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SIMULATION OF THE FRACTIONAL BROWNIAN PROCESS WITH GIVEN ACCURACY AND RELIABILITY

Random process theory is being used more and more in various fields of science due to the high computing power of modern computers. However, it's often important to know how much we can rely on the models we use. This paper examines the modelling of the fractional Brownian motion with given accuracy and reliability. The modelling is based on Dzhaparidze and van Zanten series representation of the fractional Brownian motion. We consider the fractional Brownian motion as input process to a time-invariant linear system with a real-valued square-integrable impulse response function, which is defined on the finite domain $\tau \in [0, T]$. We prove the theorem that gives the conditions, specifically the value of the upper limit of the summing in the model, under which the obtained model approximates fractional Brownian motion with given accuracy and reliability taking into account the response of the system. For the proof, we use the properties of square-Gaussian stochastic processes.

Keywords: ***fractional Brownian motion, simulation, accuracy, reliability, Linear time-invariant system (LTI).***

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Introduction

The importance of modeling stochastic processes is growing in various fields such as finance, medicine, physics, ecology, and sociology. With modern computing capabilities, the simulation of stochastic processes is efficient, but assessing the quality and reliability of the results is crucial to avoid distorting further considerations in the study. For this reason, methods and models are developed to approximate the modeling of stochastic processes with predetermined accuracy and reliability. In papers (Kozachenko et al., 2016; Kozachenko et al., 2005; Rozora, & Lyzhechko, 2018) the problem of simulation of Gaussian stochastic processes with given accuracy and reliability are considered, including modelling with respect to the response of a linear system.

This paper considers the simulation of fractional Brownian motion processes. In the work (Dzhaparidze, & Zanten, 2004) the series representation of fractional Brownian motion can be found, which is used to construct the mathematical model of the process. These processes are considered as input processes to a time-invariant linear system with a real-valued square-integrable impulse response function.

The aim of the research is to find conditions under which the obtained model approximates fractional Brownian motion with given accuracy and reliability taking into account the response of the system.

To simulate the fractional Brownian motion, display the process sample path, and find the necessary conditions the software environment for statistical computing and graphics R was used.

1. Fractional Brownian motion

Consider fractional Brownian motion. It is Gaussian stochastic process with continuous time $t \in [0, T]$, which starts at zero, has expectation zero for all t in $[0, T]$, and has the following covariance function:

$$R(t, s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t - s|^{2H}\right),$$

where H is a real number in $(0,1)$, called the Hurst index. The value of H determines the behavior of the fractional Brownian process:

a) if $H > \frac{1}{2}$ then the increments of the process are positively correlated;

b) if $H < \frac{1}{2}$ then the increments of the process are negatively correlated;

c) if $H = \frac{1}{2}$ then the process is a Brownian motion with independent increments (or standard Wiener process)

2. A series representation for the fractional Brownian motion

Consider fractional Brownian process B_t with Hurst index H in $C([0, T])$ space.

Let J_ν be the Bessel function of the first kind of order ν, then

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{\nu+2n}}{\Gamma(n+1)\Gamma(\nu+n+1)},$$

where $x > 0, \nu \neq -1, -2, \ldots$ and $\Gamma(z)$ denotes the Euler Gamma function $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$.

For $\nu > -1$ the Bessel function of the first kind J_ν has a countable number of real positive zeros tending to infinity. The process B_t can be represented as the following sum of a series (Dzhaparidze, & Zanten, 2004):

$$B_t = \sum_{n=1}^{\infty} \frac{\sin(x_n t)}{x_n} X_n + \sum_{n=1}^{\infty} \frac{1 - \cos(y_n t)}{y_n} Y_n,$$

where X_n, Y_n are independent zero mean Gaussian random variables with variances depending on H and n, x_n is the nth positive real zero of Bessel function of the first kind J_{-H}, y_n is the nth positive real zero of Bessel function of the first kind J_{1-H}.

The process B_t can be represented in the following form (Kozachenko, Sottinen, & Vasylyk, 2005):

$$X(t) = \sum_{k=1}^{\infty} \left(c_n \sin(x_k t)\, \xi_k + d_k (1 - \cos(y_k t)) \eta_k\right), \tag{2.1}$$

where ξ_k, η_k are independent and identically distributed zero mean Gaussian random variables with $E\xi_k^2 = E\eta_k^2 = E\xi_k\eta_k = 1$, $k > 0, l > 0$ and

$$c_k = \frac{\pi^H \sqrt{2c}}{x_k^{H+1} J_{1-H}(x_k)}, \qquad k = 1,2,\dots,$$
$$d_k = \frac{\pi^H \sqrt{2c}}{y_k^{H+1} J_{-H}(y_k)}, \qquad k = 1,2,\dots,$$
$$c = \frac{\Gamma(2H+1)\sin(\pi H)}{\pi^{2H+1}}. \tag{2.2}$$

Further in the paper, we will use the representation (3.1) for the fractional Brownian process.

