sqrt{\frac{c}{H-\alpha}}}\right)^{\frac{1}{\alpha}}=\frac{2}{\pi}\left(\frac{\frac{5c}{2H}}{4^{(1-\alpha)}\frac{c}{H-\alpha}}\right)^{\frac{1}{2\alpha}}=$$

$$=\frac{4}{\pi}\left(\frac{5(H-\alpha)}{8H}\right)^{\frac{1}{2\alpha}}\leqslant\frac{4}{\pi}\left(\frac{5}{8}\right)^{\frac{1}{2\alpha}}\leqslant\frac{4}{\pi}\left(\frac{5}{8}\right)^{\frac{1}{2H}}\leqslant\frac{4}{\pi}\left(\frac{5}{8}\right)^{\frac{1}{2}}\approx 1,00658. \tag{18}$$

Therefore from (18) follows that

$$N\geqslant\left(\frac{a_{\varphi}}{\delta}\sqrt{\frac{5c}{2H}}\right)^{\frac{1}{H}}+1. \tag{19}$$

Since N is a large number, then in (17) we have

$$\left(\frac{(\delta A_{\alpha})^{\frac{b}{\alpha}}(N+1)^{\frac{2Hb}{\alpha}}}{2^b\left(1-\frac{b}{\alpha}\right)A_0^{\frac{2b}{\alpha}}N^{\frac{(H-\alpha)b}{\alpha}}}\left(\frac{\delta\,(N+1)^H}{A_0}-1\right)^{\frac{1}{\omega-1}}+1\right)^{\frac{2}{b}}\approx$$

$$\approx \left(\frac{(\delta A_\alpha)^{\frac{b}{\alpha}} \mathsf{N}^{\frac{2Hb}{\alpha}}}{2^b \left(1-\frac{b}{\alpha}\right) A_0^{\frac{2b}{\alpha}} N^{\frac{(H-\alpha)b}{\alpha}}} \left(\frac{\delta N^H}{A_0}\right)^{\frac{1}{\omega-1}} \right)^{\frac{2}{b}}.$$

Rewrite the obtained expression in details:

$$\left(\frac{(\delta A_\alpha)^{\frac{b}{\alpha}} \mathsf{N}^{\frac{2Hb}{\alpha}}}{2^b \left(1-\frac{b}{\alpha}\right) A_0^{\frac{2b}{\alpha}} N^{\frac{(H-\alpha)b}{\alpha}}} \left(\frac{\delta N^H}{A_0}\right)^{\frac{1}{\omega-1}} \right)^{\frac{2}{b}} =$$

$$= \left(\frac{\left(\delta 2^{1-\alpha} a_\varphi \pi^\alpha \sqrt{\frac{c}{H-\alpha}}\right)^{\frac{b}{\alpha}} N^{\frac{2Hb}{\alpha}}}{2^b \left(1-\frac{b}{\alpha}\right) \left(a_\varphi \sqrt{\frac{5c}{2H}}\right)^{\frac{2b}{\alpha}} N^{\frac{(H-\alpha)b}{\alpha}}} \left(\frac{\delta N^H}{a_\varphi \sqrt{\frac{5c}{2H}}}\right)^{\frac{1}{\omega-1}} \right)^{\frac{2}{b}} =$$

$$= \pi^2 \left(\frac{H}{5}\right)^{\frac{2}{\alpha}+\frac{1}{b(\omega-1)}} 2^{\frac{4}{\alpha}+\frac{1}{b(\omega-1)}-4} (H-\alpha)^{-\frac{1}{\alpha}} \left(\frac{\delta^2}{a_\varphi^2 c}\right)^{\frac{1}{\alpha}+\frac{1}{b(\omega-1)}} \left(1-\frac{b}{\alpha}\right)^{-\frac{2}{b}} N^{2H\left(\frac{1}{\alpha}+\frac{1}{b(\omega-1)}\right)+2} =$$

$$= \pi^2 \left(\frac{H}{5}\right)^{\frac{1}{\alpha}} 2^{\left(\frac{3}{\alpha}-4\right)} (H-\alpha)^{-\frac{1}{\alpha}} \left(1-\frac{b}{\alpha}\right)^{-\frac{2}{b}} N^2 \left(\frac{N^{2H}\delta^2 2H}{5ca_\varphi^2}\right)^{\left(\frac{1}{\alpha}+\frac{1}{b(\omega-1)}\right)} =$$

