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CHOOSING THE OPTIMAL STRATEGY FOR A DISCRETE DYNAMIC COOPERATIVE GAME

Game theory steadily expands across economics, sociology, and engineering. Improvements in computing power have facilitated the creation of mathematical models, the discovery of optimal solutions for diverse and complex problems, and the handling of large datasets. In recent years, game theory concepts have also been applied in the military domain and across all levels of security, including cybersecurity. A game involves the interaction of two or more rational players or teams, each pursuing specific strategies to compete for a given reward. Game theory provides the construction of an optimal strategy for such games. Game theory provides the framework for constructing an optimal strategy for such games. Moreover, game-theoretic approaches can be extended to developing algorithms that allow system developers to predict game outcomes in favor of a group of players, using complex game structures. This article explores research in game theory. A group of players, performing actions sequentially, aims to achieve the best possible result within a limited number of steps. Collected statistical data makes it possible to analyze optimal strategies practically. Based on game theory, the Markov model is also employed as a research method to achieve better outcomes in dynamic corporate games.

K e y w o r d s : ***game theory, statistics, Markov model, cooperative games, strategy.***

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Introduction

In today's highly competitive and rapidly changing market environment, discrete dynamic cooperative games are becoming an essential tool for modeling the strategic behavior of companies and groups in the business environment. In real business, decisions are often made not continuously, but at specific intervals (for example, budgets are determined quarterly, strategic plans once a year). The discrete approach allows you to model the successive stages of decision-making clearly. The dynamism of the market environment is characterized by rapid changes in technology, legislation, and the emergence of new competitors. Dynamic games allow you to consider changes in the state of the system from stage to stage and predict their impact on the players' strategies. In a corporate environment, it is essential to consider the consequences of actions in the long term, and not just instant benefits, so with the help of discrete dynamic games, you can model situations where current decisions affect future opportunities. In addition, modern companies have access to significant amounts of statistical information (Big Data), which allows them to create more realistic dynamic models, and the application of game theory helps to optimize decisions based on the analysis of accumulated data. Game models help companies build adaptive strategies in conditions of uncertainty, when the behavior of competitors or external factors is unpredictable. Discrete dynamic games are used in economics, finance, cybersecurity, energy, logistics, and the development of IT solutions.

A dynamic discrete cooperative game is a model in game theory used to analyze strategic interactions between companies or divisions within an organization. In this model, time is viewed as discrete, that is, divided into distinct periods (e.g., quarters or years), and participants make decisions at each of these stages, taking into account both the current situation and the possible future consequences of their actions.

Let us consider the characteristics of dynamic discrete cooperative games, such as: *discreteness of time, multi-stage, and strategic interaction.*

Discreteness of time means that players make decisions at separate moments in time, which allows for modeling a sequence of strategic steps. Actions are separated by time intervals, and the results of actions accumulate or change step by step. This will enable you to structure the game, easily model strategies, and predict the consequences of actions. Facilitates the application of mathematical methods such as dynamic programming, Markov processes, and optimization theory. Corresponds to real-world scenarios where decisions are made not continuously, but periodically (for example, daily trading on the stock exchange, staged negotiations, turn-by-turn in the game).

Multi-stage (or multi-step) - game participants make decisions sequentially at several stages, and their decisions' results affect the game's further course. At each stage, players can change their strategies, adapt to new circumstances, and take into account both their own past decisions and the actions of other players and opponents. In this case, the game is divided into a certain number of stages or moves, and decisions at each stage depend on the current state of the game and the history of previous steps. The goal of such games is to optimize the total win or achieve a certain result by the end of the game, and players must take into account not only the current win but also the future consequences of their actions. Two types of multi-stage games are considered: with a fixed number of stages (finite game), when players know how many moves there will be, and with an infinite number of stages (infinite or stochastic game), when the game continues indefinitely or with a certain probability of ending at each stage.

