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ESTIMATION OF RUIN PROBABILITY FOR RANDOM SUM OF NEGATIVE BINOMIALLY DISTRIBUTED NUMBER OF SUB-GAUSSIAN CLAIMS

This paper investigates the properties of a classical risk process in which the number of insurance claims follows a negative binomial distribution, and the claim sizes belong to the class of strictly sub-Gaussian random variables of a generalized type. This class is broad and includes both standard sub-Gaussian and Gaussian random variables. The core assumption is that each random variable in this class satisfies a specific inequality that constrains its exponential growth, allowing for control over the behavior of its probabilistic tail. It is also assumed that the insurance premium collected over time is described by a continuous, monotone increasing function whose increments are bounded by another continuous, increasing function. This condition enables flexible modeling of premium growth over time.

The aim of the paper is to obtain estimates for the probability of ruin in the described risk process model. To achieve this, the approach is based on the method of metric entropy, which allows for the consideration of both the complex nature of the claim count distribution and the properties of strictly sub-Gaussian claim sizes. This approach makes it possible to derive estimates even in cases where the exact distribution of the claims is unknown, provided that they exhibit sub-Gaussian behavior.

K e y w o r d s: ***negative binomial distribution, sub-Gaussian random variable, ruin probability, claim distribution, metric entropy.***

AMS 2020 classification: 60G50, 91G05.

Introduction

Negative binomial and Poisson distributions are popular models for describing the number of claims, see e.g. (Klugman, Panjer, & Willmot, 2012; Boucher, Denuit, & Guillén, 2008; Ismail & Zamani, 2013; Shi & Valdez, 2014; David & Jemna, 2015). The Poisson distribution assumes that the variance and mean of the claims should be equal, which is often not the case for real data. The negative binomial distribution doesn't have this limitation. So, let's develop a risk model using the negative binomial number of claims on the basis of the classic Poisson model.

$$X(t) = u + ct - \sum_{i=1}^{N_t} Y_i, \quad t \geqslant 0, \tag{1}$$

where u is the initial capital of an insurance company, $c > 0$ is the intensity of receiving insurance premiums and

(I) Y_i are independent and identically distributed random variables representing the values of insurance claims, $\mathsf{E}\, Y_i = m_1$, $\mathsf{E}\, Y_i^2 = m_2$.

(II) $Y_i \in \mathrm{Sub}_\varphi(\Omega)$, i.e. Y_i belong to the space of φ-sub-Gaussian random variables with norm $\tau = \tau_\varphi(Y_i) < \infty$.

(III) N_t is a process that describes the number of insurance claims that happened between 0 and t according to negative binomial distribution, i.e. $\mathbf{P}(N_t = k) = C_{k+r-1}^k (1-\theta)^k \theta^{rt}$, where $k \in \{0, 1, 2, \ldots\}$ and $r$acts as an analog of the claims intensity, such that the expected number of claims during the interval (t, s) is proportional to its length, $\mathsf{E}(N(s) - N(t)) = \frac{rs(1-\theta)}{\theta} - \frac{rt(1-\theta)}{\theta} = \frac{r(s-t)(1-\theta)}{\theta}$. In other words, $N(s) - N(t) \sim \mathrm{NB}(r(s-t), \theta)$.

(IV) N_t and the sequence $(Y_1, Y_2, Y_3, \ldots)$ are mutually independent. And let's also assume that the number of claims on independent time intervals are independent.

The risk process described by (1) can also be expressed as:

$$X(t) = u + Ct - \sum_{i=1}^{N_t} (Y_i + \alpha) = u + (C - \alpha\mu)t - \sum_{i=1}^{N_t} Y_i, \quad t \geqslant 0, \tag{2}$$

where α is the mean value of an insurance payment.

This work focuses on the cumulative total of payments:

$$S(t) = \sum_{i=1}^{N_t} Y_i, \tag{3}$$

under the conditions outlined above.

Properties of similar sums with φ-sub-Gaussian claim sizes with Poisson or binomial number of claims are studied in papers (Yamnenko & Vasylyk, 2007; Saienko & Yamnenko, 2013) and (Yamnenko & Lamin, 2022).

The characteristics of sub-Gaussian, φ-sub-Gaussian and strictly φ-sub-Gaussian random variables and processes can be found in the books (Buldygin & Kozachenko, 2000; Kozachenko, Vasylyk, & Yamnenko, 2008; Vershynin, 2018) and other studies, e.g. (Kozachenko & Kovalchuk, 1998; Giuliano Antonini, Kozachenko, & Nikitina, 2003; Kozachenko, Yamnenko, & Vasylyk, 2005; Hopkalo, Kozachenko, Vasylyk, & Sakhno, 2020).

1. φ-sub-Gaussian random values and processes

This section contains the necessary definitions for φ-sub-Gaussian random variables and processes.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a standard probability space and let $(\mathbf{T}, \rho)$ be a pseudometric (metric) compact space equipped by pseudometric (metric) ρ.