$$= \frac{\pi^2 N^2}{16} \left(\frac{8H}{5(H-\alpha)}\right)^{\frac{1}{\alpha}} \left(1-\frac{b}{\alpha}\right)^{-\frac{2}{b}} \left(\frac{N^{2H}\delta^2 2H}{5ca_\varphi^2}\right)^{\left(\frac{1}{\alpha}+\frac{1}{b(\omega-1)}\right)}. \tag{20}$$

For the sake of simplicity we put $x := \frac{\alpha}{H}$, and $y := \frac{b}{\alpha} = \frac{b}{xH}$. Now we consider new unknown parameters $(x, y) \in (0,1)^2$. The expression (20) in new notations takes the form

$$\left(\frac{\pi N}{4}\right)^2 \left(\frac{8}{5(1-x)}\right)^{\frac{1}{xH}} (1-y)^{-\frac{2}{xyH}} \left(\frac{N^{2H}\delta^2 2H}{5ca_\varphi^2}\right)^{\left(\frac{1}{y(\omega-1)}+1\right)\frac{1}{xH}}.$$

Let us use (14), in particular $G = \frac{N^{2H}\delta^2 2H}{5ca_\varphi^2}$. From (19) follows that $G > 1$. Then we get

$$\left(\frac{\pi N}{4}\right)^2 \left(\left(\frac{8}{5(1-x)}\right)^{\frac{1}{x}} (1-y)^{-\frac{2}{xy}} G^{\left(\frac{1}{y(\omega-1)}+1\right)\frac{1}{x}} \right)^{\frac{1}{H}}. \tag{21}$$

Thus, now the main task consists in finding such values of the parameters x and y, which minimize the function

$$S^G(x,y) := \left(\frac{8}{5(1-x)}\right)^{\frac{1}{x}} (1-y)^{-\frac{2}{xy}} G^{\left(\frac{1}{y(\omega-1)}+1\right)\frac{1}{x}} = \left(\frac{8}{5} \frac{G^{\left(\frac{1}{y(\omega-1)}+1\right)} (1-y)^{-\frac{2}{y}}}{(1-x)} \right)^{\frac{1}{x}}. \tag{22}$$

In order to do that we do the following:

- find partial derivatives $\frac{\partial S^G}{\partial x}$, $\frac{\partial S^G}{\partial y}$ and solve corresponding equations $\frac{\partial S^G}{\partial x} = 0$, $\frac{\partial S^G}{\partial y} = 0$;
- check the correctness of the found solutions;
- check the matrix of second derivatives for positive definiteness.

Let us take the partial derivative on x:

$$\frac{\partial S^G}{\partial x} = -\frac{\left(\frac{8}{5}\right)^{\frac{1}{x}} \left(\frac{G^{\left(\frac{1}{y(\omega-1)}+1\right)} (1-y)^{-\frac{2}{y}}}{(1-x)} \right)^{\frac{1}{x}} \ln \frac{8}{5}}{x^2} +$$

$$+ \left(\frac{8}{5}\right)^{\frac{1}{x}} \left(\frac{G^{\left(\frac{1}{y(\omega-1)}+1\right)} (1-y)^{-\frac{2}{y}}}{(1-x)} \right)^{\frac{1}{x}} \left(\frac{1}{(1-x)x} - \frac{\ln\left(\frac{G^{\left(\frac{1}{y(\omega-1)}+1\right)} (1-y)^{-\frac{2}{y}}}{1-x} \right)}{x^2} \right) =$$

$$= -\frac{S^G(x,y) \ln \frac{8}{5}}{x^2} + S^G(x,y) \left(\frac{1}{(1-x)x} - \frac{\ln\left(\frac{G^{\left(\frac{1}{y(\omega-1)}+1\right)} (1-y)^{-\frac{2}{y}}}{(1-x)} \right)}{x^2} \right) =$$

$$= \frac{S^G(x,y)}{x^2}\left(\frac{x}{(1-x)} - \ln\left(\frac{8}{5}\frac{G^{\left(1+\frac{1}{y(\omega-1)}\right)}(1-y)^{-\frac{2}{y}}}{1-x}\right)\right) =$$

$$= \frac{S^G(x,y)}{x^2}\left(\frac{1}{1-x} + \ln(1-x) - \ln\left(\frac{8G}{5}\left(\frac{G^{\frac{1}{(\omega-1)}}}{(1-y)^2}\right)^{\frac{1}{y}}\right) - 1\right). \tag{23}$$

