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ON PROPERTIES OF THE HASSE DIAGRAMS OF NP-CRITICAL POSETS OF ORDER LESS THAN 8

M. Kleiner proved that a poset has finite representation type if and only if it does not contain subposets of the form (1, 1, 1, 1), (2, 2, 2), (1, 3, 3), (1, 2, 5) and (И, 4), which are called the critical posets. L. Nazarova proved that a poset is tame if and ohly if it does not contain subposets of the form (1, 1, 1, 1, 1), (1, 1, 1, 2), (2, 2, 3), (1, 3, 4), (1, 2, 6), (И, 5), called supercritical. The critical and supercritical posets are also (respectively) critical with respect to weakly positivity and weakly non-negativity of the associated Tits quadratic form. In the case of substitution on positivity and nonnegativity, such posets which are called P-critical and NP-critical, respectively, were described by the author together with V. Bondarenko (their number, up to isomorphism and duality, is 75 and 115). This article studies combinatorial properties of the Hasse diagrams of NP-critical posets.

Keywords: *poset, critical and supercritical poset, duality, Hasse diagram, minimax equivalence.*

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Introduction

In the work M. Kleiner proved that a poset is of finite representation type if and only if it does not contain the subposets of the form (1, 1, 1, 1), (2, 2, 2), (1, 3, 3), (1, 2, 5), (И, 4), these posets are called the critical posets (Kleiner, 1972). In the work (Drozd, 1974) Yu. A. Drozd proved that a poset has finite representation type if and only if its Tits qadratic form is weakly positive. In the work (Bondarenko, & Styopochkina, 2005) it is proved that a poset is *P*-critical (i.e. critical with respect to the positivity of the Tits quadratic form) if and only if it is minimax isomorphic to a critical poset (the minimax isomorphism was introduced in the work (Bondarenko, 2005)), all such posets were described (Bondarenko, & Styopochkina, 2005).

In the work (Nazarova, 1975), L. Nazarova proved that a poset is tame if and only if it does not contain subposets of the form (1, 1, 1, 1, 1), (1, 1, 1, 2), (2, 2, 3), (1, 3, 4), (1, 2, 6), (И, 5), these posets are called supercritical. In the work (Bondarenko, & Styopochkina, 2008) it is proved that a poset is *NP*-critical (i.e. critical with respect to the non-negativity of the Tits quadratic form) if and only if it is minimax isomorphic to a supercritical poset, all such critical posets were described in the work (Bondarenko, & Styopochkina, 2009).

In connection with the above, it is actual the study of combinatorial properties of *P*-critical and *NP*-critical posets. In this paper we continue the investigation of the paper (Bondarenko et al., 2015) on *P*-critical posets, namely, consider the case of *NP*-critical posets.

The main methods are standard methods of combinatorial analysis.

1. Main results. Let *S* be a finite poset. For *S* (without an element denoted by 0), the Tits quadratic form is defined by the following equation (Drozd, 1974):

$$q_S(z) = z_0^2 + \sum_{i\in S} z_i^2 + \sum_{i<j, i,j\in S} z_i z_i - z_0 \sum_{i\in S} z_i. \qquad (1)$$

The poset *S* is called critical with respect to nonnegativity of the Tits quadratic form or, briefly, *NP*-critical if the Tits form of any its proper subposet is non-negative but the Tits form of *S* is not non-negative (Bondarenko, & Styopochkina, 2005). Such posets were described in the work (Bondarenko, & Styopochkina, 2009) (their number, up to isomorphism and duality, is 115). The set of all *NP*-critical posets of order less than m is denoted by $NP_{<m}$.

We denote by *H*(*S*) the Hasse diagram of a poset *S*. For a class of finite posets χ we denote by $VA(\chi)$ the set of pairs (s, k) of non-negative integer numbers, such that s and k are respectively the number of vertices and edges of *H*(*X*) for an $X \in \chi$.

Theorem 1. $VA(NP_{<8})$ *consists of the following pairs*:

(5, 0), (5, 1), (5, 3), (5, 4), (5, 5), (5, 6), (7, 4), (7, 5), (7, 6), (7, 7).

From the theorem it follows the following statements.

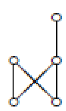
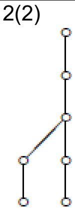
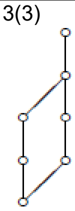
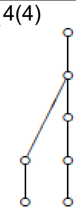
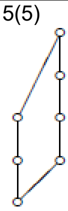
Corollary 1. *Let* (s, i), $(s, j) \in VA(NP_{<8})$ *and* $i < k < j$. *If s is not equal to* 5 (*the smallest first coordinate for the pairs of* $VA(NP_{<8})$), *then* $(s, k) \in VA(NP_{<8})$.

Corollary 2. *Let* $(s, k) \in VA(NP_{<8})$. *If* s *is not equal to* 5 (*the smallest coordinate for the pairs of* $VA(NP_{<8})$), *then* $s > k$.