Definition 1. (Buldygin & Kozachenko, 2000) A continuous even convex function φ is said to be an Orlicz N-function if it is strictly increasing for $x > 0$, $\varphi(0) = 0$ and

$$\frac{\varphi(x)}{x} \to 0 \text{ as } x \to 0 \quad \text{and} \quad \frac{\varphi(x)}{x} \to \infty \text{ as } x \to \infty.$$

Condition Q. (Guiliano, Kozachenko, & Nikitina, 2023)

We say that an N-function φ satisfies condition Q if

$$\liminf_{x\to 0} \frac{\varphi(x)}{x^2} = \alpha > 0. \tag{4}$$

It is permitted α to be infinite.

Definition 2. (Buldygin & Kozachenko, 2000) Let φ be an Orlicz N-function satisfying condition Q. The random variable ξ belongs to the space $\mathrm{Sub}_\varphi(\Omega)$ (a space of φ-sub-Gaussian random variables), if it is centered, i.e. $\mathsf{E}\,\xi = 0$, the moment generating function $\mathsf{E}\exp\{\lambda\xi\}$ exists for all $\lambda \in \mathbb{R}$ and there exists a positive constant a such that the following inequality

$$\mathsf{E}\exp(\lambda\xi) \leqslant \exp(\varphi(a\lambda)) \tag{5}$$

holds for all $\lambda \in \mathbb{R}$.

Theorem 1. *(Buldygin & Kozachenko, 2000) The space* $\mathrm{Sub}_\varphi(\Omega)$ *is a Banach space with respect to the norm*

$$\tau_\varphi(\xi) = \inf\{a \geqslant 0\colon\ \mathsf{E}\exp(\lambda\xi) \leqslant \exp(\varphi(a\lambda)),\ \lambda \in \mathbb{R}\}$$

and the inequality

$$\mathsf{E}\exp(\lambda\xi) \leqslant \exp\big(\varphi\big(\lambda\tau_\varphi(\xi)\big)\big), \tag{6}$$

holds for all $\lambda \in \mathbb{R}$*. Moreover, for all* $r > 0$ *there exists constant* $c_r > 0$ *such that* $(\mathsf{E}\,\xi^r)^{1/r} \leqslant c_r\tau_\varphi(\xi)$.

Definition 3. (Buldygin & Kozachenko, 2000) Metric entropy with regard to pseudometric (metric) ρ, or just metric entropy is a function

$$H_{(T,\rho)}(u) = H_T(u) = H(u) = \begin{cases} \log N_{(T,\rho)}(u), & \text{if } N_{(T,\rho)}(u) < +\infty \\ +\infty, & \text{if } N_{(T,\rho)}(u) = +\infty \end{cases},$$

where $N_{(T,\rho)}(u) = N_T(u) = N(u)$ denotes the least number of closed ρ balls with radius u covering the space (T, ρ).

Example. If $\mathbf{T}$ is an interval $[a, b]$ and ρ is the Euclidean distance then

$$\ln\left(\max\left\{\frac{b-a}{2u}, 1\right\}\right) \leqslant H(u) \leqslant \ln\left(\frac{b-a}{2u} + 1\right).$$

Definition 4. The random process $X = \{X(t), t \in T\}$ is called *φ-sub-Gaussian process* if for all $t \in T$ $X(t) \in \mathrm{Sub}_\varphi(\Omega)$.

Definition 5. (Kozachenko & Kovalchuk, 1998) A family of random variables Δ from the space $\mathrm{Sub}_\varphi(\Omega)$ is called *strictly* $\mathrm{Sub}_\varphi(\Omega)$, if there exists a constant $C_\Delta > 0$ such that for arbitrary finite set $I : \xi_i \in \Delta, i \in I$, and for any $\lambda_i \in \mathbb{R}$ the following inequality takes place

$$\tau_\varphi\left(\sum_{i\in I} \lambda_i\xi_i\right) \leqslant C_\Delta\left(\mathsf{E}\left(\sum_{i\in I}\lambda_i\xi_i\right)^2\right)^{1/2}. \tag{7}$$

If Δ is a family of strictly $\mathrm{Sub}_\varphi(\Omega)$ random variables, then the linear closure $\overline{\Delta}$ of the family Δ in the space $L_2(\Omega)$ is also strictly $\mathrm{Sub}_\varphi(\Omega)$ family of random variables. Linearly closed families of strictly $\mathrm{Sub}_\varphi(\Omega)$ random variables form a space of *strictly φ-sub-Gaussian* random variables. This space is denoted by $\mathrm{SSub}_\varphi(\Omega)$.

