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ON MODELING GAUSSIAN STATIONARY ORNSTEIN – UHLENBECK PROCESSES WITH GIVEN RELIABILITY AND ACCURACY IN Lp-SPACES

Even though the problem of modelling and simulation is not new it continues to be actual over time. Our computers are becoming more powerful and this allows us to use more sofisticated algorithms for more complicated problems. In this paper we constructed the model from the series decomposition of the Gaussian stationary Ornstein – Uhlenbeck process. The Ornstein – Uhlenbeck process is widely used to model reversal processes, exchange rates, asset price volatility, etc. Controlling the model's accuracy and reliability with which it approximates the real process is important for applications. For this purpose we have established the relation between the model's accuracy measured in the norm of Lp-space and reliability. The classical methods and results from the general theory of stochastic processes and sub-Gaussian spaces of random variables were used in our research. Since Gaussian stochastic processes are sub-Gaussian as well, we can utilize them. For one particular case the calculations were made in order to show how our results can be used in the particular situations. The results from our paper can help to simulate and analyse the situations which the Ornstein – Uhlenbeck process fits well.

Keywords: ***modeling with given reliability and accuracy; Gaussian stationary process, Ornstein – Uhlenbeck process, spectral density of stochastic process.***

AMS 2020 classification: 60G10, 60G15, 91-10.

Introduction

The issue of modeling and simulation of stochastic process is very vast. Simulation of different situations via different stochastic models can help to make an appropriate decision for many practical problems which are too complicated for solving by standard analytical instruments. This implies the strong demand for such methods in different application areas: from epidemiology to telecommunications and finance, see for example (Shanbhag, & Rao, 2003). The theory of modeling the stochastic processes and fields is worked out by many scientists, and a large number of papers are devoted to this topic. In particularly, the various methods for modeling Gaussian stochastic processes and fields have been thoroughly studied in the works by (Ianevych, Rozora, & Pashko, 2022; Kozachenko, Pogorilyak, Rozora, & Tegza, 2016; Rozora, Ianevych, Pashko, & Zatula, 2023) and many others.

In this research our main objective is to construct the appropriate model for an Ornstein – Uhlenbeck process. In the field of finance, this process is well known from the Vasicek interest rate model. In particular, an Ornstein – Uhlenbeck process is widely used to model reversal processes and for stochastic modeling of exchange rates. Recently, the Ornstein – Uhlenbeck process has been used in finance as a model of asset price volatility as well. So, for the real situation it is important to have possibility to simulate this kind of process and to control the possible errors measured in some norm and with some predetermined reliability. (Kozachenko, & Petranova, 2017) also studied and modelled the centered and stationary Ornstein – Uhlenbeck process but with accuracy measured in the norm of the space $C[0,T]$.

In this paper, we formulated and proved Theorem (our main result) which establishes a connection between the model's error measured in the norm of L_p-space and the given accuracy and reliability. For this we utilized the methods and results from the theory of sub-Gaussian stochastic process. In particular, we rely on the estimates for sub-Gaussian stochastic processes from the paper by (Kozachenko, & Kamenshchikova, 2009). We have also performed some calculations for one particular case to show how our theory works in practice.

1. On covariance functions and spectral densities

Let us consider in what follows a stationary Ornstein – Uhlenbeck stochastic process $X(t), t \in \mathbb{R}$, for which $\mathsf{E}(X(t)) = 0$ and the covariance function for all $t \in \mathbb{R}$ is

$$R(\tau) = \mathsf{E}(X(t+\tau)\overline{X(t)}) = \sigma^2 \exp\{-\alpha|\tau|\},\ \sigma, \alpha > 0. \tag{1}$$

According to the Bochner – Khinchin's theorem, a covariance function of a real stationary process can be presented as

$$R(\tau) = \int_{-\infty}^{\infty} \cos(\lambda\tau) f(\lambda) d\lambda, \quad \tau \in \mathbb{R}.$$

