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Деякі застосування узагальнених дробових похідних

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Some applications of generalized fractional derivatives

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У роботах Кочубея (2011) і Тоальдо (2015) були представлені нові типи диференціальних операторів, названі диференціально–згортковими операторами, або похідними типу згортки відносно функцій Бернштейна. Ці похідні узагальнюють класичні дробові похідні Капуто–Джрбашяна і Рімана–Ліувілля та дозволяють, зокрема, описувати властивості субординаторів та обернених субординаторів. У сучасній літературі інтенсивно досліджуються нові класи диференціальних рівнянь із узагальненими дробовими операторами, що можуть бути використані у багатьох теоретичних та прикладних задачах, зокрема для опису аномальних дифузій та інших складних процесів. У цій статті наведено стислий огляд основних властивостей узагальнених дробових похідних та проілюстовано їх застосування у рівняннях, що визначають деякі класи випадкових полів на сфері.

Ключові слова: узагальнені похідні типу згортки, функції Бернштейна, субординатори, власні функції, диференціальні рівняння з узагальненими дробовими операторами, випадкові початкові умови, випадкові поля на сфері

The paper presents a concise summary of main properties of generalized fractional derivatives, so-called convolution type derivatives with respect to Bernštein functions. Applications are considered to modeling time dependent random fields on the sphere as solutions to partial differential equations with the generalized fractional derivative in time and random initial condition.

Key Words: generalized convolution–type derivatives, Bernštein functions, subordinators, eigenfunctions, differential equations with generalized fractional derivatives, random initial conditions, random fields on the sphere

1 Introduction

In the papers by Kochubei [8] and Toaldo [12] the new types of differential operators are presented which are related to Bernštein functions and generalize the classical Caputo-Djrbashian and Reimann-Liuville fractional derivatives. These operators are called differential-convolution operators in [8], or convolution-type derivatives with respect to Bernštein functions in [12]. It is shown in [12] that these derivatives provide the unifying framework for the study of subordinators and their inverse processes, and, in particular, the governing equations for densities of subordinators and their inverses are obtained in terms of the convolution-type derivatives. The introduction of these derivatives has inspired numerous recent studies devoted to new types of generalized fractional equations, their probabilistic interpretation and use in various applied areas, in particular, for modeling anomalous diffusions and other complicated processes.

This paper presents a concise summary of main properties of generalized fractional derivatives. We discuss, in particular, the equations for probability densities of subordinators and their inverse processes and the eigenvalue problem for generalized fractional operators. Note that the knowledge of eigenfunctions of generalized fractional operators allows to construct the representations of solutions to corresponding partial differetial equations by using the method of separation of variables.

As applications, we consider the models of time dependent random fields on the sphere arising as solutions to partial differential equations with the generalized fractional derivative in time and random initial condition represented by a Gaussian random field.

Papers [6], [5] develop the approach to

construct time dependent random fields on the sphere through coordinates change and subordination by means of partial differential equations with fractional operators of a particular form. The representations of the fields are given by the Karhunen-Loève expansion, and also as a coordinate changed random field. In [7] generalization of the results of the mentioned papers was presented via introducing the convolution–type derivatives in time into the corresponding equations. It was suggested in [7] that considering equations with the tempered fractional derivative in time can lead to the exact formulas for terms involved in the Karhunen-Loève expansion, since the formulas for the density of the inverse tempered stable subordinator are available, e.g., in [1]. In the present paper we consider the example with the tempered fractional derivative in more detail and show that the terms in the expansions of the fields can be given in the exact form.

The paper is organized as follows. Section 2 collects the basic definitions and facts on the generalized fractional derivatives. In Section 3 we consider models of random fields on the sphere driven by equations with fractional operators.

2 Generalized fractional derivatives

In this section we review the main definitions and some important facts on the generalized fractional derivatives. (For more details see, e.g., [11, 12].)

