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Кваліфікаційна наукова
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ДИСЕРТАЦІЯ

Теорія дуальності для не увігнутих функцій корисності за умов невизначеності моделі

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*Дисертація містить результати власних досліджень. Використання ідей,
результатів і текстів інших авторів мають посилання на відповідне джерело.*

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Анотація

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У моделі фінансового ринку оптимальні інвестиційні стратегії залежать від уподобань інвестора. Однак ці вподобання можуть сильно відрізнятися залежно від різних факторів, як-от: модель ринку, в тому числі його повнота та припущення щодо множини ймовірнісних мір на ринку, властивості функції корисності інвестора, моделювання виплат, бюджетні обмеження, тощо.

У дисертаційній роботі досліджуються існування оптимальних інвестицій у загальному випадку в умовах невизначеності моделі та їхня побудова. Зокрема, розглядаються як стандартні, так і стійкі функціонали максимізації очікуваної корисності у випадку, коли функція корисності не обов'язково є увігнутою. Аналіз проводиться для загальної множини ймовірнісних мір як у повних, так і в неповних моделях ринку. Розглядається два випадки множини допустимих кінцевих виплат: зі стандартним обмеженням на капітал та з додатковим обмеження кінцевої виплати згори певною випадковою величиною.

Важливими інструментами, що використовуються для встановлення результатів, є мінімаксні тотожності, яка дозволяють змінювати порядок максимізації за стратегіями та мінімізації за множиною мір, в контексті моделі повного ринку, та теорія дуальності для моделі неповного ринку. В обох випадках використовується принцип увігнутості, який передбачає розгляд увігнутої обвідної початкової функції корисності.

У дисертаційній роботі досліджено чотири основні випадки:

1. модель повного ринку, стандартна задача максимізації функції корисності (а також її модифікація з обмеженням бюджету);
2. модель повного ринку, робастна задача максимізації функції корисності (та її модифікація з обмеженням бюджету);

3. модель неповного ринку, стандартна задача максимізації функції корисності;
4. модель неповного ринку, робастна задача максимізації функції корисності.

Вивчення неувігнутих функцій корисності є одним із найважливіших аспектів дисертації, оскільки наявні емпіричні дані свідчать про те, що люди схильні уникати ризиків, коли є гарантований виграш, та ризикувати, за високої ймовірності цілковитого програшу.

Хоча багато статей спираються на принцип увігнення, який передбачає заміну неувігнутої функції корисності її увігнутою обвідною, є й інші підходи до роботи з неувігненими функціями. Ця дисертація також пропонує підхід до розв'язування задач максимізації з неувігнутими функціями корисності через їхнє увігнення.

Дисертація починається з вивчення властивостей увігнення функцій корисності та дослідження окремого випадку увігнення для функції корисності, пов'язаної з функцією виплат для кол-опціону.

У моделі повного ринку досліджено мінімаксу рівність для не увігнутих функцій корисності. Доведено, що за певних природних припущень стандартний функціонал максимізації функції корисності для початкової функції корисності дорівнює відповідному функціоналу від увігнення початкової функції корисності. Використовуючи теореми Колмоша продемонстровано існування «найгіршої» міри у двох випадках: інфімуму математичного сподівання від початкової функції корисності та від її увігнення. Мінімаксні рівності для не увігнутих функцій корисності отримано як наслідок рівностей та нерівностей, які пов'язують робастні функціонали від початкової функції корисності та її увігнення. Крім того, у дисертації також розглянуто випадки, коли всі нерівності перетворюються на рівності, що забезпечує справедливість мінімаксної рівності для початкової не увігнутої функції корисності.

Основною метою дисертації є встановлення існування оптимального розв'язку, що, як добре відомо, рівносильно відсутності арбітражу на ринку. Це дає альтернативний спосіб доведення фундаментальної теореми фінансової математики, і, відповідно, веде до задачі вибору еквівалентної мартингальної міри.

У цьому напрямку доведено існування та єдиність оптимального розв'язку стандартної задачі максимізації не увігнутої функції корисності. Крім того, побудовано його явний вигляд за умови стандартного бюджетного обмеження. У роботі показано, що цей розв'язок також є єдиним розв'язком стандартної задачі максимізації функціоналу для увігнення початкової функції корисності. У випадку з додатковим обмеженням згори на кінцеву виплату, заданого дискретною випадковою величиною, доведено аналогічні теореми та наведено приклади, які демонструють, що отримані результати можуть не мати місця, коли розподіл випадкової величини, яка обмежує виплату, є неперервним.