When $\varphi(x) = x^2/2$ the space $\mathrm{SSub}_\varphi(\Omega)$ is called the space of *strictly sub-Gaussian* random variables and is denoted as $\mathrm{SSub}(\Omega)$. The space of jointly Gaussian random variables belongs to the space $\mathrm{SSub}(\Omega)$ and $\tau^2(\xi) = \mathsf{E}\,\xi^2$, i.e. $C_\Delta = 1$.

Definition 6. (Buldygin & Kozachenko, 2000) A random process $X = \{X(t), t \in T\}$ is a *strictly φ-sub-Gaussian process* if the corresponding family of random variables belongs to the space $\mathrm{SSub}_\varphi(\Omega)$. The defining constant of this family is called the defining constant of the process X and is denoted by C_X.

Condition Σ**.** Suppose there exists such a continuous monotonically increasing function $\sigma = \{\sigma(h), h > 0\}$, that $\sigma(h) \to 0$, as $h \to 0$, and the following inequality for increments of the process is true

$$\sup_{\rho(t,s)\leqslant h} \tau_\varphi(Y(t) - Y(s)) \leqslant \sigma(h). \tag{8}$$

If a process $X(t)$ is continuous in norm $\tau_\varphi(\cdot)$ then the function $\sigma(h) = \sup\limits_{\rho(t,s)\leqslant h} \tau_\varphi(X(t) - X(s))$ satisfies condition Σ.

2. Negative binomial random sums

In this section, we study the properties of sums whose number of terms follows a non-negative binomial distribution, as they appear in model (2). The following lemmas provide results concerning the moments of such sums and estimates for their moment-generating functions.

Lemma 1. *Let* $\left\{S(t) = \sum_{i=1}^{N_t} Y_i(t), t \geqslant 0\right\}$ *be a random process satisfying the conditions (I)–(IV). Then for all* $t \geqslant 0$, $0 \leqslant t_1 \leqslant t_2$, *and* $\lambda \in \mathbb{R}$ *the following expressions hold true*

$$\mathsf{E}\, S(t) = \frac{(1-\theta)rt}{\theta} m_1, \quad \mathsf{Var}\, S(t) = \frac{(1-\theta)rt}{\theta}(m_2 - m_1^2) + m_2 \frac{(1-\theta)rt}{\theta^2},$$
$$\mathsf{E}(S(t_2) - S(t_1)) = \frac{(1-\theta)r(t_2-t_1)}{\theta} m_1,$$
$$\mathsf{Var}(S(t_2) - S(t_1)) = \frac{(1-\theta)r(t_2-t_1)}{\theta}(m_2 - m_1^2) + m_2 \frac{(1-\theta)r(t_2-t_1)}{\theta^2}$$

and

$$\mathsf{E}\exp\{\lambda S(t)\} \leqslant \left(\frac{\theta}{1-(1-\theta)\exp\{\varphi(\lambda\tau)\}}\right)^{rt},$$
$$\mathsf{E}\exp\{\lambda(S(t_2) - S(t_1))\} \leqslant \left(\frac{\theta}{1-(1-\theta)\exp\{\varphi(\lambda\tau)\}}\right)^{r(t_2-t_1)}.$$

Proof. Recall that for a random variable with a negative binomial distribution $\xi \sim \mathrm{NB}(r,p)$, we have its moment-generating function is given by $M_\xi(\lambda) = \mathsf{E}\exp\{\lambda\xi\} = \left(\frac{p}{1-(1-p)e^\lambda}\right)^r$. Considering that $S(t)$ has the compound negative binomial distribution, from (6) we have the following:

$$\mathsf{E}\exp\{\lambda S(t)\} = \mathsf{E}\exp\left\{\lambda\sum_{i=1}^{N_t} Y_i\right\} = \mathsf{E}\left[\mathsf{E}\left[\exp\left\{\sum_{i=1}^{N_t}\lambda Y_i\right\}\Big|N_t\right]\right] =$$
$$= \sum_{n=0}^{\infty}\mathsf{E}\exp\left\{\sum_{i=1}^{n}\lambda Y_i\right\}\mathbf{P}(N_t = n) = \sum_{n=0}^{\infty}\prod_{i=1}^{n}\exp\{\lambda Y_i\}\mathbf{P}(N_t = n) = \sum_{n=1}^{\infty}(\mathsf{E}\exp\{\lambda Y_1\})^n\mathbf{P}(N_t = n) \leqslant$$
$$\leqslant \sum_{n=1}^{\infty}(\exp\{\varphi(\lambda\tau_\varphi(Y_1))\})^n\mathbf{P}(N_t = n) \leqslant \left(\frac{\theta}{1-(1-\theta)\exp\{\varphi(\lambda\tau)\}}\right)^{rt}.$$

