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Про діагональні локальні SL–групи

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On diagonal locally SL–groups

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Нехай $\mathbb{N}$ — множина натуральних чисел. Нехай $\mathbb{F}$ — поле. У роботі [1] визначені класи груп $SL_s^p(\mathbb{F})$ та $GL_s^p(\mathbb{F})$ періодичних нескінченних $\mathbb{N}\times\mathbb{N}$–матриць таких, що відповідають числу Стейніца s. У даній роботі ми вводимо ширший клас діагональних локальних SL–груп та GL–груп і вивчаємо їх зв'язок з локально матричними алгебрами. Зокрема, ми показуємо, що довільна сепарабельно–діагональна локальна SL–група (відповідно GL–група) ізоморфна групі $SL_s^p(\mathbb{F})$ (відповідно $GL_s^p(\mathbb{F})$).

Ключові слова: Локально матрична алгебра, число Стейніца, діагональне занурення, періодична матриця.

Let $\mathbb{N}$ be the set of natural numbers. Let $\mathbb{F}$ be a field. In [1], we introduced a class of groups $SL_s^p(\mathbb{F})$ and $GL_s^p(\mathbb{F})$ of periodic infinite $\mathbb{N}\times\mathbb{N}$–matrices that correspond to a Steinitz number s. In this paper we introduce a wider class of diagonal locally SL–groups and GL–groups and study their relations with locally matrix algebras. In particular, we show that every separable–diagonal locally SL–group (respectively GL–group) is isomorphic to a group $SL_s^p(\mathbb{F})$ (respectively $GL_s^p(\mathbb{F})$).

Key Words: Locally matrix algebra, Steinitz number, diagonal embedding, periodic matrix.

Статтю представив доктор фізико-математичних наук, професор А.П.Петравчук

1 Introduction

Let $\mathbb{P}$ be the set of all primes. A *Steinitz* or *supernatural* number is an infinite formal product of the form

$$\prod_{p\in\mathbb{P}} p^{r_p},$$

where $r_p \in \mathbb{N}\cup\{0,\infty\}$. The product of two Steinitz numbers

$$\prod_{p\in\mathbb{P}} p^{r_p} \quad\text{and}\quad \prod_{p\in\mathbb{P}} p^{k_p}$$

is a Steinitz number

$$\prod_{p\in\mathbb{P}} p^{r_p+k_p},$$

where we assume, that

$$k_p \in \mathbb{N}\cup\{0,\infty\},\ t+\infty=\infty+t=\infty+\infty=\infty$$

for all positive integers t. Denote by $\mathbb{SN}$ the set of all Steinitz numbers. Obviously, the set of all positive integers $\mathbb{N}$ is a subset of the set of all Steinitz numbers $\mathbb{SN}$.

The Steinitz number v *divides* u if

$$\text{there exists}\quad w\in\mathbb{SN},\quad\text{such that}\quad u=v\cdot w.$$

The divisibility relation $|$ makes $\mathbb{SN}$ into a partially ordered set with the greatest element

$$I=\prod_{p\in P} p^{\infty}$$

and the least element 1. Moreover, the poset $(\mathbb{SN}, |)$ is a complete lattice.

Steinitz numbers were introduced by Ernst Steinitz [6] in 1910 to classify algebraic extensions of finite fields.

Consider the algebra $M_{\mathbb{N}}(\mathbb{F})$ of $\mathbb{N}\times\mathbb{N}$ matrices over the ground field $\mathbb{F}$ having finitely many nonzero elements in each column.

Definition 1. Following [1], we call an $\mathbb{N}\times\mathbb{N}$ matrix *periodic* (more precisely: *n–periodic*) if it is block–diagonal $\operatorname{diag}(a,a,\ldots)$, where a is an $n\times n$ matrix.

Let $M_n^p(\mathbb{F})$ be the subalgebra of $M_{\mathbb{N}}(\mathbb{F})$ that consists of all n–periodic matrices. Clearly,

$$M_n^p(\mathbb{F}) \;\cong\; M_n(\mathbb{F}).$$

Let s be a Steinitz number. Then

$$M_s^p(\mathbb{F}) \;=\; \cup_{n|s}\, M_n^p(\mathbb{F})$$

is a subalgebra of $M_{\mathbb{N}}(\mathbb{F})$; see [1]. J.Glimm [3] defined Steinitz number of uniformly hyperfinite algebras in terms of ascending chains of matrix subalgebras. So, by the Theorem of J.Glimm [3], $M_s^p(\mathbb{F})$ is the only (up to isomorphism) unital locally matrix algebra of Steinitz number s.