Крім стандартної задачі максимізації для не увігнутої функції корисності, в дисертації також розглядається робастна задача максимізації не увігнутої функції корисності. Доведено існування та єдиність оптимального розв'язку цієї задачі, а також запропоновано метод побудови оптимального розв'язку.

Аналогічні результати отримано в моделі неповного ринку, включаючи рівності та нерівності, які пов'язують робастний функціонал від початкової функції корисності з функціоналом від її увігнення, а також наведено достатні умови, за яких мінімаксна рівність має місце для початкової не увігнутої функції корисності.

Друга частина другого основного розділу присвячена моделі неповного ринку, вона використовує теорію дуальності. По-перше, тут представлено оптимальну стратегію для стандартної задачі максимізації функціоналу функції корисності та виведено декілька властивостей цієї стратегії. По-друге, отримано аналогічні результати для робастної задачі максимізації функціоналу увігнення функції корисності. Показано, за певних умов, існування оптимального розв'язку для робастної задачі максимізації функціоналу для не увігнутої функції корисності, а також побудовано оптимальний розв'язок.

Новизна роботи полягає насамперед у розгляді функцій корисності, які не обов'язково є увігнутими, та побудові оптимальних інвестиційних стратегій для таких функцій як у повних, так і в неповних ринкових моделях. Іншим важливим оригінальним здобутком є встановлення мінімаксних рівностей та інших результатів для робастного функціоналу від не увігнутої функції корисності, зокрема у випадку додаткового обмеження згори на виплату.

Дисертаційна робота має як теоретичне, так і практичне значення. Резуль-

тати дослідження можуть бути використані для доведення або спростування мінімакських рівностей для не увігнутих функцій корисності. Вони є також важливими для побудови явного вигляду оптимальних стратегій у найбільш загальних моделях, де, по-перше, відношення інвестора до ризику може змінюватися від несхильності до прийняття, по-друге, є значна гнучкість щодо можливої множини ймовірнісних мір. Як оптимальні інвестиції, так і максимізація математичного сподівання функції корисності є одними з основних задач фінансової математики, тому отримані результати можуть бути включеними у спеціалізовані курси.

У цілому, дисертаційна робота презентує важливий підхід до задачі максимізації не увігнутих функцій корисності, які часто залишаються поза увагою в літературі. Вона проливає нове світло на центральні задачі фінансової математики і пропонує підходи, які проливають світло на оптимальні інвестиційні стратегії в умовах невизначеності моделі та вподобань інвестора, які моделюються не увігнутою функцією корисності.

Ключові слова: Оптимальні інвестиції, теорія дуальності, мінімакс рівність, стандартна задача максимізації функції корисності, робастна задача максимізації функції корисності, неувігнута максимізація, бюджетні обмеження, неопукла оптимізація, увігнення.

Annotation

Bahchedjioglou O. O. Duality theory under model uncertainty for non-concave utility functions. — Manuscript.

Doctor of Philosophy thesis, specialty "112— Statistics"— Taras Shevchenko National University of Kyiv, Ministry of Education and Science of Ukraine, Kyiv, 2023.

In a financial market model, the optimal investment strategies depend on the preferences of the investor. However, these preferences can vary greatly, depending on various factors such as market completeness, assumptions on probability measures, properties of the utility function, payoff modeling, and budget constraints, among others.

This thesis addresses the existence and construction of optimal investment strategies in a general setup under model uncertainty. Specifically, it considers both the standard and robust utility maximization functionals, assuming that the investor's utility function is not necessarily concave. The analysis is conducted under a general set of prior probability measures in both complete and incomplete market models. Additionally, two cases of admissible final endowments are examined: standard budget constraints and an additional upper bound represented by a random variable.

The main tools employed to achieve the desired results are the minimax identity, which allows for the interchangeability of maximization over strategies and minimization over measures, in the context of the complete market model, and the duality theory for the incomplete market model. In both cases, the concavification principle is utilized, which involves considering the concave envelope of the initial utility function.