Since $N_{t_1} \leqslant N_{t_2}$ for $t_1 \leqslant t_2$, similarly we get

$$\mathsf{E}\exp\{\lambda(S(t_2) - S(t_1))\} = \mathsf{E}\exp\left\{\lambda\sum_{i=N_{t_1}+1}^{N_{t_2}} Y_i\right\} = \mathsf{E}\left(\mathsf{E}\exp\left\{\lambda\sum_{i=N_{t_1}+1}^{N_{t_2}} Y_i\right\}\Big|N_{t_1}, N_{t_2}\right) =$$
$$= \sum_{k=0}^{\infty}\sum_{m=k}^{\infty}(\mathsf{E}\exp(\lambda Y_1))^{m-k}\mathbf{P}(N_{t_1} = k)\mathbf{P}(N_{t_2} = m|N_{t_1} = k) =$$
$$= \sum_{k=0}^{\infty}\sum_{m-k=0}^{\infty}(\mathsf{E}\exp(\lambda Y_1))^{m-k}\mathbf{P}(N_{t_1} = k)\mathbf{P}(N_{t_2} - N_{t_1} = m-k) =$$
$$= \sum_{k=0}^{\infty}\mathbf{P}(N_{t_1} = k)\sum_{j=0}^{\infty}(\mathsf{E}\exp(\lambda Y_1))^j\mathbf{P}(N_{t_2} - N_{t_1} = j) = \sum_{j=0}^{\infty}(\mathsf{E}\exp(\lambda Y_1))^j\mathbf{P}(N_{t_2} - N_{t_1} = j) \leqslant$$
$$\leqslant \sum_{j=0}^{\infty}\exp\{\varphi(\lambda\tau_\varphi(Y_1))\}^j\mathbf{P}(N_{t_2} - N_{t_1} = j) = \left(\frac{\theta}{1-(1-\theta)\exp\{\varphi(\lambda\tau)\}}\right)^{r(t_2-t_1)}.$$

The first two moments for a compound distribution (see, e.g. (Klugman et al., 2012)) are

$$\mathsf{E}\, S(t) = \mathsf{E}\, N_t\, \mathsf{E}\, Y_1, \quad \mathsf{E}\, S^2(t) = \mathsf{E}\, N_t\, \mathsf{Var}\, Y_1 + \mathsf{E}\, Y_1^2\, \mathsf{E}\, N_t^2, \quad \mathsf{Var}\, S(t) = \mathsf{E}\, N_t\, \mathsf{Var}\, Y_1 + \mathsf{E}\, Y_1^2\, \mathsf{Var}\, N_t,$$

therefore

$$\mathsf{E}\, S(t) = \frac{(1-\theta)rt}{\theta} m_1, \quad \mathsf{Var}\, S(t) = \frac{(1-\theta)rt}{\theta}(m_2 - m_1^2) + m_2 \frac{(1-\theta)rt}{\theta^2}.$$

and

$$\mathsf{E}(S(t_2)-S(t_1)) = \mathsf{E}(N_{t_2}-N_{t_1})\,\mathsf{E}\,Y_1 = \frac{(1-\theta)r(t_2-t_1)}{\theta}m_1,$$

$$\mathsf{Var}(S(t_2)-S(t_1)) = \mathsf{Var}\,S(t_2-t_1) = \frac{(1-\theta)r(t_2-t_1)}{\theta}(m_2-m_1^2)+m_2\frac{(1-\theta)r(t_2-t_1)}{\theta^2}.$$

□

We now examine the estimates for the probability that the sums $S(t)$ in (3) exceed a given level $x \geqslant 0$, where $f(t)$ is a continuous function representing the total insurance premiums collected over time t.

First, we consider the conditions necessary to bound the supremum of the difference $S(t)-f(t)$. Additionally, we derive an estimate for the exponential moment of this supremum.

From Lemma 1 it follows that

$$\mathsf{E}(S(t_2)-S(t_1))^2 = \mathsf{Var}(S(t_2)-S(t_1)) + [\mathsf{E}(S(t_2)-S(t_1))]^2 =$$
$$= \frac{(1-\theta)r(t_2-t_1)}{\theta^2}[\theta^2(m_2-m_1^2)+m_2+(1-\theta)r(t_2-t_1)m_1^2] \leqslant$$
$$\leqslant \frac{(1-\theta)r(t_2-t_1)\max\{t_2-t_1,1\}}{\theta^2}\left[\theta^2(m_2-m_1^2)+m_2+(1-\theta)rm_1^2\right].$$

Then let's choose

$$\sigma(h) = \kappa h^{\frac{1}{2}}\mathbf{1}_{h<1} + \kappa h\mathbf{1}_{h\geqslant 1}, \tag{9}$$

as a function $\sigma(h)$ satisfying assumption (8) for the process $S(t)$, where $\kappa = \frac{(1-\theta)r}{\theta^2}[\theta^2(m_2-m_1^2)+m_2+(1-\theta)rm_1^2]$ and $\mathbf{1}_{\cdot}$ is the indicator function.