In cooperative games that simulate the activities of companies or groups in a business environment, *strategic interaction* means a decision-making process by several participants (companies, divisions, groups), where each participant takes into account not only their own goals, but also the behavior of competitors, partners or customers. The gain of one player depends not only on his actions, but also on the actions of other players. Therefore, each participant tries to predict how others will react to his strategy and adjusts his behavior accordingly. Players can either compete (for example, for market share) or cooperate (for example, create alliances). For such games, it is important to formalize the players, namely to determine who the participants are in the game, describe the strategies that are available to each player, and find an equilibrium state where none of the players has an incentive to change their strategy unilaterally (for example, Nash equilibrium). Among the features of cooperative games, it is worth noting the important role of information (complete, incomplete, asymmetric), long-term consequences of decisions, the frequent occurrence of mixed strategies (planning actions with certain probabilities), and possible coalitions or collaborations between players. Thus, strategic interaction in cooperative games is a process in which companies or groups make decisions, taking into account the possible actions of competitors and partners, to maximize their own gain in conditions of interdependence.

1. Analysis of the latest research and publications

Recent research in the field of game theory demonstrates its effectiveness in solving various corporate problems.

The basic principles of game theory were laid down by E. Borel, A. Cournot, J. Bertrand, J. Neumann, O. Morgenstern, J. Harshani, R. Selten, T. Schelling, R. Aumann. A significant contribution to its development was made by V. Vitlinsky, V. Algin, T. Boydel, E. Wentzel, E. Vilkas, O. Granaturov, S. Nakonechny, V. Sokolov and others. However, it was thanks to J. Neumann that the final formation of this theory took place (Neumann, 1928). He managed to mathematically substantiate the general strategy for a game between two participants in conditions of minimization and maximization. The book by J. Neumann and O. Morgenstern's "Game Theory and Economic Behavior" demonstrate the possibility of applying game theory to a certain number of participants, which made it possible to use it in economics to model the behavior of enterprises in a competitive environment, that is, to form and choose a strategy for the development of enterprises. The "minimax" strategy, or minimization of maximum losses, proposed in this book is especially important. Such a strategy allows rationalizing the costs of enterprises in conditions of uncertainty (Baranovska, & Bukovskiy, 2017; Chan Kim, & Mauborgne, 2016; Dixit, & Nalebuff, 1991).

Zoryana Koval in her work proposes a methodology for selecting and evaluating enterprise strategies using game theory (Koval, 2021). This approach allows you to take into account competitors' strategies or the state of the market, which contributes to making optimal decisions in conflict situations. The author analyzes the advantages and disadvantages of using game theory methods in this area, and also explores the features of using strategy selection criteria. She considered the selection and search for opportunities to use game theory methods to analyze the effectiveness of enterprise strategies, developed methods for assessing the effectiveness of enterprise strategies by modeling the situation using game theory, and forming conclusions and recommendations on the application of game theory methods and tools in the field of assessing the effectiveness of enterprise strategies.

V. V. Kazimko explores the application of game theory to modeling information security problems (Kazimko, 2022). He notes that game theory concepts can be used to develop mechanisms that allow system developers to change the balance and predict outcomes in favor of defenders using complex game designs. Using games, it is possible to develop and analyze optimal player actions and find possible mathematical solutions to security problems. From a security perspective, the combination of cyberspace and physical space leads to cyber-physical security, or the combination of security and economics creates cyber insurance. To solve security and privacy problems in new areas, the most suitable tools are game-theoretical methods, as they provide a variety of proven mathematical methods for creating multi-user strategies using different ways to cover the confidentiality and security aspects of player interaction. The presented theories show the effectiveness and feasibility of using game theory and differential game theory in the field of information security.

2. Research purpose

The research is aimed at constructing an optimal strategy for a discrete dynamic cooperative game. In accordance with the set goal, the following main tasks are solved:

1. Analysis of the literature on research in the field of game theory, namely discrete dynamic cooperative games.
2. Construction of a mathematical model of the game.
3. Choice of strategy for the player.
4. Construction of an optimal strategy.