The thesis investigates four main cases:

1. Complete market model, standard utility maximization (+ cases with budget constraints);
2. Complete market model, robust utility maximization (+ cases with budget constraints);
3. Incomplete market model, standard utility maximization;
4. Incomplete market model, robust utility maximization.

The study of non-concave utility functions is a crucial aspect of the thesis, as the existing empirical evidence suggests that individuals tend to exhibit risk aversion in choices involving sure gains and risk-seeking behavior in choices involving highly probable losses.

While many articles rely on the concavification principle, which involves replacing the non-concave utility function with its concave envelope, there are other approaches to dealing with non-concave functions. This thesis also provides approach for solving non-concave utility maximization problems using concavification principle.

The thesis begins by studying the properties of concavified utility functions and investigating a particular case of concavification for a utility related to the call option payoff function.

In the complete market model, the minimax identity for non-concave utility functions is thoroughly investigated. It is proven that, under certain natural assumptions, the standard utility functional of the initial utility function is equal to that of its concavification. By employing a Kolmos-type argument, the existence of a “worst-case scenario” measure is demonstrated in two cases: the infimum of the expected value of the initial utility function and its concavification. The minimax identity for the initial non-concave utility function is established by deriving equalities and inequalities that relate the robust utility functional of the initial utility function to its concavification. The thesis also presents cases where all the inequalities become equalities, thus ensuring the validity of the minimax identity for the initial non-concave utility function.

The main objective of the thesis is to establish the existence of an optimal solution, which is equivalent to the absence of arbitrage opportunities. This provides an alternative way to prove the “fundamental theorem of asset pricing” and leads to the selection of an equivalent martingale measure.

Regarding this issue, the existence and uniqueness of the optimal solution to the standard non-concave utility maximization problem were proven. Moreover, its explicit form was constructed under the assumption of standard budget constraints. It was shown that this solution is also a unique optimal solution for the maximization problem of the concavified utility function. Furthermore, in the case of an additional upper bound represented by a discrete random variable, a similar theorem is proven,

and examples are provided to illustrate that the previous conclusions may not hold for a continuous random variable representing the upper bound.

In addition to the standard non-concave utility maximization problem, the thesis also addresses the robust non-concave utility maximization problem. The existence and uniqueness of the optimal solution to this problem are established, and a method for constructing the optimal solution is derived.

Similar results are obtained in the incomplete market model, including equalities and inequalities that relate the robust utility functional of the initial utility function to its concavification, as well as assumptions under which the minimax identity holds for the initial utility function.

The second part of the second main chapter focuses on the incomplete market model and utilizes the duality theory approach. Firstly, an optimal strategy for the standard utility maximization problem is presented, and several properties of this strategy are derived. Secondly, a similar theorem is proven for the robust maximization functional of the concavified utility function. Subsequently, under certain conditions, the existence of an optimal solution for the robust non-concave utility maximization problem is demonstrated, and a method for constructing the optimal solution is provided.

The novelty of this work lies primarily in the consideration of utility functions that are not necessarily concave and the construction of optimal investment strategies for such functions in both complete and incomplete market models. The study of the minimax identity and the significant results obtained for non-concave robust functionals, as well as the incorporation of budget constraints in these cases, also contribute to the originality of the thesis.

The theoretical and practical significance of the thesis is substantial. The research results can be used to validate or challenge the minimax identity for non-concave utility functions, which can make a significant contribution to the construction of explicit optimal solution strategies in general model and investor preference settings. Moreover, optimal investments and the maximization of expected utility of terminal wealth are fundamental problems in Mathematical Finance, making the progress made in this area valuable for teaching specialized courses in the field.

Overall, the thesis presents a valuable approach to the utility maximization problem within a non-concave framework, a topic that is often overlooked in the

current literature. It sheds new light on a central issue in Mathematical Finance and provides insights that can enhance our understanding of optimal investment strategies under model uncertainty and non-concave preferences.

Keywords: Optimal investments, duality theory, minimax identity, standard utility functionals, robust utility functionals, non-concave utility, non-convex optimization, constrained optimization, concave envelope.