Now we take the partial derivative on y:

$$\frac{\partial S^G}{\partial y} = \frac{\partial}{\partial y}\left(\frac{8}{5}\frac{G^{\left(\frac{1}{y(\omega-1)}+1\right)}(1-y)^{-\frac{2}{y}}}{(1-x)}\right)^{\frac{1}{x}} =$$

$$= \left(\frac{8}{5(1-x)}\right)^{\frac{1}{x}}\frac{1}{x}\left(G^{\left(\frac{1}{y(\omega-1)}+1\right)}(1-y)^{-\frac{2}{y}}\right)^{\left(\frac{1}{x}-1\right)} \times$$

$$\times\left(-\frac{G^{\left(\frac{1}{y(\omega-1)}+1\right)}\ln G\,(1-y)^{-\frac{2}{y}}}{y^2(\omega-1)} + G^{\left(\frac{1}{y(\omega-1)}+1\right)}(1-y)^{-\frac{2}{y}}\left(\frac{2}{(1-y)\,y} + \frac{2\ln(1-y)}{y^2}\right)\right) =$$

$$= \frac{1}{x}S^G(x,y)\left(-\frac{\ln G}{y^2(\omega-1)} + \frac{2}{(1-y)\,y} + \frac{2\ln(1-y)}{y^2}\right) =$$

$$= \frac{2}{xy^2}S^G(x,y)\left(-\frac{\ln G}{2(\omega-1)} + \frac{1}{1-y} - 1 + \ln(1-y)\right). \tag{24}$$

Next we solve the equations $\frac{\partial S^G}{\partial y} = 0$ and $\frac{\partial S^G}{\partial x} = 0$.
We use the facts that $(x,y) \in (0,1)^2$ and $S^G(x,y) > 0$. From (23) and (24) we get a corresponding equivalent system of equations:

$$\begin{cases} \dfrac{1}{1-x} + \ln(1-x) - \ln\left(\dfrac{8G}{5}\left(\dfrac{G^{\frac{1}{(\omega-1)}}}{(1-y)^2}\right)^{\frac{1}{y}}\right) - 1 = 0; \\ \dfrac{1}{1-y} + \ln(1-y) - \ln G^{\frac{1}{2(\omega-1)}} - 1 = 0. \end{cases} \tag{25}$$

Consider (15) with respect to $z \in (0,1)$, with the parameter $p > 1 : \frac{1}{1-z} + \ln(1-z) - \ln p - 1 = 0$. Denote by $R(p)$ the solution of this equation. It exists and is unique for $p > 1$. Really, the function $\frac{1}{1-z} + \ln(1-z) - \ln p - 1$ is strictly increasing on $z \in (0,1)$, because

$$\left(\frac{1}{1-z} + \ln(1-z) - \ln p - 1\right)'_z = \frac{z}{(1-z)^2} > 0.$$

Also

$$\frac{1}{1-0} + \ln(1-0) - \ln p - 1 = -\ln p < 0, \quad \text{and} \quad \lim_{z\to 1-}\left(\frac{1}{1-z} + \ln(1-z) - \ln p - 1\right) = +\infty.$$

Denote by x_G and y_G corresponding solutions to the system (25). Then we can rewrite them in the form

$$y_G = R\left(G^{\frac{1}{2(\omega-1)}}\right), \quad x_G = R\left(\frac{8G}{5}\left(\frac{G^{\frac{1}{\omega-1}}}{(1-y)^2}\right)^{\frac{1}{y_G}}\right).$$