3. Presentation of the basic material and substantiation of the obtained research results

Consider a dynamic discrete cooperative game. There is a team of N players who perform actions sequentially. The initial player is selected randomly, after which the players take turns deciding to perform an action or pass the move to the next player (passing the move to oneself is prohibited). The game can have a maximum of M stages. If no one has performed an action by the last stage, then the player to whom the move was passed at the last stage is obliged to perform the action.

Rules of the game. Each player has two main options: *to perform an action* or *pass the move* to another player.

Performing an action can lead to:

- winning with probability P_{win}, which brings the player 1 point and ends the game;
- losing with probability P_{loss}, which leads to the loss of 1 point and also ends the game.

Passing the move to another player. A player passes the move to one of N-1 others with probability P_{pass}, and continues the game. Passing the move to oneself is prohibited. Passing the move to a player who has already participated in the game is allowed.

Winning chances for each player are variable at each stage.

Sequence of moves. After each player's move, the next player also chooses between performing an action or passing the move. The player can pass the move to any of the other players (except himself).

Multi-stage game structure. The game takes place in M stages:

Stage 1: The first player (randomly chosen) decides to perform an action or pass the move to another player.

Stage 2: The next player makes a decision. He can perform an action or pass the move to another player, including returning the move to the previous player.

…

Stage M: In the last stage, the player is obliged to perform an action.

Stopping the game. The game stops after any win or loss.

The goal of the game. It is necessary to calculate the probability of winning for players at any stage when performing an action, which involves taking into account all possible decisions of players at previous stages. Choose a strategy for each player at each stage and build the optimal game strategy to obtain the maximum chance of success.

Mathematical formulation.

Let us denote the set of players by the vector $G = (G_1\ G_2\ G_3 \ldots G_N)$.

Probabilities of actions. As initial data, we take complete statistical data of previous games in terms of players and their quality of action execution. From these, we can form relative outcome matrices to construct the initial game matrices.

Let ${P_{win}}_j^k$ be the probability of winning at the stage k $(k \in [1;M])$ of player G_j $(j \in [1;N])$. That is, in matrix form $P_{win} = \begin{pmatrix} P_{win}^1 \\ P_{win}^2 \\ \ldots \\ P_{win}^M \end{pmatrix}$, where P_{win}^k is the vector of success probability, when performing an action at stage k $(k \in [1;M])$, for each player of dimension N.

Let ${P_{lost}}_j^k$ be the probability of losing at the stage k $(k \in [1;M])$ of player G_j $(j \in [1;N])$. In matrix form $P_{lost} = \begin{pmatrix} P_{lost}^1 \\ P_{lost}^2 \\ \ldots \\ P_{lost}^M \end{pmatrix}$, where P_{lost}^k is the vector of failure probabilities, when performing an action at stage k $(k \in [1;M])$, for each player of dimension N.

Let ${P_{pass}}_j^k$ be the probability that player G_j $(j \in [1;N])$ decides to pass the move instead of performing an action at stage k $(k \in [1;M])$.

$${P_{pass}}_j^k = 1 - {P_{win}}_j^k - {P_{lost}}_j^k.$$

In matrix form $P_{pass} = \begin{pmatrix} P_{pass}^1 \\ P_{pass}^2 \\ \ldots \\ P_{pass}^M \end{pmatrix}$, where P_{pass}^k is a vector of probabilities of passing a move for each player of dimension N, and $(k \in [1;M])$.

The probability of passing a move by player G_j $(j \in [1;N])$ to player G_i $(i \in [1;N])$ is the same for any of the stages, denoted by $P_{pass\ ji}$. Since passing a move to oneself is forbidden, then $P_{pass\ jj} = 0$. Moreover, the sum of all probabilities is equal to 1, that is:

$$\sum_{i=1}^{N} P_{pass\ ji} = 1.$$

Accordingly, for player G_j the following equality is valid at each stage:

$${P_{win}}_j^k + {P_{lost}}_j^k + {P_{pass}}_j^k \times \left(\sum_{i=1}^{N} P_{pass\ ji}\right) = 1; \text{ where } k \in [1;M] \text{ is the stage number, } j, i \in [1;N].$$

Objective:

1) Calculate the probability of winning for players at any stage when making certain moves, which involves taking into account all possible decisions of players at previous stages.