3. Fractional Brownian motion approximation

Consider a time-invariant linear system with a real-valued square-integrable impulse response function $H(\tau)$, which is defined on a finite domain $\tau \in [0, T]$. Then the response of the system to an input signal $X(t)$ has the following form:

$$Y(t) = \int_0^T H(\tau) X(t-\tau)\, d\tau, t \in [0,T]. \tag{3.1}$$

Suppose that the impulse response function $H(\tau)$ is known and that the input signal in system (3.1) is a stochastic process (2.1). Then from (2.1) and (3.1) it follows that the response of the system $Y(t)$ can be presented as

$$Y(t) = \sum_{k=0}^{\infty} \left(\xi_k m_k(t) + \eta_k s_k(t)\right), \tag{3.2}$$

where $m_k(t) = c_k \int_0^T H(\tau)\sin\left(x_k(t-\tau)\right) d\tau,\ \ s_k(t) = d_k \int_0^T H(\tau)\left(1-\cos\left(y_k(t-\tau)\right)\right) d\tau$.

Definition 3.1. A random process $X_N(t)$ is called a model of the process $X(t)$ if

$$X_N(t) = \sum_{k=1}^{N} \left(c_n \sin(x_k t)\, \xi_k + d_k (1 - \cos(y_k t))\eta_k\right).$$

If the model $X_N(t)$ is an input signal of a linear system (4.1), then the output process is given by

$$Y_N(t) = \sum_{k=0}^{N} \left(\xi_k m_k(t) + \eta_k s_k(t)\right).$$

Denote $\xi_N(t)$ as the sum of the squared differences $X(t) - X_N(t)$ and $Y(t) - Y_N(t)$:

$$\xi_N(t) = \left(X(t) - X_N(t)\right)^2 + \left(Y(t) - Y_N(t)\right)^2. \tag{3.3}$$

Definition 3.2. The model $X_N(t)$ approximates a stochastic process $X(t)$ taking into account the response of system (3.1), with given reliability $1 - \nu, \nu \in (0,1)$, and accuracy $\delta > 0$ in the space $C([0,T])$, if

$$P\left\{\sup_{t\in[0,T]} |\xi_N(t) - E\xi_N(t)| > \delta\right\} < \nu.$$

Further we will find conditions under which the model $X_N(t)$ approximates the process $X(t)$ with given accuracy and reliability by taking into account the response of the system (3.1).

4. Square-Gaussian stochastic processes

To find conditions under which the model $X_N(t)$ approximates the process $X(t)$ with given accuracy and reliability let us give some definitions of square-Gaussian stochastic processes.

Assume (T, ρ) is a compact metric space with metric ρ.

Definition 4.1. Let $\Xi = \{\xi_t, t \in T\}$ be a family of centered joint Gaussian random variables. A space $SG_\Xi(\Omega)$ is a space of square-Gaussian random variables if for any element $\eta \in SG_\Xi(\Omega)$ there exists a real-valued matrix $A_{n\times n}$ and random vector $\bar{\xi}^T = (\xi_1, \xi_2, \dots, \xi_n), \xi_k \in \Xi, k = \overline{1,n}$ such that $\eta = \bar{\xi}^T A \bar{\xi} - E\bar{\xi}^T A \bar{\xi}$, or $\eta \in SG_\Xi(\Omega)$ is a square-mean limit of sequence $\eta = \lim_{n\to\infty} \left(\bar{\xi}^T A \bar{\xi} - E\bar{\xi}^T A \bar{\xi}\right)$.

Definition 4.2. A stochastic process $\xi(t), t \in [0,T]$, is a square-Gaussian if for any fixed $t \in [0,T]$, a random variable $\xi(t)$ belongs to the space $SG_\Xi(\Omega)$ and $\sup_{t\in[0,T]} |\xi(t)| < \infty$.

The following theorem gives us conditions we will use further for modelling process $X(t)$ with given accuracy and reliability.

Theorem 4.3. (Kozachenko et al., 2016) Assume that $\xi(t), t \in [0,T]$, is a separable square-Gaussian stochastic process and

$$\sup_{|t-s|<h} \sqrt{D\left(\xi(t) - \xi(s)\right)} \le \sigma(h) = kh^\alpha, \alpha \in (0,1],$$

where k is some constant. Then, for $x > \frac{2\sqrt{2}\max\left\{\delta_0, k\left(\frac{T}{2}\right)^\alpha\right\}}{\alpha}$, the inequality

$$P\left\{\sup_{t\in[0,T]} |\xi(t)| > x\right\} < 4e^{\frac{3}{\alpha}} \exp\left\{-\frac{x}{2\sqrt{2}\delta_0}\right\} \times \left(\frac{x\alpha}{2\sqrt{2}\delta_0}\right)^{\frac{2}{\alpha}} \left(1 + \frac{2x}{\sqrt{2}\delta_0}\right)^{\frac{1}{2}},$$

holds true, where $\delta_0 = \sup_{t\in[0,T]} \left(D\left(X(t)\right)\right)^{1/2}$.