List of published works on the topic of the dissertation

Scientific works in which the main scientific results of the dissertation are published

1. Bahchedjioglou O., Shevchenko G. Minimax identity with robust utility functional for a nonconcave utility // *Modern Stochastics: Theory and Applications*. 2022. Vol. 10, no. 1. P. 19–35. Access mode: <https://doi.org/10.15559/22-VMSTA215>.
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5. Bahchedjioglou O. Characterization of minimax identity with robust non-concave utility functions for constrained case of random endowments // *Proceeding of XX International Scientific – Practical Conference “Shevchenkivska Vesna – 2022”*. Kyiv, Ukraine. April 14, 2022. C. 5.
6. Bahchedjioglou O. Properties of utility maximization functionals for non-concave utility function in complete market model // *15ème Colloque Bachelier*

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Notation

Utility functions

U – utility function;

U_c – concave envelope of U , or the smallest concave function such that $U_c \geq U$.

Sets of measures and related to them sets

\mathbb{P} – the real world measure;

\mathcal{Q} – set of measures that we consider and which satisfies certain assumptions;

\mathcal{Z} – set of Radon-Nikodym derivatives $\frac{dQ}{d\mathbb{P}}$, for $Q \in \mathcal{Q}$;

\mathcal{Q}_e – set of measures in \mathcal{Q} which are equivalent to the real world measure \mathbb{P} ;

\mathcal{Z}_e – set of Radon-Nikodym derivatives $\frac{dQ}{d\mathbb{P}}$, for $Q \in \mathcal{Q}_e$;

\mathcal{M}_e – set of all equivalent local martingale measures;

\mathcal{M}_a – set of all absolutely continuous local martingale measures;

Q^e – unique equivalent local martingale measure in a complete market.

Introduction

Relevance of the topic. Utility maximization is a fundamental concept in finance that seeks to identify the best investment strategies that would lead to the highest possible return for a given level of risk or the lowest possible risk for a given level of return. This concept is widely used in investment decision-making by both individuals and institutional investors.

One of the most significant challenges of utility maximization is the completeness of the market, which refers to the availability of all possible investment options. In practice, it is rare for a market to be fully complete, meaning that not all investment options are available to investors. Therefore, it is necessary to consider a general setup that can account for an incomplete market.

Another critical aspect of utility maximization is the properties of the utility function. The utility function describes the preferences of investors in terms of the expected return and risk of different investment options. It is essential to consider the properties of the utility function, such as its concavity, in order to make informed investment decisions.

Evidently, underlying probability measures play a significant role in utility maximization: one has to use accurate probability measures in order to make informed investment decisions. Given that in most cases, the information about underlying probabilities is limited, it is important to work a robust setting, which includes a general set of probability measure.

The thesis is devoted to the study of the existence and construction of the optimal investment strategies in the general model setup. We consider the standard and robust utility maximization functionals under model uncertainty and assumption that the investor's utility function is not necessarily concave. Moreover, this problem is considered under a general set of prior probability measures in the complete and incomplete market models. Additionally, in the spirit of Föllmer and Schied [16,

Chapter 3], we look at the two cases of admissible final endowments: the standard budget constraints and an additional upper bound given by some random variable.

In the second chapter we examined optimal investment strategies for standard and robust non-concave utility maximization problems in complete market model. We considered the very general model setup and implemented a modified constrained counterpart of the optimization problem. To obtain an optimal investment we relied on the minimax identity, concavification principle and relations between the corresponding functionals of an initial non-concave utility function and its concavifications. Throughout the chapter we studied the minimax identity for non-concave utility functions and applied obtained relations of robust functionals of initial and concavified utility functions.

The third chapter of thesis is devoted to the similar problems, however, in incomplete market model, which makes the considered model setup more general. The key tool, in obtaining the previous results in expanded model setup, is duality theory. As before we studied the minimax identity for non-concave utility functions and learned how to relate the corresponding functionals of an initial non-concave utility function and its concavifications. Moreover, presented a way to derive an optimal investment strategies for standard and robust non-concave utility maximization problems (in some cases with implementing additional assumptions in comparison with the second chapter).

Connection of work with scientific programs, plans, topics. The work was carried out within the framework of state budgetary research scientific topics: «Stochastic dynamical systems, inhomogeneous in time or random time: asymptotics and statistical analysis» (state registration number 0122U001843) and №19БФ038-01 «Exact formulas, estimates, asymptotic properties, and statistical analysis of complex evolutionary systems with many degrees of freedom» (state registration number 0119U100317) department of probability theory, statistics and actuarial mathematics, faculty of Mechanics and Mathematics Taras Shevchenko National University of Kyiv, included in the comprehensive thematic plan of research works «Modern mathematical problems of natural science, economics and finance».