Let $B \subset \mathbf{T}$ be a compact set. Put $0<\beta\leqslant 1$ be such a number that $\beta \leqslant \sigma\left(\inf_{s\in B}\sup_{t\in B}\rho(t,s)\right)$.

Lemma 2. *Let $X(t) = S(t)-f(t)$ with $\{S(t)=\sum_{i=1}^{N_t}Y_i, t\in\mathbf{T}\}$ be a random process satisfying assumptions (I)–(IV). Let's assume also that increments of continuous function $f=\{f(t),t\in\mathbf{T}\}$ are bounded by a monotone increasing function $\delta=\{\delta(s),s>0\}$: $|f(u)-f(v)|\leqslant\delta(\rho(u,v))$, and let $\{q_k,k\in\mathbb{N}\}$ be such a sequence that $q_k>1$, and $\sum_{k=1}^{\infty}\frac{1}{q_k}\leqslant 1$. Then for all $\lambda>0$ and $p\in(0,1)$ the following estimate holds true*

$$\mathsf{E}\exp\left\{\lambda\sup_{t\in B}X(t)\right\} \leqslant \sup_{u\in B}\left[\left(\frac{\theta}{1-(1-\theta)\exp\{\varphi(q_1\lambda\tau)\}}\right)^{ru/q_1}\exp\{-\lambda f(u)\}\right]\times$$
$$\times\left[\prod_{k=1}^{\infty}\left(N_B\left(\frac{\beta^2p^{2k}}{\kappa^2}\right)\right)^{\frac{1}{q_k}}\right]\left[\prod_{k=2}^{\infty}\left(\frac{\theta}{1-(1-\theta)\exp\{\varphi(q_k\lambda\tau)\}}\right)^{\frac{r\beta^2p^{2k}}{\kappa^2q_k}}\exp\left\{\lambda\delta\left(\frac{\beta^2p^{2k}}{\kappa^2}\right)\right\}\right]. \tag{10}$$

Proof. Put $\varepsilon_k = \sigma^{(-1)}\left(\beta p^k\right) = \frac{1}{\kappa^2}\beta^2p^{2k}$ and denote by V_{ε_k} set of closed balls of radii ε_k, which forms the miminal coverage of the compact B. The number of points in the set V_{ε_k} equals to $N_B(\varepsilon_k)$. From Lemma 1 and Chebyshev's inequality for any $\varepsilon>0$ we get

$$\mathbf{P}\{|S(t)-S(s)|>\varepsilon\} \leqslant \frac{1}{\varepsilon^2}\mathsf{E}\left(\sum_{n=1}^{N_t}Y_n-\sum_{k=1}^{N_s}Y_k\right)^2 = \varepsilon^{-2}[\sigma(t-s)]^2.$$

So, the processes $S(t)$ and $X(t)=S(t)-f(t)$ are continuous in probability. Therefore the set $V=\bigcup_{k=1}^{\infty}V_{\varepsilon_k}$ is a separant of X, and with probability one the following equality holds true

$$\sup_{t\in B}X(t) = \sup_{t\in V}X(t). \tag{11}$$

Consider α-mapping of the V into V_{ε_n} (see (Buldygin & Kozachenko, 2000)). Let's apply Chebyshev's inequality. It follows from (6), (9) and lemma 1 that

$$\mathbf{P}\left\{|S(t)-S(\alpha_n(t))|>p^{\frac{n}{2}}\right\} \leqslant \frac{\mathsf{E}\left(S(t)-S(\alpha_n(t))\right)^2}{p^n} = \frac{\sigma^2(|t-\alpha_n(t)|)}{p^n} < \frac{\sigma^2(\varepsilon_n)}{p^n} = \frac{[\sigma(\sigma^{(-1)}(\beta p^n))]^2}{p^n} = \beta^2p^n.$$

This last inequality means that

$$\sum_{n=1}^{\infty}P\left\{|S(t)-S(\alpha_n(t))|>p^{\frac{n}{2}}\right\}<\infty.$$

Therefore, it follows from the Borel-Cantelli lemma that $S(t)-S(\alpha_n(t))\to 0$ with probability one. The same statement is valid for the process X.

The set V is countable, therefore $X(t)-X(\alpha_n(t))\to 0$ as $n\to\infty$ for all t simultaneously. Let t be an arbitrary point from the set V. Put $t_m=\alpha_m(t)$, $t_{m-1}=\alpha_{m-1}(t_m)$, …, $t_1=\alpha_1(t_2)$ for any integer $m\geqslant 1$. Then for all $m\geqslant 2$ the following inequality takes place

$$X(t) = X(t_1)+\sum_{k=2}^{m}(X(t_k)-X(t_{k-1}))+X(t)-X(\alpha_m(t)) \leqslant \max_{u\in V_{\varepsilon_1}} X(u)+\sum_{k=2}^{m} \max_{u\in V_{\varepsilon_k}} (X(u)-X(\alpha_{k-1}(u))+X(t)-X(\alpha_m(t))).$$