5. Simulation of fractional Brownian motion with given accuracy and reliability

It is easy to show that the stochastic process $\xi_N(t) - E\xi_N(t)$ is a stochastic square-Gaussian process with $\xi_N(t)$ from (3.3). Then we can use Theorem 4.3.

Given (2.1), (3.2), (3.3), the process $\xi N(t)$ can be represented as

$$\xi_N(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left(\phi_{kl}^1 \xi_k \xi_l + \phi_{kl}^2 \xi_k \eta_l + \phi_{kl}^3 \eta_k \eta_l\right),$$

where

$$\begin{cases} \phi_{kl}^1 = \phi_{kl}^1(t) = c_k c_l \sin(x_k t)\sin(x_l t) + m_k m_l, \\ \phi_{kl}^2 = \phi_{kl}^2(t) = c_k d_l \sin(x_k t)\,(1-\cos(y_l t)) + m_k s_l, \\ \phi_{kl}^3 = \phi_{kl}^3(t) = d_k d_l (1-\cos(y_k t))(1-\cos(y_l t)) + s_k s_l. \end{cases} \tag{5.1}$$

Let us introduce the following differences

$$\Delta\phi_{kl}^1 = \phi_{kl}^1(t) - \phi_{kl}^1(s),$$
$$\Delta\phi_{kl}^2 = \phi_{kl}^2(t) - \phi_{kl}^2(s),$$
$$\Delta\phi_{kl}^3 = \phi_{kl}^3(t) - \phi_{kl}^3(s).$$

Lemma 5.1. Let $\xi_N(t)$ be a stochastic process from (3.3) (Rozora, & Lyzhechko, 2018). Then

$$E\xi_N(t) = \sum_{k=N+1}^{\infty} \left(\phi_{kk}^1(t) + \phi_{kk}^3(t)\right),$$

$$D\xi_N(t) = \sum_{k,l=N+1}^{\infty} \left(2\left(\phi_{kl}^1(t)\right)^2 + \left(\phi_{kl}^2(t)\right)^2 + 2\left(\phi_{kl}^3(t)\right)^2\right),$$

$$D\big(\xi_N(t) - \xi_N(s)\big) = \sum_{k,l=N+1}^{\infty} \left(2(\Delta\phi_{kl}^1)^2 + \left(\Delta\phi_{kl}^2\right)^2 + 2\left(\Delta\phi_{kl}^3\right)^2\right). \tag{5.2}$$

Consider the following conditions:

Condition A. There exists a constant $q > 0$ such that $|H(\tau)| < q$ on the domain $[0, T]$.

Condition B. The following integral is convergent $I_H = \int_0^T H^2(\tau)d\tau < \infty$.

Condition C. $\sum_{k=N+1}^{\infty} c_k^2 x_k^{2l} < \infty, \sum_{k=N+1}^{\infty} c_k^2 x_k^{2l} < \infty, \alpha \in (0,1]$.

Lemma 5.2. Suppose that conditions A, B, and C are satisfied. Then

$$E\xi_N(t) \le (1 + I_H T) \sum_{k=N+1}^{\infty} (c_k^2 + 4d_k^2), \tag{5.3}$$

$$\begin{aligned} D\xi_N(t) \le{}& \left(1 + \frac{1}{4}T^2 I_H^2\right)\left(\sum_{k=N+1}^{\infty} c_k^2\right)^2 + \left(8q^2 + \frac{1}{4}I_H^2\right)\left(\sum_{k=N+1}^{\infty} \frac{c_k^2}{x_k}\right)^2 + \frac{T}{2}I_H^2\left(\sum_{k=N+1}^{\infty} \frac{c_k^2}{x_k}\right)\left(\sum_{k=N+1}^{\infty} c_k^2\right) + \\ &+(16 + 3T^2 I_H^2)\left(\sum_{k=N+1}^{\infty} c_k^2\right)\left(\sum_{k=N+1}^{\infty} d_k^2\right) + (96q^2 + 9I_H^2)\left(\sum_{k=N+1}^{\infty} \frac{c_k^2}{x_k}\right)\left(\sum_{k=N+1}^{\infty} \frac{d_k^2}{y_k}\right) + \\ &+(16q^2 T + 3T I_H^2)\left(\sum_{k=N+1}^{\infty} \frac{c_k^2}{x_k}\right)\left(\sum_{k=N+1}^{\infty} d_k^2\right) + 9T I_H^2\left(\sum_{k=N+1}^{\infty} c_k^2\right)\left(\sum_{k=N+1}^{\infty} \frac{d_k^2}{y_k}\right) + \\ &+\left(1 + 2q^2T^2 + \frac{9}{4}I_H^2 T^2\right)\left(\sum_{k=N+1}^{\infty} d_k^2\right)^2 + \left(72q^2 + \frac{81}{4}I_H^2\right)\left(\sum_{k=N+1}^{\infty} \frac{d_k^2}{y_k}\right)^2 + \\ &+\left(24q^2T + \frac{27}{2}T I_H^2\right)\left(\sum_{k=N+1}^{\infty} \frac{d_k^2}{y_k}\right)\left(\sum_{k=N+1}^{\infty} d_k^2\right) := \delta_0(N), \end{aligned} \tag{5.4}$$