The purpose and objectives of the research. The aim of the dissertation is in the study of standard and robust utility functionals for initial utility function and

its concavification and solving the following problems: establishing the existence of the optimal investment strategies and construction of them under different model setups, deriving how can be related the maximizing functionals of initial utility function and its concavification and solving the described problems in general framework: both complete and incomplete market models, not necessarily concave utility function, standard budget constraint and implementation of additional upper bound on final endowments.

The object of research are the standard and robust utility functionals.

The subject of research are the optimal solutions for the standard and robust utility maximization problems for non-concave utility function and its concavification, as well as minimax identities for the robust non-concave utility functionals.

Research methods. The work uses methods of probability theory, theory of random processes, functional analysis, financial mathematics, non-convex optimization.

Scientific novelty of the obtained results. All results obtained in the dissertation work are new. The main results are as follows:

- the properties of concavified utility function are investigated;
- minimax identity for non-concave utility functions is established;
- it is proved that under some natural assumptions, the standard utility functional of the initial utility function is equal to that of its concavification;
- in a complete market model, the existence of the “worst-case scenario” measure is shown in two cases: where one looks at expected value of the initial utility function and that of its concavification;
- a chain of equations and inequalities that relate the robust utility functionals of a non-convex utility function and its concavification is established, generalizing the existing results, and several examples are given where all of the inequalities become equations and hence the minimax identity for the initial non-concave utility function holds;
- existence and uniqueness of the optimal solution to standard non-concave utility maximization problem is proved, and it is shown that the solution is also a unique optimal solution for the maximization problem of the concavified utility function;

- an explicit optimal solution is constructed under the assumption of standard budget constraints in a complete market model;
- in a constrained case, where an upper bound on the final endowment is imposed, similar results are established when the upper bound is a discrete random variable, and it is shown that the conclusions may fail when the upper bound is a continuous random variable;
- in complete market model the existence and uniqueness of the optimal solution to the robust non-concave utility maximization problem is proved and an approach is given to construct it;
- in the incomplete market model, similar results are proved: equations and inequalities that relate the robust utility functionals of a non-concave utility function and its concavification are established, and the assumption under which minimax identity holds is proposed;
- the duality theory approach in incomplete market model is used to obtain optimal investment strategies for the standard utility maximization problem, the properties of optimal strategy are derived; similar results are obtained for the robust maximization functional of the concavified utility function, in particular, the existence of an optimal solution for the robust non-concave utility maximization problem is shown and its construction is proposed.

Practical significance of the obtained results. As investors seek to maximize their wealth, they face the challenge of terminal wealth maximization, a problem that hinges on their individual preferences. This research explores various scenarios related to investor preferences and future payoff modeling, encompassing factors such as market completeness, properties of utility functions, and budget constraints. We establish the existence of optimal investment strategies for both standard and robust utility maximization problems, subject to certain assumptions in some cases. Moreover, we present a methodology for explicitly constructing optimal strategies in specific situations.

The research investigates the equations and inequalities that connect robust utility functionals of an initial non-concave utility function and its concavification. By doing so, we can verify or disprove the minimax identity for non-concave utility functions. This contribution holds significant theoretical and practical implications

as it greatly assists in formulating explicit optimal solution strategies for the general case of model and investor preferences.

Personal contribution of the acquirer. All the results of thesis were obtained by solely by its author. All four articles containing results were published. Two of them, [8, 22], are published by the author solely. The two remaining articles [7] and [6] is coauthored by the scientific supervisor, Doctor of Physical and Mathematical Sciences, Professor Georgiy M. Shevchenko, whose contribution was in proposing the research problem and in general advising.

Approbation of the results of the dissertation. The results of the dissertation were reported and discussed at international scientific conferences and meetings of scientific seminars of leading Ukrainian scientific institutions.