Let $GL_n^p(\mathbb{F})$ be the group of invertible elements of $M_n^p(\mathbb{F})$ and

$$SL_n^p(\mathbb{F}) \;=\; [GL_n^p(\mathbb{F}), GL_n^p(\mathbb{F})].$$

Clearly,

$$GL_n^p(\mathbb{F}) \cong GL_n(\mathbb{F}), \quad SL_n^p(\mathbb{F}) \cong SL_n(\mathbb{F}).$$

Let $n_1,\ n_2,\ \ldots$ be a sequence of positive integers such that $n_i | n_{i+1},\ i \geqslant 1$, and s is the least common multiple of the numbers $(n_i, i \geqslant 1)$. Then

$$GL_{n_1}^p(\mathbb{F}) \;\subset\; GL_{n_2}^p(\mathbb{F}) \;\subset\; \cdots,$$

$$\cup_{i\geqslant 1} GL_{n_i}^p(\mathbb{F}) \;=\; GL_s^p(\mathbb{F}),$$

$$SL_{n_1}^p(\mathbb{F}) \;\subset\; SL_{n_2}^p(\mathbb{F}) \;\subset\; \cdots,$$

$$\cup_{i\geqslant 1} SL_{n_i}^p(\mathbb{F}) \;=\; SL_s^p(\mathbb{F}).$$

Let $m,\ n$ be natural numbers such that n divides m.

Definition 2. We call an embedding φ_{SL} of the group $SL_n(\mathbb{F})$ into the group $SL_m(\mathbb{F})$ *diagonal* if there exists an element $g \in SL_m(\mathbb{F})$ such that

$$\varphi_{SL}(a) \;=\; g^{-1}\ \mathrm{diag}(\underbrace{a, a, \ldots, a}_{m/n})\ g.$$

Similarly, we define diagonal embeddings φ_{GL} and φ_M of the group $GL_n(\mathbb{F})$ into the group $GL_m(\mathbb{F})$ and of the algebra $M_n(\mathbb{F})$ of $n \times n$ matrices over $\mathbb{F}$ into the algebra $M_m(\mathbb{F})$:

$$\varphi_{GL}:\ GL_n(\mathbb{F}) \to GL_m(\mathbb{F}),$$

$$\varphi_{GL}(a) \;=\; g^{-1}\ \mathrm{diag}(\underbrace{a, a, \ldots, a}_{m/n})\ g,\ \ g \in GL_m(\mathbb{F});$$

$$\varphi_M:\ M_n(\mathbb{F}) \to M_m(\mathbb{F}),$$

$$\varphi_M(a) \;=\; g^{-1}\ \mathrm{diag}(\underbrace{a, a, \ldots, a}_{m/n})\ g,\ \ g \in M_m(\mathbb{F}).$$

2 Diagonal locally SL– and GL–groups

Let I be a partially ordered set such that for any two elements $i,\ j \in I$ there exists an element $k \in I$ such that $i \leqslant k, j \leqslant k$.

Definition 3. We call a group G a *diagonal locally SL–group* if there exists a family of subgroups $H_i \subseteq G,\ i \in I$, such that

1) if $i \leqslant j$ then $H_i \subseteq H_j$ and let

$$\mathrm{id}_{ij}:\ H_i \;\to\; H_j$$

denote the embedding homomorphism;

2) $\cup_{i\in I} H_i = G$;

3) for any $i \in I$ there is a fixed isomorphism

$$u_i:\ H_i \;\to\; SL_{n_i}(\mathbb{F}), \quad n_i \in \mathbb{F};$$

4) for any $i \leqslant j$

$$\varphi_{ij} \;=\; u_j\ \mathrm{id}_{ij}\ u_i^{-1} : SL_{n_i}(\mathbb{F}) \to SL_{n_j}(\mathbb{F})$$

is a diagonal embedding. In particular,

$$n_i \quad \text{divides} \quad n_j \quad \text{for} \quad i \leqslant j.$$

Similarly we define a *diagonal locally GL–group*. It is easy to see that for $i \leqslant j \leqslant k$, $i,\ j,\ k \in I$, we have

$$\varphi_{ik} \;=\; \varphi_{jk}\ \varphi_{ij}.$$

Hence, we can talk about a direct limit of groups $SL_{n_i}(\mathbb{F}),\ i \in I$, with respect to the system of homomorphisms

$$\varphi_{ij}, \quad i,\ j \in I, \quad i \leqslant j;$$

see [5]. It is straightforward that this direct limit is isomorphic to the group G.

Recall (see [4]) that an associative $\mathbb{F}$–algebra A is called a *locally matrix algebra* if for each finite subset of A there exists a subalgebra $B \subset A$ containing this subset such that $B \cong M_n(\mathbb{F})$ for some n.

A locally matrix algebra A is said to be *unital* if it contains a unit 1.