Using the Fourier transform it can be deduced that the spectral density of the process with the covariance function (1) is

$$f(\lambda) = \frac{\sigma^2 \alpha}{\pi(\alpha^2 + \lambda^2)}.$$

So, for the real-valued stationary Ornstein – Uhlenbeck stochastic process

$$R(\tau) = \sigma^2 \int_{-\infty}^{\infty} \frac{\alpha}{\pi(\alpha^2+\lambda^2)} \cos(\lambda\tau) d\lambda, \quad \tau \in \mathbb{R}.$$

According to the Karhunen theorem (see, e.g. (Lukacs, 1970)), our stochastic process $X(t), t \in \mathbb{R}$ can be presented as

$$X(t) = \int_{-\infty}^{\infty} \cos(\lambda t) d\xi_1(\lambda) + \int_{-\infty}^{\infty} \sin(\lambda t) d\xi_2(\lambda), \ t \in \mathbb{R}, \tag{2}$$

where $\xi_1(\lambda)$ and $\xi_2(\lambda)$ are independent centered Gaussian stochastic processes with orthogonal increments such that if $\lambda_1 < \lambda_2$, then

$$\mathsf{E}(\xi_i(\lambda_2) - \xi_i(\lambda_1))^2 = \int_{\lambda_1}^{\lambda_2} f(\lambda) d\lambda, \quad i = 1, 2.$$

The covariance function (1) can be represented as the sum of two integrals:

$$R(t,s) = R(t-s) = \int_{-\infty}^{\infty} f_1(t,\lambda) f_1(s,\lambda) d\lambda + \int_{-\infty}^{\infty} f_2(t,\lambda) f_2(s,\lambda) d\lambda,$$

where

$$f_1(t,\lambda) = \cos(\lambda t) \cdot \sigma \sqrt{\frac{\alpha}{\pi(\alpha^2+\lambda^2)}}; \tag{3}$$

$$f_2(t,\lambda) = \sin(\lambda t) \cdot \sigma \sqrt{\frac{\alpha}{\pi(\alpha^2+\lambda^2)}}. \tag{4}$$

In what follows we will need the results of the theorem from (Kozachenko et al., 2016).

Theorem 1. *Let $X(t), t \in T$ be a centered complex stochastic process with finite variance and covariance function $R(t,s) = \mathsf{E}\, X(t)\overline{X(s)}$. Assume also that $(\Lambda, B_\Lambda, \mu)$ is a measurable space with a σ-additive measure μ. Suppose that for all $t \in T$ functions $f_i(t,\lambda)$, $i = 1, ..., n$ belong to the space $L_2(\Lambda, \mu)$; $\{g_k(\lambda), k \in \mathbb{Z}\}$ is an orthonormalized basis in $L_2(\Lambda, \mu)$. The covariance function $R(t,s)$ is represented as*

$$R(t,s) = \sum_{i=1}^{n} \int_{\Lambda} f_i(t,\lambda) \overline{f_i(s,\lambda)} d\mu(\lambda)$$

if and only if the stochastic process admits a decomposition

$$X(t) = \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}} a_{ik}(t) \xi_{ik}, \tag{5}$$

where

$$a_{ik}(t) = \int_{\Lambda} f_i(t,\lambda) \overline{g_k(\lambda)} d\mu(\lambda),$$

ξ_{ik} are centered uncorrelated random variables, $\mathsf{E}\, \xi_{ik} = 0$, $\mathsf{E}\, \xi_{ik}\xi_{il} = \delta_{kl}$ and $\mathsf{E}\, \xi_{ik}^2 = 1$ for all $i = 1, ..., n$.