Now we check that these solutions are defined correctly. It is true, since $G > 1$, then $R\left(\sqrt{G}\right)$ exists. Also we get that $\frac{8}{5}G^{1+\frac{1}{y(\omega-1)}} > 1$. Obviously $(1-y)^{\frac{2}{y}} < 1$. Thus, $\frac{8G}{5}\left(\frac{G^{\frac{1}{\omega-1}}}{(1-y)^2}\right)^{\frac{1}{y_G}} > 1$ and x_G is also defined correctly.
Next we check whether the point (x_G, y_G) is a point of minimum of the function $S^G(x,y)$, which is defined in (22). For this, it is necessary to check the positive definiteness of the matrix of second derivatives.
Let's find the second partial derivative with respect to x using (23):

$$\frac{\partial^2 S^G}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{S^G(x,y)}{x^2}\left(\frac{1}{1-x} + \ln(1-x) - \ln\left(\frac{8G}{5}\left(\frac{G^{\frac{1}{(\omega-1)}}}{(1-y)^2}\right)^{\frac{1}{y}}\right) - 1\right)\right) =$$

$$= \frac{\partial}{\partial x}\left[\frac{S^G(x,y)}{x^2}\right]\left(\frac{1}{1-x} + \ln(1-x) - \ln\left(\frac{8G}{5}\left(\frac{G^{\frac{1}{(\omega-1)}}}{(1-y)^2}\right)^{\frac{1}{y}}\right) - 1\right) +$$

$$+\frac{S^G(x,y)}{x^2}\frac{\partial}{\partial x}\left(\frac{1}{1-x}+\ln(1-x)-\ln\left(\frac{8G}{5}\left(\frac{G^{\frac{1}{(\omega-1)}}}{(1-y)^2}\right)^{\frac{1}{y}}\right)-1\right)=$$

$$=\frac{\partial}{\partial x}\left[S^G(x,y)\right]\left(\frac{1}{1-x}+\ln(1-x)-\ln\left(\frac{8G}{5}\left(\frac{G^{\frac{1}{(\omega-1)}}}{(1-y)^2}\right)^{\frac{1}{y}}\right)-1\right)+\frac{S^G(x,y)}{x(1-x)^2}. \tag{26}$$

The second partial derivative with respect to y is equal to

$$\frac{\partial^2 S^G}{\partial y^2}=\frac{\partial}{\partial y}\left[\frac{2}{xy^2}S^G(x,y)\left(\frac{1}{1-y}+\ln(1-y)-\frac{1}{2(\omega-1)}\ln G-1\right)\right]=$$
$$=\frac{\partial}{\partial y}\left(\frac{2}{xy^2}S^G(x,y)\right)\left(\frac{1}{1-y}+\ln(1-y)-\frac{1}{2(\omega-1)}\ln G-1\right)+$$

$$+\frac{2}{xy^2}S^G(x,y)\frac{\partial}{\partial y}\left(\frac{1}{1-y}+\ln(1-y)-\frac{1}{2(\omega-1)}\ln G-1\right)=$$
$$=\frac{\partial}{\partial y}\left[\frac{2}{xy^2}S^G(x,y)\right]\left(\frac{1}{1-y}+\ln(1-y)-\frac{1}{2(\omega-1)}\ln G-1\right)+\frac{2S^G(x,y)}{xy(1-y)^2}. \tag{27}$$

The mixed derivative is equal to

$$\frac{\partial^2 S^G}{\partial x\partial y}=\frac{\partial^2 S^G}{\partial y\partial x}=\frac{\partial}{\partial x}\left(\frac{2}{xy^2}S^G(x,y)\left(\frac{1}{1-y}+\ln(1-y)-\frac{1}{2(\omega-1)}\ln G-1\right)\right)=$$
$$=\frac{\partial}{\partial x}\left[\frac{2}{xy^2}S^G(x,y)\right]\left(\frac{1}{1-y}+\ln(1-y)-\frac{1}{2(\omega-1)}\ln G-1\right). \tag{28}$$

From (26), (27), (28) and (25) we get

$$\frac{\partial^2 S^G}{\partial x^2}(x_G,y_G)=\frac{S^G(x_G,y_G)}{x_G(1-x_G)^2}>0;\quad \frac{\partial^2 S^G}{\partial y^2}(x_G,y_G)=\frac{2S^G(x_G,y_G)}{x_G y_G(1-y_G)^2}>0;\quad \frac{\partial^2 S^G}{\partial x\partial y}(x_G,y_G)=0.$$