$$\sqrt{D\big(\xi_N(t) - \xi_N(s)\big)} \le K(N)|t-s|^{\alpha}, \alpha \in (0,1],$$

where

$$\begin{aligned} K(N) = 2^{2-\alpha}\Bigg[&\sum_{k=N+1}^{\infty} c_k^2 x_k^{2\alpha} \sum_{k=N+1}^{\infty} c_k^2 + \left(\sum_{k=N+1}^{\infty} c_k^2 x_k^{\alpha}\right)^2 + 4(1+q^2T)^2 \sum_{k=N+1}^{\infty} c_k^2 x_k^{2\alpha}\, k = N + 1\infty dk2 + \\ &+4(1+q^2T)\sum_{k=N+1}^{\infty} c_k^2 x_k^{\alpha} \sum_{k=N+1}^{\infty} d_k^2 y_k^{\alpha} + \sum_{k=N+1}^{\infty} c_k^2 \sum_{k=N+1}^{\infty} d_k^2 y_k^{2\alpha} + 2\sum_{k=N+1}^{\infty} d_k^2 y_k^{2\alpha} \sum_{k=N+1}^{\infty} d_k^2 + \\ &+2\left(\sum_{k=N+1}^{\infty} d_k^2 y_k^{\alpha}\right)^2 8q^2\Bigg[\left(\sum_{k=N+1}^{\infty} c_k^2 x_k^{2\alpha-1}\right)\left(\sum_{k=N+1}^{\infty} c_k^2 x_k^{-1}\right) + \left(\sum_{k=N+1}^{\infty} c_k^2 x_k^{\alpha-1}\right)^2 + \\ &+2(1+q^2T)\left(\sum_{k=N+1}^{\infty} c_k^2 x_k^{2\alpha-1}\right)\left(\sum_{k=N+1}^{\infty} d_k^2 y_k^{-1}\right) + (3+2q^2T)\left(\sum_{k=N+1}^{\infty} c_k^2 x_k^{\alpha-1}\right)\left(\sum_{k=N+1}^{\infty} d_k^2 y_k^{\alpha-1}\right) + \\ &+\left(\sum_{k=N+1}^{\infty} c_k^2 x_k^{-1}\right)\left(\sum_{k=N+1}^{\infty} d_k^2 y_k^{2\alpha-1}\right) + \left(\sum_{k=N+1}^{\infty} d_k^2 y_k^{2\alpha-1}\right)\left(\sum_{k=N+1}^{\infty} d_k^2 y_k^{-1}\right) + \left(\sum_{k=N+1}^{\infty} d_k^2 y_k^{\alpha-1}\right)^2\Bigg] + 16q^4 + \\ &+2q^4T^2\left[\left(\sum_{k=N+1}^{\infty} d_k^2 y_k^{2\alpha-2}\right)\left(\sum_{k=N+1}^{\infty} d_k^2\right) + \left(\sum_{k=N+1}^{\infty} d_k^2 y_k^{\alpha-1}\right)^2\right]\Bigg]^{\frac{1}{2}}. \end{aligned} \tag{5.5}$$

Proof. From Lemma 5.1 follows

$$E\xi_N(t)=\sum_{k=N+1}^{\infty}\left(\phi_{kk}^1(t)+\phi_{kk}^3(t)\right).$$

Using the values $\phi_{kk}^1(t),\phi_{kk}^3(t)$ from (6.1), we have

$$E\xi_N(t)=\sum_{k=N+1}^{\infty}\left(c_k^2\sin^2(x_kt)+m_k^2(t)+d_k^2(t)(1-\cos(y_kt))^2+s_k^2(t)\right)\le\sum_{k=N+1}^{\infty}\left(c_k^2+m_k^2(t)+4d_k^2+m_k^2(t)\right).$$

Now we evaluate $m_k^2(t)+s_k^2(t)$. We obtain

$$m_k^2(t)+s_k^2(t)=c_k^2\left(\int_0^T H(\tau)\sin\big(x_k(t-\tau)\big)d\tau\right)^2+d_k^2\left(\int_0^T H(\tau)\left(1-\cos\big(y_k(t-\tau)\big)\right)d\tau\right)^2\le$$

$$\le\int_0^T H^2(\tau)d\tau\left[\int_0^T\left(c_k^2\sin^2\big(x_k(t-\tau)\big)+4d_k^2\sin^4\left(\frac{y_k}{2}(t-\tau)\right)\right)d\tau\right]\le I_HT[c_k^2+4d_k^2].$$

This proves the estimate (5.3) for $E\xi_N(t)$.