Let $f(x)$ be a Bernštein function:

$$f(x) = a + bx + \int_0^\infty \left(1 - e^{-xs}\right) \overline{\nu}(ds), \qquad (2.1)$$

$x > 0$, $a, b \geq 0$, $\overline{\nu}(ds)$ is a non-negative measure on $(0, \infty)$ (the Lévy measure for $f(x)$) such that

$$\int_0^\infty (s \wedge 1)\, \overline{\nu}(ds) < \infty.$$

The generalized Caputo-Djrbashian (C-D) derivative, or convolution-type derivative, with respect to the Bernštein function f is defined on the space of absolutely continuous functions as follows ([12], Def. 2.4):

$$\mathfrak{D}_t^f u(t) = b\frac{d}{dt}u(t) + \int_0^t \frac{\partial}{\partial t}u(t-s)\nu(s)ds, \qquad (2.2)$$

where $\nu(s) = a + \overline{\nu}(s, \infty)$ is the tail of the Lévy measure $\overline{\nu}(s)$ of the function f.

In the papers [8] and [4] the authors provided respectively the following representations of the generalized fractional derivative (notation of the authors):

$$\left(\mathbb{D}_{(k)}u\right)(t) = \frac{d}{dt}\int_0^t k(t-\tau)u(\tau)d\tau - k(t)u(0)$$

and

$$\partial_t^w f(t) = \frac{d}{dt}\int_0^t w(t-s)\big(f(s) - f(0)\big)ds,$$

where $k(\cdot)$ and $w(\cdot)$ are some fractional kernels associated with f.

In the present paper the generalized fractional derivative defined in (2.2) will be used.

In the case when $f(x) = x^\alpha, x > 0, \alpha \in (0,1)$, the derivative (2.2) becomes:

$$\mathfrak{D}_t^f u(t) = \mathfrak{D}_t^\alpha u(t),$$

where $\mathfrak{D}_t^\alpha u(t)$ is the fractional C-D derivative:

$$\mathfrak{D}_t^\alpha u(t) = \frac{d^\alpha}{dt^\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u'(s)}{(t-s)^\alpha}ds.$$

If $f(x) = x$, the convolution-type derivative coincides with the standard first-order derivative. For the Laplace transform of the derivative (2.2) the following relation holds ([12], Lemma 2.5):

$$\mathcal{L}\left[\mathfrak{D}_t^f u\right](s) = f(s)\mathcal{L}\left[u\right](s) - \frac{f(s)}{s}u(0), s > s_0,$$

for u such that $|u(t)| \leq \mathsf{M}e^{s_0 t}$, M and s_0 are some constants. Similarly to the C-D fractional derivative, the convolution type derivative can be alternatively defined via its Laplace transform.

The generalization of the classical Riemann-Liouville (R-L) fractional derivative is introduced in [12] by means of another convolution-type derivative with respect to f given by the following formula:

$$\mathbb{D}_t^f u(t) = b\frac{d}{dt}u(t) + \frac{d}{dt}\int_0^t u(t-s)\nu(s)ds.$$

The derivatives $\mathfrak{D}_t^f$ and $\mathbb{D}_t^f$ are related as follows:

$$\mathbb{D}_t^f u(t) = \mathfrak{D}_t^f u(t) + \nu(t)u(0).$$

Bernštein functions are associated in a natural way with subordinators.

Let $H(t), t \geq 0$, be a subordinator, that is, nondecreasing Lévy process. Its Laplace transform is of the form: $\mathcal{L}[H(t)](s) = \mathsf{E}e^{-sH(t)} = e^{-tf(s)}$, where the function f, called the Laplace exponent, is a Bernštein function. Consider a subordinator H with the Laplace exponent f given by (2.1), and let L be its inverse process defined as

$$L(t) = \inf \left\{ s \geq 0 : H^f(s) > t \right\}. \tag{2.3}$$

It was shown in [12] that the distribution of the inverse process L has a density $l(t,x) = \mathrm{P}\{L(t) \in dx\}/dx$ and its Laplace transform with respect to t has the form

$$\mathcal{L}_t\left(l(t,x)\right)(r) = \frac{f(r)}{r} e^{-xf(r)},$$

provided that the following condition holds: **Condition I.** $\overline{\nu}(0,\infty) = \infty$ and the tail $\nu(s) = a + \overline{\nu}(s,\infty)$ is absolutely continuous.