Conferences:

1. XX International Scientific – Practical Conference “Shevchenkivska Vesna – 2022”, April 14, 2021, Kyiv, Ukraine.
2. International Conference “Modern Stochastics: Theory and Applications V”, June 1–4, 2021, Kyiv, Ukraine.
3. Scientific conference “Actual Problems of Stochastic Analysis”, February 20–21, 2021, Tashkent, Uzbekistan.
4. XIX International Scientific – Practical Conference “Shevchenkivska Vesna – 2021”, April 15-16, 2021, Kyiv, Ukraine.
5. 15^{ème} Colloque Bachelier en Mathématiques Financières et Calcul Stochastique, January 16-21, 2023 Métabief, France.

Scientific seminars:

1. Seminar held by the department of probability theory, statistics and actuarial mathematics of faculty of Mechanics and Mathematics Taras Shevchenko National University of Kyiv (Kyiv, 2020).

Publications. The results of the dissertation were published in:

1. Four articles in peer-reviewed journals [6–8, 22];
 - two of them, [6, 7], are in the journals indexed in the Scopus and Web of Science; in SJR: Scientific Journal Rankings databases their quartiles are Q2 and Q3 correspondingly;

- the articles [8, 22] are published in peer-review Ukrainian journal (category “B”);
2. Four abstracts of reports at conferences [4, 5, 23, 24].

Structure and scope of the dissertation. The dissertation consists of an abstract in Ukrainian and English, an introduction, four chapters divided into sections and subsections, conclusions, list of references (41 items), and the appendices, first one with supplementary proofs and script, second one contains a list of publications of the acquirer on the topic of the dissertation and information about the approval of the results. The full scope of the dissertation is 137 pages, the main text takes 97 pages.

The content of the work. **The first chapter** contains a brief historical review of the literature on the subject of the dissertation and highlights the current state of the study of problems similar to those considered in the dissertation work.

In **the second chapter** firstly, in [Section 2.1](#), we introduced the model setup with all of the necessary assumptions on the model, i.e. the modelling of the payoff, assumptions on the set of probability measures and main notations that we have used throughout the work. Throughout the dissertation we consider standard and robust optimization problems with and without budget constraints in complete and incomplete market models.

Consider investor’s utility function $U : [0, \infty) \rightarrow \mathbb{R}$. We assume that U is non-constant, non-decreasing, upper semi-continuous and satisfies the mild growth condition:

$$\lim_{x \rightarrow \infty} \frac{U(x)}{x} = 0,$$

and that $U(\infty) > 0$. We set $U(x) = -\infty$ for $x < 0$.

For the standard optimization problem we aim to identify

$$\sup_{X \in \mathfrak{X}(x)} \mathbb{E}[U(X)],$$

and for the robust optimization problem we look at

$$\sup_{X \in \mathfrak{X}(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(X)].$$

Two possibilities for the set $\mathfrak{X}(x)$ of admissible final endowments are considered:

- the standard budget constraint:

$$\mathfrak{X}(x) = \{X \in L_+^1(Q^e) \mid E_{Q^e}[X] \leq x\}, x > 0,$$

- an additional upper bound:

$$\mathfrak{X}(x) = \{X \in L^1(Q^e) \mid 0 \leq X \leq W, E_{Q^e}[X] \leq x\},$$

with some random variable $W : \Omega \rightarrow [0, +\infty)$.

Afterwards we study an optimal investment strategies for both standard and robust non-concave utility maximization problems in complete market model. We considered the very general model setup and implemented a modified constrained counterpart of the optimization problem. To obtain an optimal investment we relied on the minimax identity, concavification principle and relations between the corresponding functionals of an initial non-concave utility function and its concavifications.

To ensure market completeness, the following assumption is imposed:

Assumption 2.1. *There is a unique equivalent local martingale measure, denoted by Q^e .*

Moreover, we use next conditions:

- Assumption 2.2.** (i) \mathcal{Q} is convex;
(ii) $\mathbb{P}[A] = 0$ if and only if $Q[A] = 0$ for all $Q \in \mathcal{Q}$;
(iii) The set $\mathcal{Z} := \{dQ/d\mathbb{P} \mid Q \in \mathcal{Q}\}$ is closed in $L^0(\mathbb{P})$.