Now let us prove estimate for $D\xi_N(t)$. From Lemma 6.1, it follows that we should estimate $\left(\phi_{kl}^i(t)\right)^2, i=1,2,3.$

For $\left(\phi_{kl}^1(t)\right)^2$ we obtain

$$\left(\phi_{kl}^1(t)\right)^2=c_k^2c_l^2\sin^2(x_kt)\sin^2(x_lt)+2c_kc_l\sin(x_kt)\sin(x_lt)m_k(t)m_l(t)+m_k^2(t)m_l^2(t).$$

It is clear that $\sin^2(x_kt)\sin^2(x_lt)\le 1$. Also,

$$\sin(x_kt)m_k(t)=c_k\int_0^T H(t)\sin(x_kt)\sin\big(x_k(t-\tau)\big)d\tau==\frac{c_k}{2}\int_0^T H(t)\left(\cos(x_k\tau)-\cos\big(x_k(2t-\tau)\big)\right)\le\frac{c_kq}{2}\frac{4}{x_k}=\frac{2qc_k}{x_k}.$$

Here we used such estimate

$$\int_0^T\cos(x_k\tau)d\tau=\frac{\sin(x_k\tau)}{\tau}\bigg|_0^T\le\frac{2}{x_k}.$$

$$m_k^2(t)m_l^2(t)=c_k^2\left|\int_0^T H(\tau)\sin\big(x_k(t-\tau)\big)d\,\tau\right|^2\le\frac{c_k^2I_H}{2}\int_0^T\left(1-\cos\big(2x_k(t-\tau)\big)\right)d\tau\le\frac{c_k^2I_H}{2}\left(T+\frac{1}{x_k}\right).$$

Finally, we have

$$\left(\phi_{kl}^1(t)\right)^2\le c_k^2c_l^2\left(1+\frac{8q^2}{x_kx_l}+I_H^2\left(\frac{T}{2}+\frac{1}{2x_k}\right)\left(\frac{T}{2}+\frac{1}{2x_l}\right)\right).\tag{5.6}$$

For $\left(\phi_{kl}^2(t)\right)^2$ we have

$$\left(\phi_{kl}^2(t)\right)^2=4\left[c_k^2d_l^2\sin^2(x_kt)\big(1-\cos(y_lt)\big)^2+2c_kd_l\sin(x_kt)\big(1-\cos(y_lt)\big)m_k(t)s_l(t)+m_k^2(t)s_l^2(t)\right].$$

We will use the following estimates

$$\sin^2(x_kt)\big(1-\cos(y_lt)\big)^2\le 4.$$

$$(1-\cos(y_lt))s_l(t)=d_l\int_0^T H(\tau)\left(1-\cos\big(y_l(t-\tau)\big)\right)(1-\cos(y_lt))d\tau\le$$

$$\le qd_l\int_0^T\left(1-\cos(y_lt)-\cos\big(y_l(t-\tau)\big)+\frac{1}{2}\left(\cos\big(y_l(2t-\tau)\big)+\cos(y_lt)\right)\right)d\tau\le qd_l\left(T+\frac{6}{y_l}\right).$$

$$s_l^2(t)=d_l^2\left[\int_0^T H(\tau)\left(1-\cos\big(y_l(t-\tau)\big)\right)d\tau\right]^2\le d_l^2\int_0^T H^2(\tau)d\,\tau\int_0^T\left(1-2\cos\big(y_l(t-\tau)\big)+\cos^2\big(y_l(t-\tau)\big)\right)d\tau\le$$

$$\le d_l^2I_H\int_0^T\left(1-2\cos\big(y_l(t-\tau)\big)+\frac{1}{2}\left(1+\cos\big(2y_l(t-\tau)\big)\right)\right)d\tau\le d_l^2I_H\left(\frac{3}{2}T+\frac{9}{2y_l}\right).$$

We get

$$\left(\phi_{kl}^2(t)\right)^2\le 4c_k^2d_l^2\left[4+\frac{4q^2}{x_k}\left(T+\frac{6}{y_l}\right)+\frac{I_H^2}{2}\left(T+\frac{1}{x_k}\right)\left(\frac{3T}{2}+\frac{9}{2y_l}\right)\right].\tag{5.7}$$

In the same way we obtain the estimate $\left(\phi_{kl}^3(t)\right)^2$

$$\left(\phi_{kl}^3(t)\right)^2\le d_k^2d_l^2\left[1+2q^2\left(T+\frac{6}{k}\right)\left(T+\frac{6}{y_l}\right)+I_H^2\left(\frac{3T}{2}+\frac{9}{2y_k}\right)\left(\frac{3T}{2}+\frac{9}{2y_l}\right)\right].\tag{5.8}$$

The estimate for $D\xi_N(t)$ follows from (6.6), (6.7), (6.8).