The convolution-type derivatives $\mathfrak{D}_t^f$ and $\mathbb{D}_t^f$ are useful tools which allow to study the properties of subordinators and their inverses in the unifying manner ([12, 11]). In particular, the governing equations for their densities can be given in terms of convolution-type derivatives. The density $l(t,u)$ of the inverse process L satisfies the following equation ([12], Theorem 4.1):

$$\mathbb{D}_t^f l(t,u) = -\frac{\partial}{\partial u} l(t,u),$$

subject to

$$l(t, u/b) = 0,\ l(t,0) = \nu(t),\ l(0,u) = \delta(u).$$

The following fact is important for deriving representations of solutions to various classes of partial differential equations involving generalized fractional derivatives.

Proposition 2.1. ([7]) *Let L be the inverse process for a subordinator with Bernštein function f, and assume that Condition I holds. The space Laplace transform of the density $l(t,x)$ of the inverse process L*

$$\tilde{l}(t,\lambda) = \int_0^\infty e^{-\lambda x} l(t,x)dx = \mathrm{E}e^{-\lambda L(t)} \tag{2.4}$$

is an eigenfunction of the operator $\mathfrak{D}_t^f$, that is

$$\mathfrak{D}_t^f \tilde{l}(t,\lambda) = -\lambda \tilde{l}(t,\lambda). \tag{2.5}$$

Remark 2.1. The proof of the above Proposition has been derived, by different approaches, in [3, 8, 11]. In particular, in [3], the authors consider the time-changed Poisson process N_{L_t} (with L_t being an inverse subordinator independent of the process N) and show that its marginal distributions $p_x(t) = \mathrm{P}(N_{L_t} = x)$, $x = 0, 1, \ldots$, satisfy the difference-differential equations

$$\mathfrak{D}_t^f p_x(t) = -\lambda\left[p_x(t) - p_{x-1}(t)\right]$$

with initial condition $p_x(0) = 1, x = 0$ and $p_x(0) = 0, x \geq 1$. Then (2.5) is deduced as a consequence of the above equation, since

$$p_0(t) = \mathrm{P}(N_{L_t} = 0) = \int_0^\infty e^{-\lambda x} l(t,x)dx = \tilde{l}(t,\lambda).$$

Tempered fractional derivative is defined by choosing in (2.2) the following Bernštein function:

$$g(x) = (x+\beta)^\alpha - \beta^\alpha,\ \alpha \in (0,1), \beta > 0, \tag{2.6}$$

the corresponding Lévy measure is given by the formula:

$$\overline{\nu}(ds) = \frac{1}{\Gamma(1-\alpha)} \alpha e^{-\beta s} s^{-\alpha-1} ds;$$

and its tail is

$$\nu(s) = \frac{1}{\Gamma(1-\alpha)} \alpha \beta^\alpha \Gamma(-\alpha, s),$$

where

$$\Gamma(\alpha, s) = \int_s^\infty e^{-z} z^{\alpha-1} dz$$

is the upper incomplete Gamma function defined for all $\alpha, s \in \mathbb{R}$ which is real-valued for $s \geq 0$.

The corresponding subordinator H is called the tempered stable subordinator.

The generalized convolution-type derivative (2.2), for g given by (2.6), becomes:

$$\begin{aligned}\mathfrak{D}_t^g u(t) &= \frac{\alpha\beta^\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{\partial}{\partial t} u(t-s)\Gamma(-\alpha, s)ds \\ &:= \mathcal{D}_t^{\alpha,\beta} u(t). \end{aligned} \tag{2.7}$$

Equations with tempered fractional derivatives represent currently a topic of intensive studies.

In particular, the following fact presented in [2] concerns the question on eigenfunctions of tempered fractional operators.

