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Математичне моделювання напруженого стану в'язкопружної напівплощини з включеннями

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Mathematical modeling of the stressed state of a viscoelastic half-plane with inclusions

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Розглянуто застосування методу граничних інтегральних рівнянь при дослідженні напруженого стану плоских в'язкопружних тіл із включеннями. Метод базується на використанні комплексних потенціалів та апарату узагальнених функцій. Отримано аналітичний розв'язок задачі для напівплощини із довільними за формою включеннями. Для чисельного дослідження зміни напруженого стану залежно від часу та геометрії включень було розроблено дискретний аналог системи гранично-часових інтегральних рівнянь.

Ключові слова: в'язкопружність, плоске в'язкопружне тіло, комплексні потенціали, метод граничних інтегральних рівнянь, в'язкопружні характеристики областей, резольвентні оператори.

The application of the method of boundary integral equations is considered for studying the stress state of flat viscoelastic bodies with inclusions. The method is based on the use of complex potentials and the apparatus of generalized functions. An analytical solution of the problem is obtained for a half-plane with inclusions of arbitrary shape. For a numerical study of the change in the stress state depending on the time and geometry of the inclusions, a discrete analogue of the system of boundary-time integral equations has been developed.

Key Words: viscoelasticity, flat viscoelastic body, complex potentials, method of boundary integral equations, viscoelastic characteristics of regions, resolvent operators.

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The method of boundary integral equations is one of the most popular methods for solving various problems in the theory of elasticity and viscoelasticity. Using this method, one can quite simply and accurately take into account infinitely distant boundaries, reduce the dimension of the problem by one, and also reveal the interactions of contacting areas in a natural way.

Let us consider a viscoelastic half-plane occupying a region D and containing a finite number of inclusions D_p of arbitrary shape ($p = \overline{1, n}$, where n is the number of inclusions). The outer contour goes to infinity, and the inclusions are bounded by piecewise-smooth contours. The half-plane is subjected to distributed tangential and normal loads [8].

To determine the stress state of the given region, we use the statement of the second main problem of the theory of elasticity for inhomogeneous bodies in movements [3], as well as the approach proposed for studying the stress state of a plane viscoelastic piecewise-isotropic body [8].

Let us assume that there are no mass forces, and also that the unknown densities of potentials are the stresses on the contours of inclusions. Then, based on the properties of the generalized functions, and due to Maxwell's theorem, the expressions for the components of the vector of movements can be written as:

$$u_k(x,t) = \int_{\Gamma_0} g_i(\xi,t)\, U_k^i(x,\xi,t)\, d\gamma(\xi) - - \frac{\overline{E}_p - \overline{E}_0}{\overline{E}_p} \int_{\Gamma_p} \sigma_{ij}^{(p)}(\xi,t)\, n_j(\xi)\, U_k^i(x,\xi,t)\, d\gamma(\xi), \tag{1}$$

where $\overline{E}_0$ and $\overline{E}_p$ are viscoelastic operators of homogeneous regions belonging to the class of resolvent operators [5, 6];

$$U_k^i(x,\xi,t) = \dot{U}_k^i(x,\xi,t) + \widehat{U}_k^i(x,\xi,t),$$

$\dot{U}_k^i(x,\xi,t)$ is the fundamental solution for problems of two-dimensional flat-strain state of a viscoelastic infinite body, and $\widehat{U}_k^i(x,\xi,t)$ is an additional term, which provides that the condition $g_k^i(x,\xi,t) = 0$ is satisfied $\forall x \in \Gamma$ and $\forall \xi \in D$; $g_i(\xi,t)$ — given densities along the contour Γ_0; $\sigma_{ij}^{(p)}(\xi,t)\, n_j(\xi)$ — unknown densities of potentials along the contours Γ_p.

Using the Cauchy relations, Hooke's law, and formulas (1), the stress tensor components can be expressed in the following form:

$$\sigma_{ij}(x,t) = \left(1 + \frac{\overline{E}_q - \overline{E}_0}{\overline{E}_0} S\big(D_q\big)\right) \times \times \left(\int_{\Gamma_0} g_k(\xi,t)\, U_{ij}^k(x,\xi,t)\, d\gamma(\xi) - - \frac{\overline{E}_p - \overline{E}_0}{\overline{E}_p} \int_{\Gamma_p} \sigma_{kl}^{(p)}(\xi,t)\, n_l(\xi)\, U_{ij}^k(x,\xi,t)\, d\gamma(\xi)\right), \tag{2}$$

where $\overline{E}_q$ is viscoelastic operator of homogeneous region belonging to the class of resolvent operators; $S\big(D_q\big)$ is the characteristic function of the region D_q and $U_{ij}^k(x,\xi,t)$ are stresses arising in a viscoelastic homogeneous body occupying region D under the action of unit concentrated forces at the point $\xi \in D$.