It follows from (2) that the stochastic process $X(t)$ can be represented as a sum of two processes

$$X(t) = X_\Lambda(t) + X^\Lambda(t), \tag{6}$$

where

$$\begin{aligned} X_\Lambda(t) &= \int_{-\Lambda}^{\Lambda} \cos(\lambda t) d\xi_1(\lambda) + \int_{-\Lambda}^{\Lambda} \sin(\lambda t) d\xi_2(\lambda); \\ X^\Lambda(t) &= \int_{|\lambda| \geqslant \Lambda} \cos(\lambda t) d\xi_1(\lambda) + \int_{|\lambda| \geqslant \Lambda} \sin(\lambda t) d\xi_2(\lambda). \end{aligned}$$

Then the covariance function for $X_\Lambda(t)$ can be written as

$$R_\Lambda(t,s) = \int_{-\Lambda}^{\Lambda} f_1(t,\lambda) f_1(s,\lambda) d\lambda + \int_{-\Lambda}^{\Lambda} f_2(t,\lambda) f_2(s,\lambda) d\lambda,$$

where the functions $f_1(t,\lambda)$ and $f_2(t,\lambda)$ are defined in (3)-(4).

Let us apply Theorem 1 to the process $X_\Lambda(t)$ imposing a trigonometric orthonormalized basis in the space $L_2([-\Lambda, \Lambda])$

$$\left\{ a_0 = \frac{1}{\sqrt{2\Lambda}}, a_{k1} = \frac{1}{\sqrt{\Lambda}} \cos\left(\frac{k\pi}{\Lambda}\lambda\right), a_{k2} = \frac{1}{\sqrt{\Lambda}} \sin\left(\frac{k\pi}{\Lambda}\lambda\right), \ k \geqslant 1 \right\}.$$

We obtain the following series expansion of the process X_t convergent in the mean square sense:

$$X_\Lambda(t) = a_0(t)\xi_0 + \sum_{k=1}^{\infty}\left[a_{1k\Lambda}(t)\xi_{1k} + a_{2k\Lambda}(t)\xi_{2k}\right], \tag{7}$$

where the random variables $\{\xi_0, \xi_{1k}, \xi_{2k},\ k = 1, 2, \dots\}$ are such that $\mathsf{E}\,\xi_{ik} = 0$, $\mathsf{E}\,\xi_{ik}\xi_{il} = \delta_{kl}$ for $i = 1, 2$ and $\mathsf{E}\,\xi_{ik}\xi_{il} = 0$ for $i \neq j$,

$$\begin{aligned}
a_{0\Lambda}(t) &= \frac{1}{\sqrt{2\Lambda}}\int_{-\Lambda}^{\Lambda}\cos(\lambda t)\cdot\sigma\sqrt{\frac{\alpha}{\pi(\alpha^2+\lambda^2)}}d\lambda;\\
a_{1k\Lambda}(t) &= \frac{1}{\sqrt{\Lambda}}\int_{-\Lambda}^{\Lambda}\cos(\lambda t)\cdot\sigma\sqrt{\frac{\alpha}{\pi(\alpha^2+\lambda^2)}}\cos\left(\frac{k\pi}{\Lambda}\lambda\right)d\lambda;\\
a_{2k\Lambda}(t) &= \frac{1}{\sqrt{\Lambda}}\int_{-\Lambda}^{\Lambda}\sin(\lambda t)\cdot\sigma\sqrt{\frac{\alpha}{\pi(\alpha^2+\lambda^2)}}\sin\left(\frac{k\pi}{\Lambda}\lambda\right)d\lambda.
\end{aligned}$$

Remark 1. Since the stochastic process $X(t)$ is Gaussian, the random variables $\{\xi_0, \xi_{1k}, \xi_{2k},\ k = 0, 1, 2, \dots\}$ in the equation (7) are Gaussian and independent.

2. Building the stochastic process model

In this paper, we consider modeling in the space $L_p([0,T])$. The representation (5) can be used to build a model of a stochastic process X with a given accuracy and reliability in different Banach spaces. Let us assume that the Gaussian stochastic process $X(t),\ t \in \mathbb{R}$ can be decomposed in the mean square sense as follows

$$X(t) = \sum_{k=0}^{\infty} a_k(t)\xi_k,$$

where $\xi_k, k = 0, 1, 2, \dots$ are independent normally distributed random variables with $\mathsf{E}\,\xi_k = 0, \mathsf{E}\,\xi_k^2 = 1$.