Proposition 2.2. ([2, Lemma 1]) *The solution to the tempered relaxation equation*

$$\mathcal{D}_t^{\alpha,\beta} u(t) = -\beta^\alpha u(t) \tag{2.8}$$

with initial condition $u(0) = 1$ is given by

$$\varphi(t;\alpha,\beta) = \Gamma(\alpha, \beta t)/\Gamma(\alpha), \tag{2.9}$$

thus, $\varphi(t;\alpha,\beta)$ is an eigenfunction of the operator $\mathcal{D}_t^{\alpha,\beta}$ corresponding to the eigenvalue β^α.

In the next section we consider generalized fractional derivatives for the case $a = b = 0$ in (2.1) and assuming Condition I to hold.

3 Equations with generalized fractional derivatives for modeling random fields on the sphere

Let $T(x)$, $x \in \mathbb{S}^2_1$, be a real-valued, zero-mean, isotropic Gaussian random field on the unit sphere $\mathbb{S}^2_1$. We can write the spectral representation

$$T(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} a_{lm} Y_{lm}(x), \tag{3.1}$$

where

$$a_{lm} = \int_{\mathbb{S}^2} T(x) Y^*_{lm}(x) \mu(dx) \tag{3.2}$$

are Fourier random coefficients, Y_{lm} are spherical harmonics, $\mu(dx)$ is the Lebesgue measure on the unit sphere $\mathbb{S}^2_1$. Convergence in (3.1) holds in the mean square sense, both in $L^2(dP \times \mu(dx))$ and in $L^2(dP)$ for fixed $x \in \mathbb{S}^2_1$ (see, e.g., [9]).

We recall that for a fixed l the spherical harmonics Y_{lm} (or linear combination of them) solve the eigenvalue problem

$$\Delta_{\mathbb{S}^2_1} Y_{lm} = -\mu_l Y_{lm}, \quad l \geq 0,\ |m| \leq l \tag{3.3}$$

where the eigenvalues are given by

$$\mu_l = l(l+1)$$

and $\Delta_{\mathbb{S}^2_1}$ is the spherical Laplace operator (called also Laplace-Beltrami operator).

In [7] models of random fields on the sphere are presented which are driven by equations involving the generalized fractional derivative in time $\mathfrak{D}^f_t$ and the fractional Laplacian $\psi(-\Delta_{\mathbb{S}^2_1})$ associated with Bernštein functions f and ψ respectively.

Let the function f correspond to the subordinator H, which inverse process L possesses the density l with Laplace transform $\widetilde{l}$ as introduced in Proposition 2.1 in formula (2.4).

Theorem 3.1. ([7, Theorem 1]) *The solution in $L^2(dP \times d\lambda)$ to the fractional equation*

$$\left(\gamma - \psi(-\Delta_{\mathbb{S}^2_1}) + \mathfrak{D}^f_t\right) X_t(x) = 0, \tag{3.4}$$
$$x \in \mathbb{S}^2_1,\ t \geq 0, \gamma > 0,$$

with initial condition $X_0(x) = T(x)$ is a time-dependent random field on the sphere $\mathbb{S}^2_1$ written as

$$X_t(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} a_{lm}\, \widetilde{l}(t, \gamma + \psi(\mu_l)) Y_{lm}(x) \tag{3.5}$$

where a_{lm} are given by (3.2).

The generalized Laplace operator $\psi(-\Delta_{\mathbb{S}^2_1})$ is defined in terms of the transition semigroup of the subordinate rotational Brownian motion $B^\psi(t) = B(F(t))$, where F is a subordinator with the Laplace exponent ψ. This covers, in particular, the case of fractional Laplace operator $(-\Delta_{\mathbb{S}^2_1})^\alpha$. We refer to [5, 6, 7] for more detail on this operator. To make the paper self-contained, some definitions are given in Appendix.