The positive definiteness of the matrix of second derivatives follows from Sylvester's criterion:

$$\triangle_1=\frac{\partial^2 S^G}{\partial x^2}>0;$$

$$\triangle_2=\left|\begin{pmatrix}\frac{\partial^2 S^G}{\partial x^2} & \frac{\partial^2 S^G}{\partial x\partial y}\\ \frac{\partial^2 S^G}{\partial x\partial y} & \frac{\partial^2 S^G}{\partial y^2}\end{pmatrix}\right|_{(x_G,y_G)}=\begin{vmatrix}\frac{\partial^2 S^G}{\partial x^2}(x_G,y_G) & 0\\ 0 & \frac{\partial^2 S^G}{\partial y^2}(x_G,y_G)\end{vmatrix}=$$
$$=\frac{\partial^2 S^G}{\partial x^2}(x_G,y_G)\frac{\partial^2 S^G}{\partial y^2}(x_G,y_G)>0.$$

Note the monotonicity of the function $S^G(x_G,y_G)$ of the parameter G.

$$\frac{d}{dG}S^G(x_G,y_G)=\frac{\partial S^G}{\partial x}(x_G,y_G)\frac{\partial x_G}{\partial G}+\frac{\partial S^G}{\partial y}(x_G,y_G)\frac{\partial y_G}{\partial G}+\frac{\partial S^G}{\partial G}(x_G,y_G)=$$

$$=\frac{\partial S^G}{\partial G}(x_G,y_G)=\frac{\partial}{\partial G}\left(\left(\frac{8}{5}\frac{G^{\left(\frac{1}{y(\omega-1)}+1\right)}(1-y)^{-\frac{2}{y}}}{1-x}\right)^{\frac{1}{x}}\right)=$$

$$=\frac{1}{x}\left(\frac{8}{5}\frac{G^{\left(\frac{1}{y(\omega-1)}+1\right)}(1-y)^{-\frac{2}{y}}}{1-x}\right)\frac{8}{5}\left(\frac{1}{y(\omega-1)}+1\right)\frac{G^{\frac{1}{y(\omega-1)}}(1-y)^{-\frac{2}{y}}}{1-x}=$$

$$=\frac{1}{x}\left(\frac{1}{y(\omega-1)}+1\right)\left(\frac{8}{5}\right)^{\frac{1}{x}}\left((1-y)^{-\frac{2}{y}}\right)^{\frac{1}{x}}\frac{1}{(1-x)^{\frac{1}{x}}}\left(G^{\left(\frac{1}{y(\omega-1)}+1\right)}\right)^{\frac{1}{x}-1}=$$
$$=\frac{1}{x}\left(\frac{1}{y(\omega-1)}+1\right)S^G(x_G,y_G)\frac{1}{G}>0.$$

Let us transform the first equation from (25):

$$\frac{x_G}{1-x_G}+\ln(1-x_G)=\ln\left(\frac{8G}{5}\left(\frac{G^{\frac{1}{\omega-1}}}{(1-y_G)^2}\right)^{\frac{1}{y_G}}\right)\Longleftrightarrow$$
$$\Longleftrightarrow\frac{8G}{5}\left(\frac{G^{\frac{1}{\omega-1}}}{(1-y_G)^2}\right)^{\frac{1}{y_G}}=\exp\left\{\frac{x_G}{1-x_G}\right\}(1-x_G).$$

Then we can write the function $S^G(x_G,y_G)$ in the form:

$$S^G(x_G,y_G)=\left(\frac{8}{5}\frac{G^{\left(\frac{1}{y_G(\omega-1)}+1\right)}(1-y_G)^{-\frac{2}{y_G}}}{1-x_G}\right)^{\frac{1}{x_G}}=\left(\frac{e^{\frac{x_G}{1-x_G}}(1-x_G)}{(1-x_G)}\right)=e^{\frac{1}{1-x_G}}. \tag{29}$$