The following notations is used for brevity:

$\mathcal{Z}_e := \{dQ/d\mathbb{P} \mid Q \in \mathcal{Q}_e\}$, where \mathcal{Q}_e denotes the set of measures in \mathcal{Q}
that are equivalent to \mathbb{P} ;

$$C(x) = \{g \in L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}) \mid 0 \leq g \leq X_T \text{ for some } X \in \mathfrak{X}(x)\}.$$

In particular,

$$g \in C(x) \iff E_{Q^e}(g) \leq x.$$

In [Section 2.2](#), we consider a strictly increasing, strictly concave and twice continuously differentiable function $U : [0, \infty) \rightarrow \mathbb{R}$, which has vanishing derivative

this part (until considering the constrained case) we consider the utility function $U: (0, \infty) \rightarrow \mathbb{R}$ that is strictly increasing, continuously differentiable and satisfying a mild growth condition.

Lemma 2.20. *The function $U_c(\cdot)$ is strictly increasing and continuously differentiable on $(0, \infty)$.*

Lemma 2.21. *Suppose Assumption 2.2 holds and that for all $Q \in \mathcal{Q}_e$: $u_Q^c(x) < \infty$ for some $x > 0$.*

Additionally, assume that $\lim_{x \rightarrow \infty} \frac{u_Q^c(x)}{x} = 0$ for each $Q \in \mathcal{Q}_e$. Then, for any $x > 0$, there exists some $\widehat{g} \in C(x)$ and $\widehat{Q} \in \mathcal{Q}$ such that

$$u^c(x) = \inf_{Q \in \mathcal{Q}} E_Q[U_c(\widehat{g})] = E_{\widehat{Q}}[U_c(\widehat{g})] = u_{\widehat{Q}}^c(x).$$

Conditions are given under which all of the inequalities in Theorem 2.13 become equations and, thus, the minimax identity holds for the initial non-concave utility function U .

Theorem 2.23. *Suppose that all assumptions from Theorem 2.13 hold.*

Then, the following two equations are equivalent:

- (i) $\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)] = \sup_{g \in C(x)} E_{\widehat{Q}}[U(g)],$ for $\widehat{Q} \in \mathcal{Q}_e$.
- (ii) $\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)] = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)].$

Assumption 2.25. *Assume that the set \mathcal{Q} from Lemma 2.21 can be replaced by the set \mathcal{Q}_e , i.e. for any $x > 0$, there exists some $\widehat{g} \in C(x)$ and $\widehat{Q}_0 \in \mathcal{Q}_e$ such that*

$$u^c(x) = \inf_{Q \in \mathcal{Q}} E_Q[U_c(\widehat{g})] = E_{\widehat{Q}_0}[U_c(\widehat{g})] = u_{\widehat{Q}_0}^c(x).$$

Further on, $\widehat{Q}_0 \in \mathcal{Q}_e$ denotes a measure such that

$$u^c(x) = u_{\widehat{Q}_0}^c(x).$$

Theorem 2.26. *Suppose that all assumptions from Lemma 2.21 and Theorem 2.13 are fulfilled and that Assumption 2.25 holds. Additionally, assume that at least one of the items below holds*

- (i) *Exists such a measure $\widehat{Q} \in \mathcal{Q}_e$ that for all $g \in C(x)$: $\inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)] = E_{\widehat{Q}}[U_c(g)];$*

(ii) For any sequence $g_n \in C(x)$ such that $\lim_{n \rightarrow \infty} E_{\widehat{Q}_0}[U_c(g_n)] = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)]$ it holds that $\lim_{n \rightarrow \infty} E_{\widehat{Q}_0}[U_c(g_n)] = \lim_{n \rightarrow \infty} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g_n)]$.

Then, we have

$$\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)] = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)].$$

Hence, all inequalities in the [Theorem 2.13](#) are equalities.

Theorem 2.27. Suppose that all assumptions from [Theorem 2.13](#) hold. Additionally, assume that $\frac{dQ^e}{dP}$ has a continuous distribution. Also, suppose that for some non-negative λ the maximizer for $u^c(x)$ satisfies $g^* \in -\partial V(\lambda \cdot \frac{dQ^e}{dP})$.

Then g^* is also a maximizer for $u(x)$. Moreover, we have

$$\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)] = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)].$$

Hence, all inequalities in the [Theorem 2.13](#) become equalities.

We consider two cases of admissible final endowments: the standard budget constraint and additional upper bound (which has not been considered before in such model setup).