Note that the essential component of the proof of Theorem 3.1 is the knowledge of eigenfuntions for the operators involved into the equation. For the generalized convolution-type derivative we have (see Proposition 2.1):

$$\mathfrak{D}^f_t \widetilde{l}(t, \lambda) = -\lambda\, \widetilde{l}(t, \lambda), \quad \lambda > 0, \tag{3.6}$$

and for the generalized Laplace operator $\psi(-\Delta_{\mathbb{S}^2_1})$ we know that ([6]):

$$\psi(-\Delta_{\mathbb{S}^2_1}) Y_{lm}(x) = -\psi(\mu_l) Y_{lm}(x). \tag{3.7}$$

Example 3.1. *Equations with fractional Caputo-Djrbashian derivative.* If $f(x) = x^\beta$, that is, L is the inverse process for stable subordinator, the derivative $\mathfrak{D}^f_t$ becomes the fractional Caputo-Djrbashian derivative and the Laplace transform $\widetilde{l}(t, \mu)$ is given by the Mittag-Leffler function: $\widetilde{l}(t, \mu) = E_\beta(-t^\beta \mu)$. The one-parameter Mittag–Leffer function is defined as

$$E_\beta(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\beta k + 1)},\ x \in \mathbb{R},\ \beta > 0.$$

Theorem 3.1 reduces to first part of Theorem 1 in [6].

Example 3.2. *Equations with tempered fractional derivative.* It was suggested in [7] that equation (3.4) can be considered with the tempered fractional derivative in time, and, correspondingly, in the representation of the solution (3.5) we will have the Laplace transform $\widetilde{l}(t, \lambda)$ of the density of the inverse tempered stable subordinator, expressions for this density are given, e.g., in [1].

We present here all the details for the case of equation (3.4) with the tempered fractional derivative and give the representation of solution in the explicit form.

Following [10], define the R-L tempered fractional derivative of order $0 < \beta < 1$ as follows:

$$\mathbb{D}^{\beta,\lambda}_t u(t) = e^{-\lambda t} \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{e^{\lambda s} u(s) ds}{(t-s)^\beta} - \lambda^\beta u(t),$$

and the C-D tempered fractional derivative of order $0 < \beta < 1$:

$$\mathfrak{D}_t^{\beta,\lambda} u(t) = \mathbb{D}_t^{\beta,\lambda} u(t) - \frac{u(0)}{\Gamma(1-\beta)} \int_t^\infty e^{-\lambda r} \beta r^{-\beta-1}\, dr. \tag{3.8}$$

Let $H_\lambda(x)$ be the tempered stable subordinator with the Laplace exponent

$$f_\lambda(s) = (s+\lambda)^\beta - \lambda^\beta.$$

Consider the inverse process $L_\lambda(t)$ and denote its density by $g_\lambda(t,x)$.

It was established in [10, Lemma 3.3.] that the Laplace transform $\widetilde{g}_\lambda(t,\mu) = \mathsf{E}e^{-\mu L_\lambda(t)}$ is a solution to the eigenvalue problem for $\mathfrak{D}_t^{\beta,\lambda}$:

$$\mathfrak{D}_t^{\beta,\lambda} \widetilde{g}_\lambda(t,\mu) = -\mu \widetilde{g}_\lambda(t,\mu)$$

with $\widetilde{g}_\lambda(0,\mu) = 1$. Moreover, it was shown in [10, Theorem 3.4.] that for any $\lambda, \mu > 0$, $\mu \neq \lambda^\beta$ the function $\widetilde{g}_\lambda(t,\mu)$ can be written in the form:

$$\widetilde{g}_\lambda(t,\mu) = \frac{\mu}{\pi} \int_0^\infty \frac{e^{-t(r+\lambda)}}{r+\lambda} \Phi(r,1)\, dr, \tag{3.9}$$

where

$$\Phi(r,1) = \frac{r^\beta \sin(\beta\pi)}{r^{2\beta}\sin^2(\beta\pi) + (\mu - \lambda^\beta + r^\beta \cos(\beta\pi))^2}.$$

We present in the next theorem the solution to the equation (3.4) with tempered fractional derivative (3.8).