Thus, the condition (17) is satisfied if

$$\left(S^G(x_G,y_G)\right)^{\frac{1}{H}}\frac{\pi^2N^2}{8}\exp\left[-\frac{1}{q}\left(\sqrt{G}-1\right)^q\right]\leqslant\nu\Longleftrightarrow$$
$$\Longleftrightarrow\frac{\pi^2N^2}{8}\exp\left[\frac{1}{H(1-x_G)}-\frac{1}{q}\left(\sqrt{G}-1\right)^q\right]\leqslant\nu.$$

This completes the proof of the theorem. □

4. Simulation results for the optimized models

Using Theorem 3, we construct models $\tilde{Z}$ of φ-sub-Gaussian generalized fractional Brownian motion Z with the function $\varphi(x)=\frac{x^2}{\omega}\mathbb{I}_{\{|x|<1\}}+\frac{|x|^\omega}{\omega}\mathbb{I}_{\{|x|\geqslant 1\}}$, $\omega\geqslant 2$, such that $\tilde{Z}$ approximates Z with given reliability $1-\nu=0.99$ and accuracy $\delta=0.01$ in the space $C([0,1])$.

The examples of such models for several values of the Hurst index H and two cases of ω are presented below.

1. $H=\frac{7}{8}$, $\omega=3$. In this case we have

- $c=0.0264278$, $\mu=3.47172\times10^{-10}$, $N\geqslant 1042.53$, $G=253.316$, $x_G=0.941959$, $y_G=0.728925$,
- $\alpha=0.824214$, $b=0.600791$.

Thus, it is enough to set $N=1043$, and as a result we get the following model approximating generalized fractional Brownian motion with parameter $H=\frac{7}{8}$ with reliability 0.99 and accuracy 0.01 in $C([0,1])$ (Fig. 1).

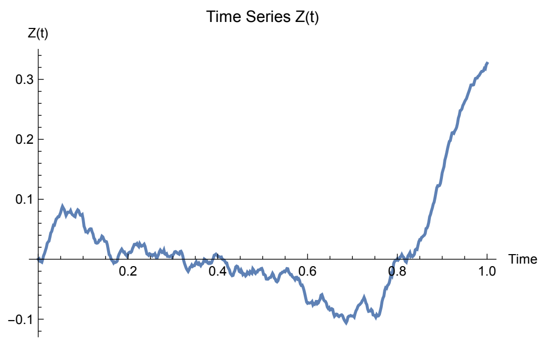

Fig. 1. The model of φ-GFBM with parameter $H=\frac{7}{8}$

2. $H=\frac{8}{9}$, $\omega=3$. In this case we have:

- $c=0.0234107$, $\mu=8.9692\times10^{-10}$, $N\geqslant 853.88$, $G=247.095$, $x_G=0.941809$, $y_G=0.728297$,
- $\alpha=0.837164$, $b=0.609704$.

Thus, it is enough to set $N=854$, and as a result we get the following model, which approximates the process of generalized fractional Brownian motion with parameter $H=\frac{8}{9}$ with reliability 0.99 and accuracy 0.01 in $C([0,1])$ (Fig. 2).

3. $H=\frac{9}{10}$, $\omega=3$. In this case we have:

- $c=0.021007$, $\mu=1.9722\times10^{-9}$, $N\geqslant 726.468$, $G=242.174$, $x_G=0.941687$, $y_G=0.727786$,
- $\alpha=0.847518$, $b=0.616812$.

Thus, it is enough to set $N=727$, and as a result we get the following model, which approximates the process of generalized fractional Brownian motion with parameter $H=\frac{9}{10}$ with reliability 0.99 and accuracy 0.01 in $C([0,1])$ (Fig. 3).