2) Choose a strategy for each player at each stage.

3) Construct the optimal game strategy to obtain the maximum winning chances.

Solution Approach.

To calculate the final probability of a player winning at the last stage, one can plot all possible scenarios and apply a decision tree with conditional probabilities at each step. This will ensure an accurate calculation of the success probability.

The probability of a move reaching a player at a given stage and the formation of a payoff matrix.

In order for the game to reach a player, it is necessary that the players at the previous stages pass the move on without performing any action. This scheme is a first-order Markov chain. We will write this sequence in a transition probability matrix, where the matrix data represent the probability of a move passing from one player to another and, finally, achieving a result. Thus, a matrix of probabilities of a move passing from one player to another was created, where the rows correspond to the player passing the move, and the columns to the player receiving the move. Since a player cannot pass the move to himself, the central diagonal of the matrix will be zero.

$$T = \begin{pmatrix} 0 & p_{12} & p_{13} & \ldots & p_{1N} \\ p_{21} & 0 & p_{23} & \ldots & p_{2N} \\ p_{31} & p_{32} & 0 & \ldots & p_{3N} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ p_{N1} & p_{N2} & p_{N3} & \ldots & 0 \end{pmatrix}.$$

For a given transition matrix, the statement that the sum of the probabilities in each row is equal to 1 is valid:

$$\sum_{i=1}^{N} p_{ij} = 1, \qquad j \in [1;N]$$

where $i \in [1;N]$ is the row number of the player who passes the move, $j \in [1;N]$ is the column number of the player who receives the move.

To determine the probability of a player getting a move at a certain stage, we define the initial state vector.

$$V^1 = (p_1 \quad p_2 \quad p_3 \quad \ldots \quad p_N),$$

where the elements are the probabilities of the player getting the first move.

Let's consider the probabilities of the player getting a move at the second stage, for this we multiply the initial state vector by the transition matrix. The resulting vector will be the resulting vector.

$$V^2 = (V^1 \circ P^1_{pass}) \times T,$$

where the elements are the probabilities of the player getting a move in the second stage. Using the Hadamard product, the vectors V^1 and P^1_{pass} are multiplied element by element.

For the stage k, the equality will take the form:

$$V^k = (V^{k-1} \circ P^{k-1}_{pass}) \times T = (V^1 \circ P^1_{pass} \circ P^2_{pass} \circ \ldots \circ P^{k-1}_{pass}) \times T^{k-1}.$$

To determine the probability of winning players at the stage *k* (W^k), we calculate as the Hadamard product the probability of reaching the player's move at the stage k and the probability of winning the player when performing an action at this stage:

$$W^k = V^k \circ P^k_{win}.$$

Therefore, the total probability of winning for players at all stages can be represented as a matrix: $W = \begin{pmatrix} W^1 \\ W^2 \\ \ldots \\ W^M \end{pmatrix}$, where W^k is a vector of success probability for each player of dimension *N* at stage *k* ($k \in [1; M]$).

The matrix found is the odds of winning for players at any stage when performing an action that involves taking into account all possible decisions of players at previous stages.

Choosing a strategy for a player at each stage.

To find the optimal strategy, you need to decide when players should perform an action and when to pass the move to the next player in order to maximize the overall success probability.

Note that:

- At each stage, players have a choice: perform an action or pass the move.
- The optimal strategy will depend on the probabilities of winning and losing each player at each stage, as well as on the probabilities of passing the move.

Determining the criteria for choosing an action or passing the move at stage *k* for player G_j:

If player $G_j$received a move at stage *k*, the expected gain from performing the action for him is W_j^k, where the probability of receiving a move at this stage and the probability of winning are taken into account.