Definition 1. A stochastic process $X_N(t)$ is called a model of a process $X(t)$ if

$$X_N(t) = \sum_{k=0}^{N} a_k(t)\xi_k.$$

Let $\|\cdot\|$ denote the norm in the functional Banach space B.

Definition 2. A model $X_N(t)$ approximates a stochastic process $X(t)$ with a given accuracy $\delta > 0$ and a reliability $1-\nu,\ \nu \in (0,1)$ in a Banach space B if

$$P\left\{\|X_N(t) - X(t)\| > \delta\right\} \leqslant \nu. \tag{8}$$

To build a model with an accuracy δ and a reliability $1-\nu$ in the given space, it is necessary to find N for which the inequality (8) holds true. For the process $X(t)$ with the covariance function (1) on the interval $[0,T]$, we will use the following finite sum as a model

$$X_N(t) = a_{0\Lambda}(t)\xi_0 + \sum_{k=1}^{N}\left[a_{1k\Lambda}(t)\xi_{1k} + a_{2k\Lambda}(t)\xi_{2k}\right]. \tag{9}$$

3. Taking into account the the model's reliability and accuracy in the space Lp ([0, T])

For taking into account the predetermined accuracy we need to have at least some estimates for the probability of the model's error be larger than some value.

Let us prove an auxiliary lemma first. It then can be utilized for proving our main result.

Lemma 1. *For a centered stationary Gaussian process $X(t),\ t \in [0,T]$ with covariance function $R(\tau) = \sigma^2\exp\{-\alpha|\tau|\}$, $\sigma, \alpha > 0$ and the model $X_N(t)$, defined in (9), the inequality is true:*

$$\sup_{t\in[0,T]}\left(\mathsf{E}\left[X(t) - X_N(t)\right]^2\right)^{1/2} \leqslant B_{N,\Lambda}, \tag{10}$$

where

$$B_{N,\Lambda} = \left[\sigma^2 - \frac{2\sigma^2}{\pi}\arctan\frac{\Lambda}{\alpha} + \frac{8\Lambda\alpha\sigma^2}{N\pi^3}\left(T\cdot\ln\left|\Lambda + \sqrt{\alpha^2+\Lambda^2}\right| - T\cdot\ln\alpha - \frac{1}{\sqrt{\alpha^2+\Lambda^2}} + \frac{1}{\alpha}\right)^2\right]^{\frac{1}{2}} \tag{11}$$

Proof. The equation (6) implies that

$$\sup_{t\in[0,T]}\mathsf{E}\left[X(t) - X_N(t)\right]^2 = \sup_{tin[0,T]}\mathsf{E}\left[X_\Lambda(t) + X^\Lambda(t) - X_N(t)\right]^2 = \sup_{t\in[0,T]}\mathsf{E}\left[X^\Lambda(t)\right]^2 + \sup_{t\in[0,T]}\mathsf{E}\left[X_\Lambda(t) - X_N(t)\right]^2.$$

Let us estimate each term separately.

$$\mathsf{E}\left[X^{\Lambda}(t)\right]=\mathsf{E}\left[\int_{|\lambda|\geqslant\Lambda}\cos(\lambda t)d\xi_1(\lambda)+\int_{|\lambda|\geqslant\Lambda}\sin(\lambda t)d\xi_2(\lambda)\right]^2=\int_{|\lambda|\geqslant\Lambda}\cos^2(\lambda t)\,\mathsf{E}\,d\xi_1(\lambda)+\int_{|\lambda|\geqslant\Lambda}\sin^2(\lambda t)d\xi_2(\lambda)=$$

$$=\int_{|\lambda|\geqslant\Lambda}dF(\lambda)=\int_{|\lambda|\geqslant\Lambda}f(\lambda)d\lambda=\int_{|\lambda|\geqslant\Lambda}\frac{\sigma^2\alpha}{\pi(\alpha^2+\lambda^2)}d\lambda$$

and also

$$\int_{\Lambda}^{\infty}\frac{d\lambda}{\alpha^2+\lambda^2}=\frac{1}{\alpha}\arctan\frac{\lambda}{\alpha}\Big|_{\Lambda}^{\infty}=\frac{\pi}{2\alpha}-\frac{1}{\alpha}\arctan\frac{\Lambda}{\alpha}.$$