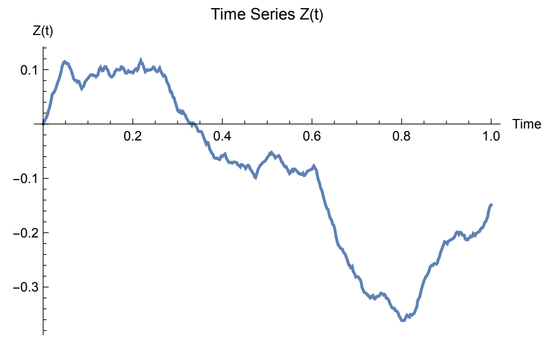

Fig. 2. The model of φ-GFBM with parameter $H = \frac{8}{9}$

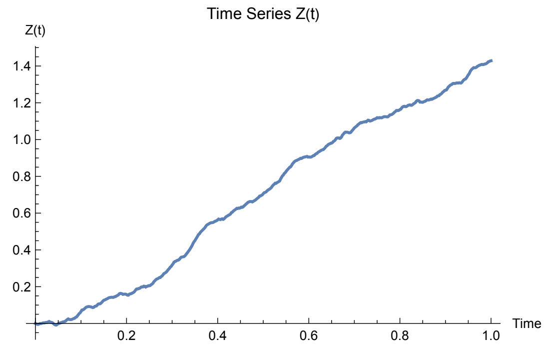

Fig. 3. The model of φ-GFBM with parameter $H = \frac{9}{10}$

4. $H = \frac{15}{16}$, $\omega = 3$.
In this case we have:

- $c = 0.0129786$, $\mu = 4.00608 \times 10^{-8}$, $N \geqslant 406.697$, $G = 225.521$, $x_G = 0.941249$, $y_G = 0.725958$,
- $\alpha = 0.882421$, $b = 0.640601$.

Thus, it is enough to set $N = 407$, and as a result we get the following model, which approximates the process of generalized fractional Brownian motion with parameter $H = \frac{15}{16}$ with reliability 0.99 and accuracy 0.01 in $C\left([0,1]\right)$ (Fig. 4).

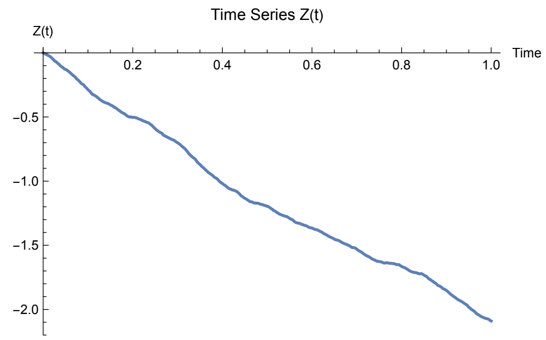

Fig. 4. The model of φ-GFBM with parameter $H = \frac{15}{16}$

5. $H = \frac{3}{4}$, $\omega = 3$. Now, let us take reliability $1 - \nu = 0.95$ and accuracy $\delta = 0.05$. This should reduce the number of terms in the model.
In this case we have:

- $c = 0.0537337$, $\mu = 2.84388 \times 10^{-6}$, $N \geqslant 732.124$ (for comparison, for reliability $1 - \nu = 0.99$ and accuracy $\delta = 0.01$ we have got $N \geqslant 6865.58$),
- $G = 276.498$, $x_G = 0.942482$, $y_G = 0.731111$, $\alpha = 0.706862$, $b = 0.516794$.

Thus, it is enough to set $N = 733$, and as a result we get the following model, which approximates the process of generalized fractional Brownian motion with parameter $H = \frac{3}{4}$ with reliability 0.95 and accuracy 0.05 in $C\left([0,1]\right)$ (Fig. 5).

6. Let us consider how the behaviour of the simulated trajectory changes in the case when $\omega = 5$, that is, when
$\varphi(x) = \begin{cases} \frac{|x|^5}{5}, & |x| \geqslant 1; \\ \frac{x^2}{5}, & |x| < 1. \end{cases}$

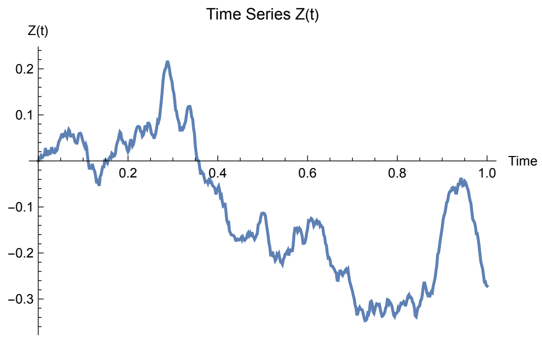

Fig. 5. The model of φ-GFBM with parameter $H = \frac{3}{4}$

The following example is given for $H = \frac{8}{9}$, reliability $1 - \nu = 0.99$ and accuracy $\delta = 0.01$.
In this case we have:

- $c = 0.0234107$, $\mu = 1.32915 \times 10^{-6}$, $N \geqslant 1284.88$, $G = 510.929$, $x_G = 0.937944$, $y_G = 0.644661$,
- $\alpha = 0.833728$, $b = 0.537471$.