Theorem 1. *Let A be a unital locally matrix $\mathbb{F}$–algebra. Then the group A^* of invertible elements of A is a diagonal locally GL–group. The commutator subgroup $[A^*, A^*]$ is a diagonal locally SL–group.*

Proof. Let I be the set of all subalgebras $B \subset A$ such that $1 \in B$ and B is isomorphic to a matrix algebra

$$M_{n(B)}(\mathbb{F}), \quad n(B) \in \mathbb{N}.$$

Let

$$u_B : B \to M_{n(B)}(\mathbb{F})$$

be an isomorphism. The set I is partially ordered by inclusion. If $B,\ B' \in I$ and $B \subseteq B'$ then by the Theorem of H.M.Wedderbun (see [2]) the embedding

$$u_{B'}\ \mathrm{id}_{B,B'}\ u_B^{-1} : M_{n(B)}(\mathbb{F}) \to M_{n(B')}(\mathbb{F})$$

is diagonal. The restriction of this embedding to groups of invertible elements yields diagonal embeddings

$$\varphi_{GL} : GL_{n(B)}(\mathbb{F}) \ \to \ GL_{n(B')}(\mathbb{F}),$$

$$\varphi_{SL} : SL_{n(B)}(\mathbb{F}) \ \to \ SL_{n(B')}(\mathbb{F}).$$

This completes the proof of the Theorem. □

Theorem 2. For an arbitrary diagonal locally SL–group (respectively GL–group) G there exists a unital locally matrix algebra A such that

$$G \ \cong \ [A^*, A^*] \text{ (respectively } G \ \cong \ A^*).$$

Proof. Suppose that G is diagonal locally SL–group, $\{H_i\}_{i\in I}$ is a locally system of subgroups,

$$u_i : \ H_i \ \to \ SL_{n_i}(\mathbb{F})$$

are isomorphisms and for each pair $i, j \in I$, $i \leqslant j$, the embedding

$$u_j\ \mathrm{id}_{ij}\ u_i^{-1}$$

is diagonal.

Any diagonal embedding

$$SL_n(\mathbb{F}) \to SL_m(\mathbb{F})$$

uniquely extends to a diagonal embedding of matrix algebras $M_n(\mathbb{F}) \to M_m(\mathbb{F})$. Let

$$\varphi_{ij} : M_{n_i}(\mathbb{F}) \to M_{n_j}(\mathbb{F})$$

be the diagonal embedding that extends

$$u_j\ \mathrm{id}_{ij}\ u_i^{-1}.$$

For $i,\ j,\ k \in I$, $i \leqslant j \leqslant k$, the embeddings

$$\varphi_{jk}\,\varphi_{ij} \quad \text{and} \quad \varphi_{ik}$$

coincide on $SL_{n_i}(\mathbb{F})$, hence

$$\varphi_{jk}\,\varphi_{ij} = \varphi_{ik}.$$

The system of algebras $M_{n_i}(\mathbb{F})$, $i \in I$, and homomorphisms

$$\varphi_{ij}, \quad i \leqslant j,$$

satisfy the conditions for a direct limit (see [5]).

Let's denote this direct limit of matrix algebras as A. Clearly, A is a locally matrix algebra. Since all embeddings φ_{ij} map an identity matrix to an identity matrix it follows that the algebra A is unital.

It is easy to see that the group of invertible elements A^* is the direct limit of groups $GL_{n_i}(\mathbb{F})$ and the commutator subgroup $[A^*, A^*]$ is the direct limit of the groups $SL_{n_i}(\mathbb{F})$, hence

$$[A^*, A^*] \cong G.$$

For a diagonal locally GL–group the arguments are similar. This completes the proof of the Theorem. □

Let G be a diagonal locally SL–group (respectively GL–group), $\{H_i\}_{i\in I}$ is a corresponding local system of subgroups,

$$u_i : \ H_i \ \to \ SL_{n_i}(\mathbb{F}), \quad i \in I,$$

are isomorphisms (respectively

$$u_i : \ H_i \ \to \ GL_{n_i}(\mathbb{F})$$

are also isomorphisms).

Definition 4. The least common multiple of all numbers n_i, $i \in I$, is called the *Steinitz number* $\mathbf{st}(G)$ of the group G.

J.Glimm [3] assigned a Steinitz number to an arbitrary unital countable–dimensional locally matrix algebra and showed that two such algebras A, B are isomorphic if and only if their Steinitz numbers $\mathbf{st}(A)$ and $\mathbf{st}(B)$ are equal.

3 On separable diagonal locally SL– and GL–groups

Definition 5. A *partially ordered set* I is called *separable* if there exists a countable subset $I_0 \subset I$ such that for an arbitrary element $i \in I$ there exists an element $j \in I_0$ and $i \leqslant j$.

Definition 6. We call a *diagonal locally SL–group* (respectively *GL–group*) *separable* if the corresponding partially ordered set I is separable.

Theorem 3. Let G be a separable diagonal locally SL–group (respectively GL–group) of Steinitz number s. Then

$$G \cong SL_s^p(\mathbb{F})$$

(respectively

$$G \cong GL_s^p(\mathbb{F})).$$

Proof. If the partially ordered set I is separable then the algebra A in the proof of Theorem 2 is countable–dimensional. Moreover, it is easy to see that the Steinitz number of the algebra A is equal to s. By Glimm's Theorem [3]

$$A \cong M_s^p(\mathbb{F}).$$

Hence

$$G \ \cong \ [A^*, A^*] \ \cong \ SL_s^p(\mathbb{F})$$

if G is a diagonal locally SL–group and

$$G \ \cong \ A^* \ \cong \ GL_s^p(\mathbb{F})$$

if G is a diagonal locally GL–group. This completes the proof of the Theorem. □

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