Finally, we should estimate $\left(D\big(\xi_N(t)-\xi_N(s)\big)\right)^{\frac{1}{2}}$. From Lemma 6.1, it follows that it is enough to evaluate

$$\left(\Delta\phi_{kl}^i\right)^2=\left(\phi_{kl}^i(t)-\phi_{kl}^i(s)\right)^2, i=1,2,3.$$

First let us estimate $|\Delta\phi_{kl}^1|$

$$|\Delta\phi_{kl}^1| = |c_k c_l(\sin(x_k t)\sin(x_l t) - \sin(x_k s)\sin(x_l s)) + m_k(t)m_l(t) - m_k(s)m_l(s)| \le$$
$$\le c_k c_l(|\sin(x_k t)||\sin(x_l t) - \sin(x_l s)| - |\sin(x_l t)||\sin(x_k t) - \sin(x_k s)|) + |m_k(t)||m_l(t) - m_l(s)| +$$
$$+|m_l(t)||m_k(t) - m_k(s)|.$$

Since in (Kozachenko et al., 2016) the following inequality is proved

$$|\sin(h)| \le h^\alpha, \alpha \in (0,1],$$

we have

$$|\sin(x_l t) - \sin(x_l s)| = 2\left|\cos\left(\frac{x_l}{2}(t+s)\right)\sin\left(\frac{x_l}{2}(t-s)\right)\right| \le 2\left|\sin\left(\frac{x_l}{2}(t-s)\right)\right| \le 2^{1-\alpha}x_l^\alpha|t-s|^\alpha.$$

In the same way we get

$$|\cos(x_l t) - \cos(x_l s)| \le 2^{1-\alpha}x_l^\alpha|t-s|^\alpha.$$

Then we obtain

$$|m_k(t) - m_k(s)| \le c_k\int_0^T |H(\tau)||\sin(x_k(t-\tau)) - \sin(x_k(s-\tau))|d\tau \le$$
$$\le 2c_k q\left|\sin\left(\frac{x_k}{2}(t-s)\right)\right|\left|\int_0^T\left|\cos\left(\frac{x_k}{2}(t+s-2\tau)\right)\right|d\tau \le 2c_k q x_k^\alpha 2^{-\alpha}|t-s|^\alpha\frac{2}{x_k} \le\right.$$
$$\le 2^{2-\alpha}qc_k x_k^{\alpha-1}|t-s|^\alpha.$$

$$|m_k(t)| = c_k\left|\int_0^T H(\tau)\sin(x_k(t-\tau))d\tau\right| \le qc_k\left|\int_0^T \sin(x_k(t-\tau))d\tau\right| \le \frac{2qc_k}{x_k}.$$

From the inequalities above we get the estimate we need

$$|\Delta\phi_{kl}^1| \le c_k c_l 2^{1-\alpha}|t-s|^\alpha(x_k^\alpha + x_l^\alpha)\left(1+\frac{4q^2}{x_k x_l}\right). \tag{5.9}$$

For $|\Delta\phi_{kl}^2|$ we have

$$|\Delta\phi_{kl}^2| = \left|2\left[c_k d_l\left(\sin(x_k t)(1-\cos(y_l t)) - \sin(x_k s)(1-\cos(y_l s))\right) + m_k(t)s_l(t) - m_k(s)s_l(s)\right]\right| =$$
$$= 2|[c_k d_l(\sin(x_k t) - \sin(x_k s) - \sin(x_k t)\cos(y_l t) + \sin(x_k s)\cos(y_l s)) + m_k(t)s_l(t) - m_k(s)s_l(s)]| \le$$
$$\le 2c_k d_l(|\sin(x_k t) - \sin(x_k s)| + |\sin(x_k t)||\cos(y_l t) - \cos(y_l s)| + |\cos(y_l t)||\sin(x_k t) - \sin(x_k s)|) +$$
$$+|m_k(t)||s_l(t) - s_l(s)| + |s_l(t)||m_k(t) - m_k(s)|].$$

Since

$$|s_l(t) - s_l(s)| \le d_l\int_0^T |H(\tau)||1-\cos(y_l(t-\tau)) - 1 + \cos(y_l(s-\tau))|d\tau \le$$
$$\le 2qd_l\left|\sin\left(\frac{y_l}{2}(t-s)\right)\right|\left|\int_0^T\left|\sin\left(\frac{y_l}{2}(t++s-2\tau)\right)\right|d\tau \le 2^{2-\alpha}qd_l y_l^{\alpha-1}|t-s|^\alpha,\right.$$

$$|s_l(t)| = d_l\left|\int_0^T H(\tau)\left(1-\cos(y_l(t-\tau))\right)d\tau\right| \le qd_l\left|\int_0^T\left(1-\cos(y_l(t-\tau))\right)d\tau\right| \le qd_l\left(T+\frac{2}{y_l}\right),$$

we obtain

$$|\Delta\phi_{kl}^2| \le c_k d_l 2^{3-\alpha}|t-s|^\alpha\left[(1+q^2T)x_k^\alpha + \frac{1}{2}y_l^\alpha + 2q^2\frac{x_k^\alpha + y_l^\alpha}{x_k y_l}\right]. \tag{5.10}$$