Therefore it follows from (11) that with probability one

$$\sup_{t\in B} X(t) = \sup_{t\in V} X(t) \leqslant \lim_{m\to\infty} \inf \Big(\max_{u\in V_{\varepsilon_1}} X(u) + \sum_{k=2}^{m} \max_{u\in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u))) \Big).$$

Next, applying Hölder's inequality and Fatou's lemma, we obtain for all $\lambda > 0$ that

$$\mathsf{E}\exp\Big\{\lambda \sup_{t\in B} X(t)\Big\} \leqslant \mathsf{E}\liminf_{m\to\infty}\exp\Big\{\lambda\Big(\max_{u\in V_{\varepsilon_1}} X(u) + \sum_{k=2}^{m} \max_{u\in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u)))\Big)\Big\} \leqslant$$

$$\leqslant \liminf_{m\to\infty}\mathsf{E}\exp\Big\{\lambda\Big(\max_{u\in V_{\varepsilon_1}} X(u) + \sum_{k=2}^{m} \max_{u\in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u)))\Big)\Big\} \leqslant$$

$$\leqslant \liminf_{m\to\infty}\Big(\mathsf{E}\exp\{q_1\lambda \max_{u\in V_{\varepsilon_1}} X(u)\}\Big)^{\frac{1}{q_1}} \prod_{k=2}^{m}\Big(\mathsf{E}\exp\Big\{q_k\lambda \max_{u\in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u)))\Big\}\Big)^{\frac{1}{q_k}} \leqslant$$

$$\leqslant \left(\mathsf{E}\exp\left\{q_1\lambda \max_{u\in V_{\varepsilon_1}} X(u)\right\}\right)^{\frac{1}{q_1}} \prod_{k=2}^{\infty}\left(\mathsf{E}\exp\left\{q_k\lambda \max_{u\in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u)))\right\}\right)^{\frac{1}{q_k}} = I_1\cdot\prod_{k=2}^{\infty} I_k.$$

Let's look at each multiplier I_k from the above separately. It follows from the Lemma 1 that

$$\mathsf{E}\exp\{q_1\lambda S(u)\} \leqslant \left(\frac{\theta}{1-(1-\theta)\exp\{\varphi(q_1\lambda\tau)\}}\right)^{rt}$$

and

$$\mathsf{E}\exp\{q_k\lambda\,(S(u) - S(\alpha_{k-1}(u)))\} \leqslant \left(\frac{\theta}{1-(1-\theta)\exp\{\varphi(q_k\lambda\tau)\}}\right)^{r|u-\alpha_{k-1}(u)|} \leqslant \left(\frac{\theta}{1-(1-\theta)\exp\{\varphi(q_k\lambda\tau)\}}\right)^{r\varepsilon_k}.$$

Using these inequalities, we get

$$I_1 \leqslant \Big(\sum_{u\in V_{\varepsilon_1}} \mathsf{E}\exp\{q_1\lambda S(u)\}\exp\{-q_1\lambda f(u)\}\Big)^{\frac{1}{q_1}} \leqslant$$

$$\leqslant \left(\sum_{u\in V_{\varepsilon_1}} \max_{u\in V_{\varepsilon_1}} \left(\frac{\theta}{1-(1-\theta)\exp\{\varphi(q_1\lambda\tau)\}}\right)^{ru} \exp\{-q_1\lambda f(u)\}\right)^{\frac{1}{q_1}} \leqslant$$

$$\leqslant \big(N_B(\varepsilon_1)\big)^{\frac{1}{q_1}} \sup_{u\in B}\left[\left(\frac{\theta}{1-(1-\theta)\exp\{\varphi(q_1\lambda\tau)\}}\right)^{ru/q_1} \exp\{-\lambda f(u)\}\right].$$

Due to the assumption on boundedness of increments of the function f

$$f(u) - f(\alpha_{k-1}(u)) \leqslant \delta(\rho(u, \alpha_{k-1}(u))) \leqslant \delta(\varepsilon_k).$$

Then in the similar way as for I_1 we get the following estimate for $k \geqslant 2$

$$I_k \leqslant \Big(N_B(\varepsilon_k) \max_{u\in V_{\varepsilon_k}} \mathsf{E}\exp\{q_k\lambda(S(u) - S(\alpha_{k-1}(u)))\}\exp\{-q_k\lambda\,(f(u) - f(\alpha_{k-1}(u)))\}\Big)^{\frac{1}{q_k}} \leqslant$$

$$\leqslant (N_B(\varepsilon_k))^{\frac{1}{q_k}}\left(\max_{u\in V_{\varepsilon_k}}\left(\frac{\theta}{1-(1-\theta)\exp\{\varphi(q_k\lambda\tau)\}}\right)^{r\varepsilon_k}\exp\{q_k\lambda\delta(\varepsilon_k)\}\right)^{\frac{1}{q_k}} \leqslant$$