Therefore,

$$\sup_{t\in[0,T]}\mathsf{E}\left[X^{\Lambda}(t)\right]^2=\frac{2\sigma^2\alpha}{\pi}\left(\frac{\pi}{2\alpha}-\frac{1}{\alpha}\arctan\frac{\Lambda}{\alpha}\right)=\sigma^2-\frac{2\sigma^2}{\pi}\arctan\frac{\Lambda}{\alpha},$$

which implies that $\sup\limits_{t\in[0,T]}\mathsf{E}\left[X^{\Lambda}(t)\right]^2\to 0$ as $\Lambda\to\infty$. For the second term in (11) we have

$$\mathsf{E}\left[X_{\Lambda}(t)-X_N(t)\right]^2=\sum_{k=N+1}^{\infty}\left[a_{1k\Lambda}^2(t)+a_{2k\Lambda}^2(t)\right].$$

Let us estimate $|a^2_{1k\Lambda(t)}|$:

$$|a_{1k\Lambda}(t)|=\sqrt{\frac{\alpha}{\Lambda\pi}}2\sigma\left|\int_0^{\Lambda}\frac{\cos(\lambda t)}{\sqrt{\alpha^2+\lambda^2}}\cos\left(\frac{k\pi}{\Lambda}\lambda\right)d\lambda\right|=\left|\begin{matrix}dv=\cos\left(\frac{k\pi}{\Lambda}\lambda\right)d\lambda, & v=\frac{\Lambda}{k\pi}\sin\left(\frac{k\pi}{\Lambda}\lambda\right)\\ u=\frac{\cos(\lambda t)}{\sqrt{\alpha^2+\lambda^2}}, & du=\left(-\frac{\sin(\lambda t)\cdot t}{\sqrt{\alpha^2+\lambda^2}}-\frac{\lambda\cos(\lambda t)}{(\alpha^2+\lambda^2)^{3/2}}\right)d\lambda\end{matrix}\right|=$$

$$=2\sigma\sqrt{\frac{\alpha}{\Lambda\pi}}\left|\frac{cos(\lambda t)}{\sqrt{\alpha^2+\lambda^2}}\cdot\frac{\Lambda\sin\left(\frac{k\pi}{\Lambda}\lambda\right)}{k\pi}\Big|_0^{\Lambda}+\int_0^{\Lambda}\frac{\Lambda\sin\left(\frac{k\pi}{\Lambda}\lambda\right)}{k\pi}\cdot\left(-\frac{\sin(\lambda t)\cdot t}{\sqrt{\alpha^2+\lambda^2}}-\frac{\lambda\cos(\lambda t)}{(\alpha^2+\lambda^2)^{3/2}}\right)d\lambda\right|\leqslant$$

$$\leqslant 2\sigma\sqrt{\frac{\alpha}{\Lambda\pi}}\cdot\frac{\Lambda}{k\pi}\left(t\int_0^{\Lambda}\frac{\left|-\sin(\lambda t)\sin\left(\frac{k\pi}{\Lambda}\lambda\right)\right|}{\sqrt{\alpha^2+\lambda^2}}d\lambda+\int_0^{\Lambda}\frac{\left|-\cos(\lambda t)\sin\left(\frac{k\pi}{\Lambda}\lambda\right)\right|}{\sqrt{\alpha^2+\lambda^2}}d\lambda\right)\leqslant$$

$$\leqslant\sqrt{\frac{\alpha\Lambda}{\pi}}\cdot\frac{2\sigma}{k\pi}\left(t\int_0^{\Lambda}\frac{d\lambda}{\sqrt{\alpha^2+\lambda^2}}+\int_0^{\Lambda}\frac{\lambda d\lambda}{(\alpha^2+\lambda^2)^{3/2}}\right)=$$