To determine the unknown densities, multiply both sides of (2) by $n_j(x)$ and obtain the system of boundary-time integral equations:

$$\sigma_{ij}^{(q)}(x,t)\, n_j(x) = \frac{2\overline{E}_q}{\overline{E}_q + \overline{E}_0} \times \times \left(\int_{\Gamma_0} g_k(\xi,t)\, U_{ij}^k(x,\xi,t)\, n_j(x)\, d\gamma(\xi) - - \frac{\overline{E}_p - \overline{E}_0}{\overline{E}_p} \times \times \int_{\Gamma_p} \sigma_{kl}^{(p)}(\xi,t)\, n_l(\xi)\, U_{ij}^k(x,\xi,t)\, n_j(x)\, d\gamma(\xi)\right). \tag{3}$$

In order to apply the integral representations (1) and (2) for the semi-infinite region, it is necessary to set additional conditions for the functions used. These conditions concern the behavior of functions at infinity and are defined as regularity conditions [1]. In this case, the functions $u_k(x,t)$ and $\sigma_{ij}(x,t)$ behave at infinity as fundamental solutions:

$$u_k(x,t) \;\sim\; U_k^i(x,\xi,t) = \begin{cases} o\left(\ln\rho + 1\right), & i = k, \\ o\left(1\right), & i \neq k; \end{cases}$$

$$\sigma_{ij}(x,t) \;\sim\; U_{ij}^k(x,\xi,t) = o\left(\rho^{-1}\right).$$

Note that the expressions for fundamental movements and corresponding stresses for the case of an elastic half-plane are given in [1].

Based on this, we write integral representations for the components of the movement vector and the stress tensor:

$$u_k(x,t) = \int_{-\infty}^{+\infty}\int_0^{+\infty} X_i(\xi,t)\, U_k^i(x,\xi,t)\, d\xi_2\, d\xi_1 - - \frac{\overline{E}_p - \overline{E}_0}{\overline{E}_p} \int_{D_p} X_i(\xi,t)\, U_k^i(x,\xi,t)\, dS(\xi) + + \int_{-\infty}^{+\infty} g_i(\xi_1,t)\, U_k^i(x,\xi,t)\, d\xi_1 - - \frac{\overline{E}_p - \overline{E}_0}{\overline{E}_p} \int_{\Gamma_p} \sigma_{ij}^{(p)}(\xi,t)\, n_j(\xi)\, U_k^i(x,\xi,t)\, d\gamma(\xi), \tag{4}$$

$$\sigma_{ij}(x,t) = \left(1 + \frac{\overline{E}_q - \overline{E}_0}{\overline{E}_p} S\big(D_q\big)\right) \times \times \left(\int_{-\infty}^{+\infty}\int_0^{+\infty} X_k(\xi,t)\, U_{ij}^k(x,\xi,t)\, d\xi_2\, d\xi_1 - - \frac{\overline{E}_p - \overline{E}_0}{\overline{E}_p} \int_{D_p} X_k(\xi,t)\, U_{ij}^k(x,\xi,t)\, dS(\xi) + + \int_{-\infty}^{+\infty} g_k(\xi_1,t)\, U_{ij}^k(x,\xi,t)\, d\xi_1 - - \frac{\overline{E}_p - \overline{E}_0}{\overline{E}_p} \int_{\Gamma_p} \sigma_{kl}^{(p)}(\xi,t)\, n_l(\xi)\, U_{ij}^k(x,\xi,t)\, d\gamma(\xi)\right). \tag{5}$$

In the absence of mass forces, the expressions for movements (4) and stresses (5) take the following form:

$$u_k(x,t)=\int_{-\infty}^{+\infty} g_i(\xi_1,t)\,U_k^i(x,\xi,t)\,d\xi_1 - \\ -\frac{\overline{E}_p-\overline{E}_0}{\overline{E}_p}\int_{\Gamma_p}\sigma_{ij}^{(p)}(\xi,t)\,n_j(\xi)\,U_k^i(x,\xi,t)\,d\gamma(\xi), \tag{6}$$

$$\sigma_{ij}(x,t)=\left(1+\frac{\overline{E}_q-\overline{E}_0}{\overline{E}_0}\,S\big(D_q\big)\right)\times \\ \times\left(\int_{-\infty}^{+\infty} g_k(\xi_1,t)\,U_{ij}^k(x,\xi,t)\,d\xi_1 - \right. \\ \left. -\frac{\overline{E}_p-\overline{E}_0}{\overline{E}_p}\int_{\Gamma_p}\sigma_{kl}^{(p)}(\xi,t)\,n_l(\xi)\,U_{ij}^k(x,\xi,t)\,d\gamma(\xi)\right). \tag{7}$$

Using relations (3), we write down the equations that allow us to determine the stresses on the contours of the inclusions:

$$\sigma_{ij}^{(q)}(x,t)\,n_j(x)=\frac{2\overline{E}_q}{\overline{E}_q+\overline{E}_0}\times \\ \times\left[\int_{-\infty}^{+\infty} g_k(\xi_1,t)\,U_{ij}^k(x,\xi,t)\,d\xi_1 - \right. \\ \left. -\frac{\overline{E}_p-\overline{E}_0}{\overline{E}_p}\int_{\Gamma_p}\sigma_{kl}^{(p)}(\xi,t)\,n_l(\xi)\,U_{ij}^k(x,\xi,t)\,d\gamma(\xi)\right]\cdot \\ \cdot n_j(x),\ x\in\Gamma_q. \tag{8}$$

For the numerical solution of the system (8), the contours of the inclusions were discretized by linear elements, which are characterized by the coordinates of their midpoints.