Theorem 3.2. *The solution in $L^2(dP \times d\lambda)$ to the fractional equation*

$$\left(\gamma - \psi(-\Delta_{\mathbb{S}_1^2}) + \mathfrak{D}_t^{\beta,\lambda}\right) X_t(x) = 0, \quad x \in \mathbb{S}_1^2,\ t \geq 0, \gamma > 0, \tag{3.10}$$

with initial condition $X_0(x) = T(x)$ given by (3.1) and tempered fractional derivative $\mathfrak{D}_t^{\beta,\lambda}$ given by (3.8) , is a time-dependent random field on the sphere $\mathbb{S}_1^2$ written as

$$X_t(x) = \sum_{l=0}^\infty \sum_{m=-l}^{+l} a_{lm} \widetilde{g}_\lambda(t, \gamma + \psi(\mu_l)) Y_{lm}(x), \tag{3.11}$$

where a_{lm} are given by (3.2) and the function $\widetilde{g}_\lambda(t,\mu)$ is given by (3.9).

Analogously to [7], the expressions for the covariance function and higher order moments of the field (3.11) can be calculated and given in terms of the angular power spectrum of the random initial condition field T and the function $\widetilde{g}_\lambda(t,\mu)$ given by (3.9), thus, all the characteristics of the field (3.11) can be evaluated explicitly. In particular, we have the following expressions for the moments of the field (3.11):

$$\mathbf{E}[(X_t(x))^n] = \sum_{l_1=0}^\infty \cdots \sum_{l_n=0}^\infty \Big(\prod_{j=1}^n \widetilde{g}_\lambda(t, \gamma + \psi(\mu_{l_j})) \Big) \times \sqrt{\frac{\prod_{j=1}^n (2l_j+1)}{(4\pi)^n}}\, \mathbf{E}[a_{l_1 0} \cdots a_{l_n 0}],$$

$$\mathbf{E}[(X_t(x))^2] = \sum_{l=0}^\infty \frac{2l+1}{4\pi} (\widetilde{g}_\lambda(t, \gamma + \psi(\mu_l)))^2 C_l,$$

where the coefficients a_{lm}, $l \geq 0$, $|m| \leq l$, are from the representation of initial condition field (3.1) and the power spectrum $C_l = \mathbf{E}|a_{lm}|^2$, $l \geq 0$.

We refer to [7] for more details on calculation of characteristic of the field (3.11).

Appendix. Generalized fractional Laplacian on the sphere

Consider $f \in L^2(\mathbb{S}_1^2) = L^2(\mathbb{S}_1^2, \mu)$, where μ is the Lebesgue measure on the unit sphere $\mathbb{S}_1^2$: $\mu(dx) = \mu(d\vartheta, d\varphi) = d\varphi\, d\vartheta \sin\vartheta$ with $x = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$, $\vartheta \in [0,\pi]$, $\varphi \in [0, 2\pi)$.

The set of spherical harmonics $\{Y_{lm} : l \geq 0,\ m = -l, \ldots, +l\}$ represents an orthogonal basis for the space $L^2(\mathbb{S}_1^2)$. For a fixed l the spherical harmonics solve the eigenvalue problem

$$\Delta_{\mathbb{S}_1^2} Y_{lm} = -\mu_l Y_{lm}, \quad l \geq 0,\ |m| \leq l,$$

where $\mu_l = l(l+1)$ and the operator

$$\Delta_{\mathbb{S}_1^2} = \frac{1}{\sin^2\vartheta} \frac{\partial^2}{\partial\varphi^2} + \frac{1}{\sin\vartheta} \frac{\partial}{\partial\vartheta} \left(\sin\vartheta \frac{\partial}{\partial\vartheta} \right), \quad \vartheta \in [0,\pi],\ \varphi \in [0,2\pi).$$