$$=\sqrt{\frac{\alpha\Lambda}{\pi}}\cdot\frac{2\sigma}{k\pi}\left(t\cdot\ln\left|\lambda+\sqrt{\alpha^2+\lambda^2}\right|-\frac{1}{\sqrt{\alpha^2+\lambda^2}}\right)\Big|_0^{\Lambda}=\sqrt{\frac{\alpha\Lambda}{\pi}}\cdot\frac{2\sigma}{k\pi}\left(t\cdot\ln\left|\Lambda+\sqrt{\alpha^2+\Lambda^2}\right|-t\cdot\ln\alpha-\frac{1}{\sqrt{\alpha^2+\Lambda^2}}+\frac{1}{\alpha}\right).$$

Similarly, we can get an estimate for $|a^2_{2k\Lambda(t)}|$. Then,

$$\sup_{t\in[0,T]}\mathsf{E}\left[X_{\Lambda}(t)-X_N(t)\right]^2\leqslant\sum_{k=N+1}^{\infty}2\left[\sqrt{\frac{\alpha\Lambda}{\pi}}\cdot\frac{2\sigma}{k\pi}\left(t\cdot\ln\left|\Lambda+\sqrt{\alpha^2+\Lambda^2}\right|-t\cdot\ln\alpha-\frac{1}{\sqrt{\alpha^2+\Lambda^2}}+\frac{1}{\alpha}\right)\right]^2\leqslant$$

$$\leqslant\frac{8\Lambda\alpha\sigma^2}{\pi^3}\left(T\cdot\ln\left|\Lambda+\sqrt{\alpha^2+\Lambda^2}\right|-T\cdot\ln\alpha-\frac{1}{\sqrt{\alpha^2+\Lambda^2}}+\frac{1}{\alpha}\right)^2\cdot\sum_{k=N+1}^{\infty}\frac{1}{k^2}\leqslant$$

$$\leqslant\frac{1}{N}\frac{8\Lambda\alpha\sigma^2}{\pi^3}\left(T\cdot\ln\left|\Lambda+\sqrt{\alpha^2+\Lambda^2}\right|-T\cdot\ln\alpha-\frac{1}{\sqrt{\alpha^2+\Lambda^2}}+\frac{1}{\alpha}\right)^2.$$

Obviously $\sup\limits_{t\in[0,T]}\mathsf{E}\left[X_{\Lambda}(t)-X_N(t)\right]^2\to 0$ as $N\to\infty$. So,

$$\sup_{t\in[0,T]}\mathsf{E}\left[X_{\Lambda}(t)-X_N(t)\right]^2\leqslant\sigma^2-\frac{2\sigma^2}{\pi}\arctan\frac{\Lambda}{\alpha}+\frac{1}{N}\frac{8\Lambda\alpha\sigma^2}{\pi^3}\left(T\ln\left|\Lambda+\sqrt{\alpha^2+\Lambda^2}\right|-T\ln\alpha-\frac{1}{\sqrt{\alpha^2+\Lambda^2}}+\frac{1}{\alpha}\right)^2=:B^2_{N,\Lambda}$$

□

A Gaussian process is a special case of a sub-Gaussian process. Therefore we can use the needed estimate from the paper by (Kozachenko, & Kamenshchikova, 2009) in our case as well.

Theorem 2. *Let* $\{\mathbf{T}, \mu\}$ *be a measurable space,* $\xi = \{\xi(t),\ t \in \mathbf{T}\}$ *be a strictly sub-Gaussian stochastic process with* $\mathsf{E}\,\xi^2(t) < \infty$*. Let there exist a Lebesgue integral*

$$\int_{\mathbf{T}} \left(\mathsf{E}\,\xi^2(t)\right)^{\frac{p}{2}} d\mu(t) < \infty,$$

then $\int_{\mathbf{T}} |\xi(t)|^p d\mu(t)$ *exists with probability one and for any* $\varepsilon > cp^{p/2}$

$$P\left\{\int_{\mathbf{T}} |\xi(t)|^p d\mu(t) > \varepsilon\right\} \leqslant 2\exp\left\{-\frac{\varepsilon^{2/p}}{2c^{2/p}}\right\}, \tag{12}$$

where $c = \int_{\mathbf{T}} \left(\mathsf{E}\,\xi^2(t)\right)^{\frac{p}{2}} d\mu(t)$*.*

Now we can state our main result.