Thus, it is enough to set $N = 1285$, and as a result we get the following model, which approximates the process of generalized fractional Brownian motion with parameter $H = \frac{8}{9}$, $\omega = 5$ with reliability 0.99 and accuracy 0.01 in $C([0,1])$ (Fig. 6).

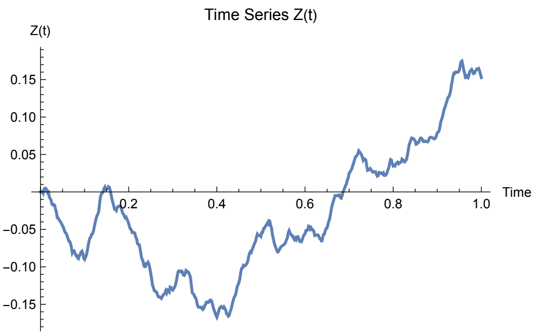

Fig. 6. The model of φ-GFBM with parameter $H = \frac{8}{9}$, $\omega = 5$

Here we see that with a larger value of the parameter ω, larger fluctuations of the simulated process are observed.

Remark 2. All calculations and simulations were performed in Wolfram Mathematica. The authors can share the script of the code with an interested reader.

Discussion and conclusions

In Tab. 1, we compare the results obtained in our work with the results obtained in the article (Vasylyk & Lovytska, 2021). We can conclude that we have found the minimum necessary number of terms N for the models in case of the optimal values of the parameters. As expected, in our case the obtained values of N are significantly smaller than the corresponding numbers for the models with the values of parameters that are not optimal.

Table 1

Comparison of the results

H	ω	$1-\nu$	δ	N in the article (Vasylyk & Lovytska, 2021)	N in the optimized model
$\frac{7}{8}$	3	0.99	0.01	1362	1043
$\frac{8}{9}$	3	0.99	0.01	1111	854
$\frac{9}{10}$	3	0.99	0.01	942	727
$\frac{15}{16}$	3	0.99	0.01	523	407
$\frac{3}{4}$	3	0.95	0.05	1006	733
$\frac{8}{9}$	5	0.99	0.01	1693	1285

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ОПТИМІЗАЦІЯ МОДЕЛІ φ-СУБГАУССОВОГО УЗАГАЛЬНЕНОГО ДРОБОВОГО БРОУНІВСЬКОГО РУХУ

Методи моделювання випадкових процесів використовують у багатьох наукових сферах, й особливу увагу приділяють дробовому броунівському руху. Ученими отримано багато результатів щодо моделювання гауссового дробового броунівського руху. У пропонованій статті розглянуто моделювання процесів із більш загального класу, а саме моделювання строго φ-субгауссового узагальненого дробового броунівського руху, визначеного на відрізку $[0, 1]$. Спочатку в роботі наведено певну попередню інформацію, необхідну для розуміння результатів нашого дослідження. В основній частині роботи для строго φ-субгауссового узагальненого дробового броунівського руху з деякою заданою функцією φ знайдено оптимальні значення параметрів, необхідних для моделювання із заданою надійністю та точністю в просторі $C([0; 1])$. Це зумовлює значне зменшення кількості доданків, необхідних для забезпечення заданої надійності та точності нашої моделі. Також у роботі наведено деякі результати розрахунків та приклади моделювання. Як й очікувалося, отримані значення кількості доданків у моделі є значно меншими за відповідні числа для моделей зі значеннями параметрів, що не є оптимальними.

К л ю ч о в і с л о в а: ***φ-субгауссовий узагальнений дробовий броунівський рух, оптимізація моделі, надійність і точність моделі, моделювання, випадковий процес.***

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