$$\leqslant (N_B(\varepsilon_k))^{\frac{1}{q_k}}\left(\frac{\theta}{1-(1-\theta)\exp\{\varphi(q_k\lambda\tau)\}}\right)^{r\varepsilon_k/q_k}\exp\{\lambda\delta(\varepsilon_k)\} =$$

$$= \left(N_B\left(\frac{\beta^2 p^{2k}}{\kappa^2}\right)\right)^{\frac{1}{q_k}}\left(\frac{\theta}{1-(1-\theta)\exp\{\varphi(q_k\lambda\tau)\}}\right)^{\frac{r\beta^2p^{2k}}{\kappa^2 q_k}}\exp\left\{\lambda\delta\left(\frac{\beta^2p^{2k}}{\kappa^2}\right)\right\}.$$

This is where the assertion of the lemma follows. □

Applying Chebyshev's inequality to Lemma 2, one can obtain an upper estimate of the ruin probability for our risk model.

Theorem 2. *Let $X(t)=S(t)-f(t)$, where $\{S(t)=\sum_{i=1}^{N_t} Y_i, t\in B\}$ is a cumulative claim process satisfying assumptions (I)–(IV). Assume also that increments of continuous premium function $f=\{f(t), t\in \mathbf{T}\}$ are bounded by such a continuous monotone increasing function $\delta=\{\delta(s), s>0\}$ that $|f(u)-f(v)|\leqslant \delta(\rho(u,v))$, and let $r=\{r(u), u\geqslant 1\}$ be such a non-decreasing monotone continuous function that $r(u)>0$ as $u>1$, and $s(t)=r(\exp\{t\}), t\geqslant 0$, is a convex function. Then for all $0\leqslant \beta\leqslant \min\{\sigma(b-a)/2), 1\}$ the following estimates hold true*

$$\mathsf{P}\left\{\sup_{t\in B}(S(t)-f(t))>x\right\}\leqslant Z_r(\beta,x),\quad \mathsf{P}\left\{\inf_{t\in B}(S(t)-f(t))<-x\right\}\leqslant Z_r(\beta,x),$$

$$\mathsf{P}\left\{\sup_{t\in B}|S(t)-f(t)|>x\right\}\leqslant 2Z_r(\beta,x),$$

where

$$Z_r(\beta,x)=\inf_{p\in(0,1)} r^{(-1)}\left(\frac{1}{\beta p}\int_0^{\beta p} r\left(N_B\left(u^2/\kappa^2\right)\right)du\right)\times$$

$$\times\inf_{\lambda>0}\exp\left\{\eta_\varphi(\lambda,p)+\sum_{k=2}^{\infty}\left[\frac{r\beta^2p^{3k-1}(1-p)}{\kappa^2}\left(\ln\theta-\ln\left(1-(1-\theta)\exp\left\{\varphi\left(\frac{\lambda\tau}{(1-p)p^{k-1}}\right)\right\}\right)\right)+\lambda\delta\left(\frac{\beta^2p^{2k}}{\kappa^2}\right)\right]-\lambda x\right\},$$

$$\eta_\varphi(\lambda,p)=\sup_{u\in B}\Big[(\ln\theta-\ln(1-(1-\theta)\exp\{\varphi(\lambda\tau/(1-p))\}))(1-p)ru-\lambda f(u)\Big].$$

Proof. Let's apply Lemma 2 for the sequence $q_k=\frac{1}{(1-p)p^{k-1}}$, where $p\in(0,1)$. Since

$$\ln\sup_{u\in B}\left[\left(\frac{\theta}{1-(1-\theta)\exp\{\varphi(\lambda\tau/(1-p))\}}\right)^{(1-p)ru}\exp\{-\lambda f(u)\}\right]=$$

$$=\sup_{u\in B}\Big[(\ln\theta-\ln(1-(1-\theta)\exp\{\varphi(\lambda\tau/(1-p))\}))(1-p)ru-\lambda f(u)\Big]=\eta_\varphi(\lambda,p),$$

then

$$\mathsf{E}\exp\left\{\lambda\sup_{t\in B}X(t)\right\}\leqslant\exp\left\{\eta_\varphi(\lambda,p)+\sum_{k=1}^{\infty}(1-p)p^{k-1}H_B\left(\frac{\beta^2p^{2k}}{\kappa^2}\right)+\right.$$

$$\left.+\sum_{k=2}^{\infty}\left[\frac{r\beta^2p^{3k-1}(1-p)}{\kappa^2}\left(\ln\theta-\ln\left(1-(1-\theta)\exp\left\{\varphi\left(\frac{\lambda\tau}{(1-p)p^{k-1}}\right)\right\}\right)\right)+\lambda\delta\left(\frac{\beta^2p^{2k}}{\kappa^2}\right)\right]\right\}.\quad(12)$$