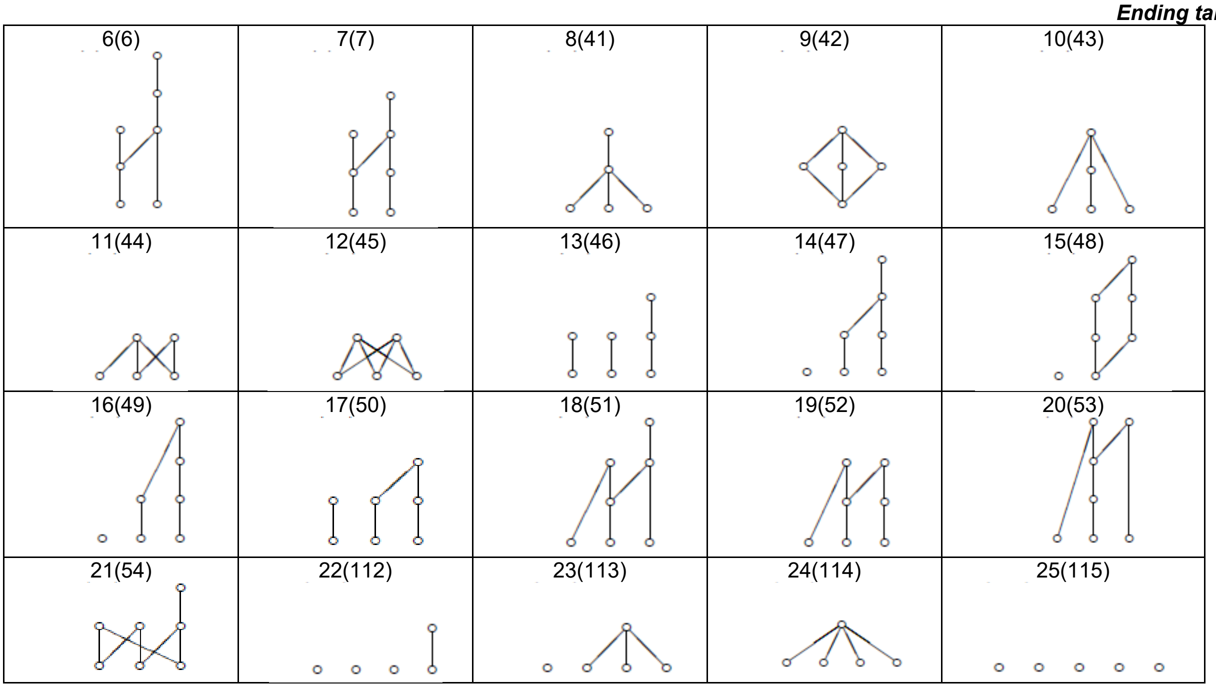
***NP*-critical posets of order less than 8.** The *NP*-critical posets were classified in the work (Bondarenko, & Styopochkina, 2008) and those of order less than 8 are given (up to isomorphism and duality) by the following Table 1.

Table 1

***NP*-critical posets of order less than 8**

1(1)	2(2)	3(3)	4(4)	5(5)
				

Ending table 1



The numbers in parentheses for the Hasse diagrams in the table 2 indicate the numbers in the general table of the work (Bondarenko, & Styopochkina, 2009).

2. Proofs. The theorem follows from the following table in which one indicates the *NP*-critical posets numbers, their numbers of vertices and edges.

Table 2

The set of pairs $VA(NP_{<8})$

	Vertices	Edges		Vertices	Edges		Vertices	Edges
1	5	5	9	5	6	18	7	6
2	7	6	10	5	4	19	7	6
3	7	7	11	5	5	20	7	6
4	7	6	12	5	6	21	7	7
5	7	7	13	7	4	22	5	1
6	7	6	14	7	5	23	5	3
7	7	6	15	7	6	24	5	4
8	5	4	16	7	5	25	5	0
			17	7	5			

3. Examples. Consider Theorem 1 for the case of critical sets of order less than 7. It follows from Theorem 1 that $VA(NP_{<8})$ *consists of the following pairs*:

(5, 0), (5, 1), (5, 3), (5, 4), (5, 5), (5, 6).

Discussion and conclusions

In this article we study combinatorial aspects of *NP*-critical of order less than 8. The obtained results (together with the corresponding research methods) will be used in the study of combinatorial aspects of other classes of posets.

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ПРО ВЛАСТИВОСТІ ДІАГРАМИ ХАССЕ *NP*-КРИТИЧНИХ ЧАСТКОВО ВПОРЯДКОВАНИХ МНОЖИН ПОРЯДКУ, МЕНШОГО ЗА 8

М. Клейнер довів, що частково впорядкована (скорочено ч. в.) множина має скінченний зображувальний тип тоді і лише тоді, коли вона не містить ч. в. підмножин вигляду (1, 1, 1, 1), (2, 2, 2), (1, 3, 3), (1, 2, 5) і (И, 4), яких називають критичними ч. в. множинами. Л. Назарова з'ясувала, що ч. в. множина є ручною тоді і лише тоді, коли вона не містить ч. в. підмножин вигляду множин (1, 1, 1, 1, 1), (1, 1, 1, 2), (2, 2, 3), (1, 3, 4), (1, 2, 6) і (И, 5), яких називають суперкритичними. Критичні і надсуперкритичні ч. в. множини є також (відповідно) критичними відносно слабкої додатності і слабкої невід'ємності асоційованої квадратичної форми Тітса. У випадку заміни на додатність та невід'ємність такі ч. в. множини, яких називають відповідно Р-критичними та NP-критичними, описані автором разом з В. Бондаренком (їх число, з точністю до ізоморфізму і дуальності, дорівнює 75 і 115).

У пропонованій статті досліджено комбінаторні властивості діаграм Хассе NP-критичних ч. в. множин.

Ключові слова: *ч. в. множина, критична та суперкритична ч. в. множина, дуальність, діаграма Хассе, мінімаксна еквівалентність.*

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