In accordance with [2, 8], unknown densities of potentials for n-th boundary element of the p-th inclusion were approximated using the function $f\big(x_n^{(p)},\xi,t\big)$. For inclusions bounded by contours that do not contain angular points, the density of potential along each boundary element was assumed to be constant.

Moreover, as it turned out, the function $f\big(x_n^{(p)},\xi,t\big)$, which approximates the stresses on each boundary element, is equal to 1 and does not depend on time t if the nodal point $x_n^{(p)}$ belongs to an ordinary boundary element, and $f\big(x_n^{(p)},\xi,t\big)=\left(\frac{r}{d}\right)^{s_1(t)-1}$ if the nodal point $x_n^{(p)}$ belongs to an angular boundary element, where d is the length of this element, a_i $(i=\overline{1,2})$ are the coordinates of the angular point, $r=\left[(\xi_i-a_i)\,(\xi_i-a_i)\right]^{\frac{1}{2}}$.

The unknown densities of potentials for n-th boundary element of the p-th inclusion can be represented as follows:

$$\sigma_{kl}^{(p)}(\xi,t)\,n_l(\xi)\ = \\ =\ A_{kl}\left(x_n^{(p)},t\right)f\left(x_n^{(p)},\xi,t\right)n_l\left(x_n^{(p)}\right), \tag{9}$$

where $A_{kl}\left(x_n^{(p)},t\right)$ are unknown constants; $n_l\left(x_n^{(p)}\right)$ — components of the outward normal vector to the contour of the inclusion at the point $x_n^{(p)}$.

Unknown constants $A_{kl}\left(x_n^{(p)},t\right)$ are determined from a system of linear algebraic equations for a given $t=t_s$:

$$A_{ij}\left(x_n^{(q)},t_s\right)f\left(x_n^{(q)},\xi,t_s\right)n_j\left(x_n^{(q)}\right)=\frac{2\overline{E}_q}{\overline{E}_q+\overline{E}_0}\times \\ \times\left[\left(\sum_{m=0}^{M} g_k\left(\xi_m^*,t\right)\int_{\xi'_m}^{\xi'_{m+1}} U_{ij}^k\left(x_n^{(q)},\xi,t_s\right)d\gamma(\xi) - \right.\right. \\ -\sum_{p=1}^{N}\frac{\overline{E}_p-\overline{E}_0}{\overline{E}_p}\sum_{m=1}^{M_p} A_{kl}\left(x_m^{(p)},t_s\right)n_l\left(x_m^{(p)}\right)\times \\ \left.\times\int_{\xi_m^{(p)}}^{\xi_{m+1}^{(p)}} f\left(x_m^{(p)},\xi,t_s\right)U_{ij}^k\left(x_n^{(q)},\xi,t_s\right)d\gamma(\xi)\right)\times \\ \left.\times\ n_j\left(x_n^{(q)}\right)\right], \tag{10}$$

where $x_n^{(p)}$ are coordinates of the n-th nodal point of the p-th inclusion; M_p is the number of boundary elements on the p-th inclusion; $\xi_n^{(p)}$, $\xi_{n+1}^{(p)}$ — coordinates of the n-th boundary element of the p-th inclusion; ξ'_m, ξ'_{m+1} — coordinates of the m-th discrete element belonging to contour Γ; ξ_m^* is the middle of this element $(m=\overline{1,M})$.

Integrals over segments $\left[\xi_m^{(p)},\,\xi_{m+1}^{(p)}\right]$ should be understood in the sense of Cauchy principal value [4].

Discrete analogs for the stress tensor components have the form

$$\sigma_{ij}(x,t_s)=\left(1+\frac{\overline{E}_q-\overline{E}_0}{\overline{E}_0}\,S\big(D_q\big)\right)\times \\ \times\left(\sum_{m=0}^{M} g_k\left(\xi_m^*,t\right)\int_{\xi'_m}^{\xi'_{m+1}} U_{ij}^k(x,\xi,t_s)\,d\gamma(\xi) - \right. \\ -\sum_{p=1}^{N}\frac{\overline{E}_p-\overline{E}_0}{\overline{E}_p}\sum_{m=1}^{M_p} A_{kl}\left(x_m^{(p)},t_s\right)n_l\left(x_m^{(p)}\right)\times$$

$$\times \int_{\xi_m^{(p)}}^{\xi_{m+1}^{(p)}} f\left(x_m^{(p)}, \xi, t_s\right) U_{ij}^k(x, \xi, t_s)\, d\gamma(\xi)\Bigg), \ x \in D.$$

Thus, the constructed mathematical model with its subsequent numerical implementation makes it possible to determine the stress state of a viscoelastic half-plane with arbitrary inclusions, taking into account the rheological parameters of materials at any given time.

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