In the same way it can be shown the estimate for $|\Delta\phi_{kl}^3|$

$$|\Delta\phi_{kl}^3| = \left|2\left[d_k d_l\left((1-\cos(y_k t))(1-\cos(y_l t)) - (1-\cos(y_k s))(1-\cos(y_l s))\right) + s_k(t)s_l(t) - s_k(s)s_l(s)\right]\right| \le$$
$$\le d_k d_l(|\cos(y_l t) - \cos(y_l s)| + |\cos(y_k t) - \cos(y_k s)| + |\cos(y_k t)||\cos(y_l t) - \cos(y_l s)| +$$
$$+|\cos(y_l t)||\cos(y_k t) - \cos(y_k s)|) + |s_k(t)||s_l(t) - s_l(s)| + |s_l(t)||s_k(t) - s_k(s)|,$$

and

$$|\Delta\phi_{kl}^3| \le d_k d_l 2^{2-\alpha}|t-s|^\alpha\left((y_k^\alpha + y_l^\alpha)\left(1+\frac{2q^2}{y_k y_l}\right) + q^2T(y_k^{\alpha-1} + y_l^{\alpha-1})\right). \tag{5.11}$$

Finally, if we put (5.9), (5.10), and (5.11) into (5.2), we will have

$$\sqrt{D(\xi_N(t) - \xi_N(s))} \le K(N)|t-s|^\alpha, \alpha \in (0,1],$$

where $K(N)$ is from (5.5). Lemma 5.2 is proved.

The results of Lemma 5.2 can be used to approximate a fractional Brownian motion process with given accuracy and reliability.

Theorem 5.3. Assume that conditions A, B, and C are satisfied. The model $X_N(t)$ approximates a fractional Brownian motion process $X(t)$ taking into account the response of system (3.1) with given reliability $1-\nu, \nu \in (0,1)$, and accuracy $\delta > 0$ in the space $C([0,T])$, if for N, the following inequalities hold true:

$$\max\left\{\delta_0(N), K(N)\left(\frac{T}{2}\right)^\alpha\right\} < \frac{\alpha\delta}{2\sqrt{2}},$$

$$e^{\frac{3}{\alpha}}\exp\left\{-\frac{\delta}{2\sqrt{2}\delta_0(N)}\right\}\times\left(\frac{\delta\alpha}{2\sqrt{2}\delta_0(N)}\right)^{\frac{2}{\alpha}}\left(1+\frac{2\delta}{\sqrt{2}\delta_0(N)}\right)^{\frac{1}{2}}<\nu,$$

where $\delta_0(N)$ and $K(N)$ are from (5.4) and (5.5) respectively.

Proof. The proof of the theorem follows from (2.1), Theorem 4.3, and Lemma 5.2.

6. Software simulation

Consider the following case $\alpha=\frac{1}{2}, T=1$, and the following impulse response function $H(\tau)=e^{-\tau}$.

Let us estimate the series

$$\sum_{k=N+1}^{\infty} c_k^2 x_k^d,\ \sum_{k=N+1}^{\infty} d_k^2 y_k^d,\ d\le 2\alpha.$$

From (Watson, 1944) we have

$$x_k\sim y_k\sim \pi k,\ k\to\infty,$$

and for large $|x|$ we have the following asymptotic relation for the Bessel function J_ν for $\nu>-1$

$$J_\nu^2(x)+J_{\nu+1}^2(x)\sim\frac{2}{\pi x}, n\to\infty.$$

Additionally, in (Kozachenko, Sottinen, & Vasylyk, 2005) the following asymptotic relations were proved

$$J_{1-H}^2(x_k)\sim\frac{2}{\pi x_k},\qquad J_{-H}^2(y_k)\sim\frac{2}{\pi y_k},\qquad k\to\infty.$$

Therefore, we obtain

$$c_k^2\sim d_k^2\sim\frac{c}{k^{2H+1}}, k\to\infty.$$

Hence, we have

$$\sum_{k=N+1}^{\infty} c_k^2 x_k^d\sim\sum_{k=N+1}^{\infty}\frac{c}{k^{2H+1}}(\pi k)^d=c\pi^d\sum_{k=N+1}^{\infty}\frac{1}{k^{2H+1-d}}=c\pi^d\sum_{k=N+1}^{\infty}\int_{k-1}^{k}\frac{dx}{k^{2H+1-d}}\le c\pi^d\sum_{k=N+1}^{\infty}\int_{k-1}^{k}\frac{dx}{x^{2H+1-d}}=$$
$$=c\pi^d\int_N^\infty\frac{dx}{x^{2H+1-d}}=\frac{c\pi^d}{N^{2H-d}(2H-d)}, \tag{6.1}$$

where c is from (2.2).