The criterion for choosing to perform an action will be the correspondence of the value of W_j^k to the maximum value of the probability of winning for this stage. That is, if $W_j^k = max\,\{W_i^m, \forall\, i \in [1, N], \forall\, m \in [k, M]\}$ is the maximum value among the elements of the payoff matrix vectors *W*, of all rows of stages starting from the *k*.

If this value is less, the player must pass the move to the next player, i.e. continue the game. The priority player for passing the move will be the player with the maximum odds of winning at the next stage. If such a player is the player passing the move, then pass the move to the next player with the highest probability.

$$G_{next} = max_i\{W_i^{k+1}, \forall\, i \in [1, N], i \neq next, k < M\}.$$

When receiving a move at the last stage *M*, the player is obliged to perform an action.

After constructing the vectors of success probability at each stage, it is possible to determine the potentially most optimal players at each stage. Accordingly, the transfer of the move to the next stage, namely the choice of the player, can be chosen precisely according to this criterion.

Calculation of the optimal strategy.

Using the above criteria, we can determine when each player should pass the move or perform an action to maximize the overall odds of winning.

To achieve a cooperative win, it is necessary to determine the stage and the player with the maximum probability of winning, and construct the optimal path to reach the given player at the given stage for him to perform the action.

$$\max_{i,k}\{W_i^k, \forall\, i \in [1, N], \forall\, k \in [1, M]\}.$$

From the maximum value of the probability of winning, the index *i* will correspond to the player G_i, and *k* - to the stage at which he should perform the action.

Discussions and conclusions

We have formulated and proposed a solution to a dynamic discrete cooperative game. The strategies of the players and the team are determined for the optimal solution of this game.

This dynamic discrete cooperative game is the basic part for building a mathematical model of the game and building optimal strategies for players. Using the constructed outcome matrices (Martyniuk, & Tsurkan, 2024), it is possible to form input data for the considered game. Since probabilistic data are formed based on the analysis of players' actions during previous similar games, they will clearly set the initial conditions for the game. In further studies, it is important to consider the qualitative impact of performing an action or passing a move on the constructed game, as well as the dynamics of changing the initial data for the game according to the mathematical model.

The complete game model can be broken down into parts that can be represented and solved using the game in question. Also, an important area of research will be the practical use. What areas of problems can be represented by such games, and how can we form the initial data for them?

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ВИБІР ОПТИМАЛЬНОЇ СТРАТЕГІЇ ДЛЯ ДИСКРЕТНОЇ ДИНАМІЧНОЇ КООПЕРАТИВНОЇ ГРИ

З кожним роком відбувається розширення використання теорії ігор в економіці, соціології, інженерних науках. Розвиток обчислювальних ресурсів дає змогу будувати математичні моделі та знаходити оптимальні стратегії для різноманітних задач і різних рівнів їхньої складності, обробляти великі об'єми інформації. Останнім часом концепції теорії ігор застосовують також до військової сфери, до всіх рівнів безпеки, включно з кіберпростором. Гра передбачає дії двох або більше раціональних гравців чи команд, що мають певну стратегію і змагаються за певну винагороду. Теорія ігор забезпечує побудову оптимальної стратегії для таких ігор. Крім того, теоретико-ігрові підходи можна поширити на розроблення алгоритмів, що допомагають розробникам систем передбачати результати ігор на користь групи гравців, використовуючи складні ігрові конструкції.

У пропонованій статті розглянуто дослідження теорії ігор. Група гравців, виконуючи послідовно дії, прагнуть досягнути максимального результату за обмежену кількість кроків. Використання зібраних статистичних даних дає змогу на практичному рівні розглянути оптимальні стратегії. Також для досягнення кращих результатів у динамічних кооперативних іграх із використанням теорії ігор як метод дослідження застосовано модель Маркова.

Ключові слова: ***теорія гри, статистика, модель Маркова, кооперативні ігри, стратегія.***

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