Theorem 3. *Let* $X(t),\ t \in [0, T]$ *be a stationary centered Gaussian stochastic process with the covariance function* $B(\tau) = \sigma^2 \exp\{-\alpha|\tau|\}$*,* $\sigma, \alpha > 0$*. The model* $X_N(t)$ *defined by (9) approximates the stochastic process* $X(t)$ *with a given accuracy* $\delta > 0$ *and a reliability* $1 - \nu$*,* $\nu \in (0, 1)$ *in the Banach space* $L_p([0, T])$ *if*

$$B_{N,\Lambda} \leqslant \frac{\delta}{T^{1/p} \max\left(\sqrt{-2\ln\frac{\nu}{2}}, \sqrt{p}\right)}, \tag{13}$$

where $B_{N,\Lambda}$ *is defined in (10).*

Proof. Since Gaussian processes are strictly sub-Gaussian as well, we can use the results from Theorem 2 for the process measuring the model's error. Let us consider a measurable space $\{T, \mu\} = \{[0, T], dt\}$. First, let us show that the conditions of Theorem 2 are fulfilled. We need to prove that there exists an integral

$$\int_T \left(\mathsf{E}\,\xi^2(t)\right)^{\frac{p}{2}} d\mu(t) < \infty.$$

The Lemma 1 implies that

$$\begin{aligned} c = \int_{\mathbf{T}} \left(\mathsf{E}\,\xi^2(t)\right)^{\frac{p}{2}} d\mu(t) = \int_0^T \left(\mathsf{E}\left(X(t) - X_N(t)\right)^2\right)^{\frac{p}{2}} dt \leqslant \int_0^T \left(\sup_{t\in[0,T]} \sqrt{\mathsf{E}\left(X(t) - X_N(t)\right)^2}\right)^p \leqslant \\ \leqslant \int_0^T B_N^p dt \leqslant B_{N,\Lambda}^p T < \infty. \end{aligned} \tag{14}$$

From the inequality (12) and the estimate (14) for the constant c we have

$$\begin{aligned} P\left\{\left(\int_{\mathbf{T}} |X(t) - X_N(t)|^p\, d\mu(t)\right)^{\frac{1}{p}} > \delta\right\} = P\left\{\int_{\mathbf{T}} |X(t) - X_N(t)|^p\, d\mu(t) > \delta^p\right\} \leqslant \\ \leqslant 2\exp\left\{-\frac{\delta^2}{2c^{2/p}}\right\} \leqslant 2\exp\left\{-\frac{\delta^2}{2(B_{N,\Lambda}^p \cdot T)^{2/p}}\right\} = 2\exp\left\{-\frac{\delta^2}{2B_{N,\Lambda}^2 \cdot T^{2/p}}\right\}. \end{aligned} \tag{15}$$

Under the conditions of the Theorem 2 it follows that the latter inequality is true only if

$$\varepsilon > cp^{p/2} \Leftrightarrow \delta^p > B_{N,\Lambda}^p T p^{p/2} \Leftrightarrow B_{N,\Lambda} < \frac{\delta}{\sqrt{p}T^{1/p}}.$$

This condition is satisfied, since under our assumptions

$$B_{N,\Lambda} \leqslant \frac{\delta}{T^{1/p} \max\left(\sqrt{-2\ln\frac{\nu}{2}}, \sqrt{p}\right)} \leqslant \frac{\delta}{\sqrt{p}T^{1/p}}.$$