In the same way we prove

$$\sum_{k=N+1}^{\infty} d_k^2 y_k^d\le\frac{c\pi^d}{N^{2H-d}(2H-d)}. \tag{6.2}$$

From the representation of $H(\tau)$ and conditions A and B obtain $q=1, I_H=\frac{1}{2}-\frac{1}{2e^2}$

Now let us estimate $K(N)$ and $\delta_0(N)$ from (5.4) and (5.5) respectively using (6.1), (6.2)

$$\delta_0(N)=(21.375-2.75e^{-2}+1.375e^{-4})\frac{c^2}{4N^{4H}H^2}+(183.375-14.75e^{-2}+7.375e^{-4})\frac{c^2}{\pi^2N^{4H+2}(2H+1)^2}+$$
$$\left.+(46.5-13e^{-2}+6.5e^{-4})\frac{c^2}{\pi N^{4H+1}(4H^2+2H)}\right]^{\frac{1}{2}},$$

$$K(N)=2^{1.5}\left[\left(\frac{5}{H(H-0.5)}+\frac{11}{(2H-0.5)^2}\right)\frac{c^2\pi}{N^{4H-1}}+\left(\frac{7}{2H(2H+1)}+\frac{6.25}{(2H+0.5)^2}+\frac{1}{16H(H+0.5)}\right)\frac{8c^2}{\pi N^{4H-1}}+\right.$$
$$\left.+\left(\frac{1}{(H+1)(H+0.5)}+\frac{4}{(2H+1.5)^2}\right)\frac{14c^2}{\pi^3N^{4H+3}}\right].$$

Now, let us provide an example of software modeling for a fractional Brownian process. For different values of accuracy and reliability we will search for such N, that satisfy Theorem 5.3. Then, we will simulate the process using those values and construct the sample path of the model. For example, for accuracy $\delta=0.1$

And reliability $1-\nu=0.95$ the cutting level *N* = 150. If we enlarge accuracy $\delta=0.35$ and decrease reliability $1-\nu=0.75$ the cutting level will be equal *N* = 38.

The figure 1 displays the trajectory of the model $X_{150}(t)$, which approximates a fractional Brownian process with Hurst index $H=0.8$ with reliability 0.95 and accuracy of 0.1, taking into account the response of the system.

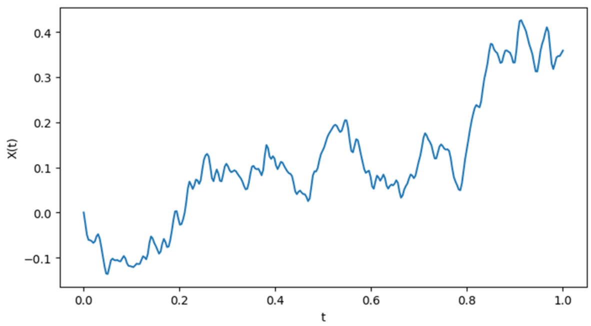

Fig. 1. The sample path of the simulated model $X_{150}(t)$ that approximates the process $X(t)$ with accuracy 0.1 and reliability 0.95, taking into account the response of the system

Discussion and conclusions

In this paper, we investigated the modelling of fractional Brownian motion with given accuracy and reliability. The process was considered as input process to a time-invariant linear system with real-valued square-integrable impulse response function, which is defined on the finite domain $\tau \in [0, T]$. Therefore, while the modelling the response of this system was taken into account.

To obtain conditions for modelling with given accuracy and reliability taking into account the response of the system, we proved the theorem which gave us the value of the upper limit of the summing in the model for such modelling.

Using the results of the obtained theorem, we performed software simulation of the fractional Brownian motion, presented the sample path of the modeled process, and the table with the conditions of simulation corresponding to given accuracy and reliability.

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МОДЕЛЮВАННЯ ДРОБОВОГО БРОУНІВСЬКОГО ПРОЦЕСУ ІЗ ЗАДАНОЮ ТОЧНІСТЮ ТА НАДІЙНІСТЮ

Теорія випадкових процесів дедалі частіше знаходить застосування в різних галузях науки завдяки високій обчислювальній потужності сучасних обчислювальних пристроїв. Однак часто важливо знати, наскільки ми можемо покладатися на моделі, які використовуємо. У пропонованій статті розглянуто методи моделювання дробового броунівського процесу із заданою точністю та надійністю. Моделювання ґрунтується на представленні дробового броунівського процесу як суми ряду. Ми розглядаємо дробовий броунівський процес як вхідний процес інваріантної в часі лінійної системи з дійсною квадратично-інтегрованою імпульсною функцією виходу, яка визначена на скінченній області $\tau \in [0, T]$. Доведено теорему, що надає умови, зокрема значення верхньої межі суми ряду, яким представлено модель, за яких отримана модель наближає дробовий броунівський рух із заданою точністю та надійністю з урахуванням виходу системи. Для доведення використано властивості квадратично-гауссівських випадкових процесів. Побудовані моделі можуть широко використовуватись як у природничих, так і в соціальних науках для апроксимації процесів з коротко-, а також із довготривалою залежністю.

Ключові слова: ***дробовий броунівський рух, моделювання, точність, надійність, лінійна однорідна система.***

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