Properties of $r(t)$ imply

$$\exp\left\{\sum_{k=1}^{\infty}(1-p)p^{k-1}H_B\left(\frac{\beta^2p^{2k}}{\kappa^2}\right)\right\}=r^{(-1)}\left(r\Big(\exp\Big\{\sum_{k=1}^{\infty}(1-p)p^{k-1}\ln N_B\Big(\frac{\beta^2p^{2k}}{\kappa^2}\Big)\Big\}\Big)\right)\leqslant$$

$$\leqslant r^{(-1)}\left(\sum_{k=1}^{\infty}(1-p)p^{k-1}r\left(N_B\left(\sigma^{(-1)}\left(\beta p^k\right)\right)\right)\right)\leqslant r^{(-1)}\left(\frac{1}{\beta p}\int_0^{\beta p}r\left(N_B\left(\sigma^{(-1)}(u)\right)\right)du\right),\quad(13)$$

where the function $\sigma(u)$ is defined in (9). Then the statement of the theorem follows from the Lemma 2, inequalities (12), (13) and the Chebyshev's one. □

Corollary 1. *Let $f(t)=ct$ describes the linear incoming flow of premiums during time interval $[0,T]$. Then $\delta(u)=cu$ and*

$$\eta_\varphi(\lambda,p)=\Big[(\ln\theta-\ln(1-(1-\theta)\exp\{\varphi(\lambda\tau/(1-p))\}))(1-p)r-\lambda c\Big]T$$

if $(\ln\theta-\ln(1-(1-\theta)\exp\{\varphi(\lambda\tau/(1-p))\}))(1-p)r-\lambda c>0$ and $\eta_\varphi(\lambda,p)=0$ elsewhere.

Corollary 2. *Let $\varphi(u)=u^2/2$, i.e. claim sizes are sub-Gaussian random variables. Let also $f(t)=ct, t\in[0,T]$ as in Corollary 1. Then*

$$\eta_\varphi(\lambda,p)=\left[\left(\ln\theta-\ln\left(1-(1-\theta)\exp\left\{\frac{\lambda^2\tau^2}{2(1-p)^2}\right\}\right)\right)(1-p)r-\lambda c\right]T$$

if $\left(\ln\theta-\ln\left(1-(1-\theta)\exp\left\{\frac{\lambda^2\tau^2}{2(1-p)^2}\right\}\right)\right)(1-p)r-\lambda c>0$ and $\eta_\varphi(\lambda,p)=0$ elsewhere.

Discussion and conclusions

The results of Lemma 2 can be extended to other sequences q_k, allowing for alternative estimates of the ruin probability. By specifying the exact form of the Orlicz N-function φ deeper optimization in Theorem 2 can be achieved. Of particular interest is the space of sub-Gaussian random variables, where $\varphi(x)=x^2/2$.

Authors' contribution: Rostyslav Yamnenko – conceptualization and theoretical foundations of research; Volodymyr Levchenko – making the mathematical calculations.

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ОЦІНЮВАННЯ ЙМОВІРНОСТІ БАНКРУТСТВА ДЛЯ ВИПАДКОВОЇ СУМИ НЕГАТИВНО БІНОМІАЛЬНО РОЗПОДІЛЕНОГО ЧИСЛА СУБГАУССОВИХ ПОЗОВІВ

У пропонованій роботі досліджено властивості класичного процесу ризику, в якому кількість страхових подій має негативний біноміальний розподіл, а розміри позовних вимог належать до класу узагальнених строго субгауссових випадкових величин. Цей клас є досить широким й охоплює як стандартні субгауссові, так і гауссові випадкові величини. В основі такого підходу лежить вимога, щоб кожна випадкова величина в цьому класі задовольняла певну нерівність, яка обмежує її експоненціальне зростання, що дає змогу контролювати поведінку її ймовірнісного хвоста. При цьому також передбачається, що страховий дохід, який надходить упродовж часу, є неперервною монотонно зростаючою функцією, прирости якої не перевищують значення іншої неперервної зростаючої функції. Така умова сприяє гнучкому регулюванню зміні доходу з плином часу.

Метою роботи є отримання оцінок ймовірності банкрутства для описаної моделі процесу ризику. Для досягнення цієї мети використовується підхід, заснований на методі метричної ентропії. Цей метод дає змогу враховувати як складну природу розподілу кількості страхових подій, так і властивості строго субгауссових розмірів позовних вимог. Застосування такого підходу робить можливим отримання оцінок і в ситуаціях, коли точний вигляд розподілу позовів невідомий, але відомо, що вони мають субгауссову поведінку.

Ключові слова: ***негативний біноміальний розподіл, субгауссова випадкова величина, ймовірність банкрутства, розподіл страхових позовів, метрична ентропія.***

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