By Definition (2), the model $X_N(t)$ approximates the stochastic process $X(t)$ with the accuracy δ and the reliability $1 - \nu$ if

$$P\left\{\left(\int_{\mathbf{T}} |X(t) - X_N(t)|^p\, d\mu(t)\right)^{\frac{1}{p}} > \delta\right\} \leqslant \nu.$$

Substituting the estimate (15) into the last inequality, we obtain

$$2\exp\left\{-\frac{\delta^2}{2B_{N,\Lambda}^2 \cdot T^{2/p}}\right\} \leqslant \nu \;\Leftrightarrow\; B_{N,\Lambda} \leqslant \frac{\delta}{T^{1/p}\sqrt{-2\ln\frac{\nu}{2}}}. \tag{16}$$

The relationship (16) is valid since, according to the condition (13),

$$B_{N,\Lambda} \leqslant \frac{\delta}{T^{1/p} \max\left(\sqrt{-2\ln\frac{\nu}{2}}, \sqrt{p}\right)} \leqslant \frac{\delta}{T^{1/p}\sqrt{-2\ln\frac{\nu}{2}}}.$$

□

Remark 2. The value $B_{N,\Lambda}$ links together the model parameters N and Λ. It balances the interval of the integration $[-\Lambda, \Lambda]$ in (6) and the number of terms in the model (9).

Remark 3. It becomes clear from (10) that

$$B_{N,\Lambda} \to 0, \text{ as } \frac{\Lambda}{N} \to 0, \ \ N, \Lambda \to \infty.$$

Thus, it is possible to find N for which the inequality (13) holds.

Example. In the case when $\sigma = \alpha = 1$, $p = 2$, $t \in [0, 10]$, for the given accuracy $\delta = 0.1$ and the reliability $1 - \nu = 0.05$, we have found the value $N = 14 \cdot 10^9$ which corresponds to the interval $[-\Lambda, \Lambda] = [-1000, 1000]$.

Discussion and conclusions

The results of this paper fill certain gaps in what has been done before by other authors mentioned in the Section 1. The inequality obtained in Theorem 3 makes it possible to control the quality of the model determined in terms of reliability and accuracy in the L_p-norm.

Authors' contribution: Tetiana Ianevych – conceptualization and theoretical foundations of research; Olga Vasylyk – analysis of sources, preparation of literature review; Julia Doshchuk – making the mathematical calculations.

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ПРО МОДЕЛЮВАННЯ ГАУССОВОГО СТАЦІОНАРНОГО ПРОЦЕСУ ОРНШТЕЙНА – УЛЕНБЕКА ІЗ ЗАДАНОЮ ТОЧНІСТЮ ТА НАДІЙНІСТЮ В Lp-ПРОСТОРІ

Незважаючи на те, що задачі моделювання та симулювання процесів не є новими, вони залишаються актуальними і дотепер. Наші комп'ютери стають потужнішими, і це дає нам змогу використовувати складніші алгоритми для більш складних задач. У статті ми побудували модель на основі розкладу гауссового стаціонарного процесу Орнштейна – Уленбека в ряд. Процес Орнштейна – Уленбека широко використовують для моделювання реверсивних процесів, обмінних курсів, коливань цін на активи тощо. Контроль точності та надійності моделі, з якими вона апроксимує реальний процес, важливий для застосувань. Із цією метою ми встановили співвідношення між точністю моделі, виміряної в нормі Lp-простору, та надійністю. Класичні методи та результати із загальної теорії стохастичних процесів і субгауссових просторів випадкових величин були використані у нашому дослідженні. Оскільки стохастичні процеси Гаусса також є субгауссовими, ми можемо їх використовувати. Для одного конкретного випадку були зроблені розрахунки, щоб показати, як наші результати можна застосовувати в конкретних ситуаціях. Результати статті допоможуть змоделювати та проаналізувати ситуації, для яких характерний процес Орнштейна – Уленбека.

Ключові слова: ***моделювання із заданою точністю та надійністю, гауссовий стаціонарний процес, процес Орнштейна – Уленбека, спектральна щільність випадкового процесу.***