The spherical harmonics are defined as

$$Y_{lm}(\vartheta,\varphi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} Q_{lm}(\cos\vartheta) e^{im\varphi},$$

where the associated Legendre functions

$$Q_{lm}(z) = (-1)^m (1-z^2)^{m/2} \frac{d^m}{dz^m} Q_l(z),$$

and the Legendre polynomials Q_l are given by the Rodrigues' formula

$$Q_l(z) = \frac{1}{2^l l!}\frac{d^l}{dz^l}(z^2-1)^l.$$

For $f \in L^2(\mathbb{S}_1^2)$ we have the representation

$$f(x) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_{lm}Y_{lm}(x), \quad x \in \mathbb{S}_1^2,$$

which holds in the L^2 sense, where

$$\begin{aligned} f_{lm} &= \int_{\mathbb{S}_1^2} f(x)Y_{lm}^*(x)\,dx \\ &= \int_0^{2\pi}\int_0^{\pi} f(\vartheta,\varphi)Y_{lm}^*(\vartheta,\varphi)\sin\vartheta\,d\vartheta\,d\varphi, \\ &|m| \le l,\ l = 0,1,2,\dots. \end{aligned}$$

(see, e.g., the Peter-Weyl representation theorem on the sphere in [9]).

The angular power spectrum of f is defined as

$$f_l = \sum_{|m|\le l}|f_{lm}|^2, \quad l = 0,1,2,\dots. \tag{3.12}$$

Define the generalized fractional Laplace operators on the sphere following [5], [6].

Let $F(t)$, $t \ge 0$ be a Lévy subordinator with the Laplace exponent

$$\Psi(\lambda) = b\lambda + \int_0^{\infty}\left(1-e^{-\lambda z}\right)M(dz),\ b \ge 0,\ \lambda \ge 0,$$

with M being the corresponding Lévy measure.

Let B_t, $t \ge 0$, be a Brownian motion on the unit sphere $\mathbb{S}_1^2$. Its transition density can be written as follows ([5, 6]):

$$\begin{aligned} Pr\{x+B_t \in dy\}/dy &= Pr\{B_t \in dy \,|\, B_0 = x\}/dy \\ &= \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} e^{-t\mu_l}\mathcal{Y}_{l,m}(y)\mathcal{Y}_{l,m}^*(x). \end{aligned}$$

Consider the the initial-value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta_{\mathbf{S}_1^2}u, & x \in \mathbf{S}_1^2,\ t > 0 \\ u(x,0) = f(x) \end{cases}$$

for $f \in L^2(\mathbf{S}_1^2)$. The solution to the above problem can be written as follows:

$$\begin{aligned} u(x,t) &= P_t f(x) = \mathbb{E}f(x+B_t) \\ &= \int_{\mathbf{S}_1^2} f(y)Pr\{x+B_t \in dy\} \\ &= \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} e^{-t\mu_l}\mathcal{Y}_{l,m}(x)f_{lm}. \end{aligned}$$

In [5] the following operator acting on $f \in L^2(\mathbf{S}_1^2)$ was introduced:

$$\Psi(-\Delta_{\mathbf{S}_1^2})f(x) := \int_0^{\infty}\left(P_t f(x) - f(x)\right)M(dt).$$

It was shown in [5] (see also [6]) that

$$\Psi(-\Delta_{\mathbf{S}_1^2})Y_{lm}(x) = -\Psi(\mu_l)Y_{lm}(x),$$

thus, $Y_{lm}(x)$ are the eigenfunctions of the operator $\Psi(-\Delta_{\mathbf{S}_1^2})$ with the eigenvalues $-\Psi(\mu_l)$.

Basing on the above fact, the action of the operator $\Psi(-\Delta_{\mathbf{S}_1^2})$ can be also defined by means of a series representation as given below.

Let us consider the space of functions

$$H^s(\mathbb{S}_1^2) = \left\{ f \in L^2(\mathbb{S}_1^2) : \sum_{l=0}^{\infty}(2l+1)^{2s}f_l < \infty \right\},$$

where f_l is the angular spectrum of f given by (3.12).

Definition. Let $f \in H^s(\mathbb{S}_1^2)$ and $s > 5/4$. Then

$$\Psi(-\Delta_{\mathbf{S}_1^2})f(x) := \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} f_{lm}Y_{lm}(x)\Psi(\mu_l).$$

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