From now on we return to the first assumption on the utility function, i.e. that investor's utility function $U : [0, \infty) \rightarrow \mathbb{R}$ such that $U(\infty) > 0$, is non-constant, non-decreasing, upper semi-continuous and satisfies the mild growth condition, $U(x) = -\infty$ for $x < 0$.

We extend previous results by showing how to compare the robust utility maximization functionals of initial non-concave utility functions and its concavification. Furthermore, we proceed in proving the minimax identity for the general case of non-concave utility functions. The crucial step for obtaining the mentioned results with implementing an additional upper bound is the use of the regular conditional distribution which sheds new light on the possible approaches for solving the optimization problem.

In the constrained case of the study the minimax identity for the robust non-concave utility functional in a complete market model we additionally to the previous part consider an upper bound for the set $\mathfrak{X}(x)$ of admissible final endowments:

$$\mathfrak{X}^W(x) = \{X \in L^1(Q^e) \mid 0 \leq X \leq W, E_{Q^e}[X] \leq x\},$$

Lemma 2.37. *Under the Assumption 2.29,*

$$\inf_{Q \in \mathcal{Q}_e} \sup_{X \in \mathfrak{X}_x^W} E_Q[U^{W(\omega)}(X)] = \inf_{Q \in \mathcal{Q}_e} \sup_{X \in \mathfrak{X}_x^W} E_Q[U_c^{W(\omega)}(X)].$$

In Section 2.4 we construct an optimal investment strategy for non-concave utility maximization problems. We started with the standard utility maximization problem and in Section 2.4.1 constructed the optimal investment strategy.

We defined the (one-sided) inverse of $U'_c(\cdot)$:

$$i(y) = \begin{cases} (U'_c)^{-1}(y) \\ \cdot \mathbb{I}\{y \in \mathbb{R}^+ \setminus \{\cup_{i \in \mathbb{N}} [U'_c(b_i^+), U'_c(a_i^-)] \cup [U'_c(c_i^+), U'_c(c_i^-)]\}\} \\ + \sum_i a_i \cdot \mathbb{I}\{y \in [U'_c(a_i^+), U'_c(a_i^-)]\} \\ + \sum_i b_i \cdot \mathbb{I}\{y \in [U'_c(b_i^+), U'_c(b_i^-)]\} \\ + \sum_i c_i \cdot \mathbb{I}\{y \in [U'_c(c_i^+), U'_c(c_i^-)]\}, & y < U'_c(0^+); \\ 0, & y \geq U'_c(0^+). \end{cases}$$

Theorem 2.39. *Assume that $\frac{dQ^e}{d\mathbb{P}}$ is continuous and that*

$$h(\lambda) = E_{Q^e} \left[i \left(\lambda \cdot \frac{dQ^e}{d\mathbb{P}} \right) \right] < \infty, \text{ for all } \lambda > 0. \quad (1)$$

Then, for any initial capital $x > 0$, there exists a unique optimal solution to the problem (2.28) given by $X^* = i(\lambda^* \cdot \frac{dQ^e}{d\mathbb{P}})$, where $\lambda^* > 0$ is such that $E_{Q^e} [i(\lambda^* \cdot \frac{dQ^e}{d\mathbb{P}})] = x$.

Remark 2.40. *Solution X^* is also a unique optimal solution for the maximization problem of concavified utility function, i.e.*

$$\sup_{X \in \mathfrak{X}(x)} \mathbb{E}[U_c(X)] = \max_{X \in \mathfrak{X}(x)} \mathbb{E}[U_c(X)] = \mathbb{E}[U_c(X^*)].$$

Section 2.5 is devoted to the study of optimization strategies in the constrained setup, i.e. we assume that there is an upper bound on the endowment, given by a random variable $W : \Omega \rightarrow [0, +\infty)$. The set of admissible payoff is then given by

$$\mathfrak{X}^W := \{X \in L^0(\mathbb{P}) \mid 0 \leq X \leq W \text{ } \mathbb{P}\text{-a.s.}\}.$$

Our goal is to maximize $\mathbb{E}[U(X)]$ for $X \in \mathfrak{X}_x^W$ where the budget set \mathfrak{X}_x^W is defined by

$$\mathfrak{X}_x^W := \{X \in L^1(Q^e) \mid 0 \leq X \leq W, E_{Q^e}[X] \leq x\}.$$

For each $k > 0$ denote

$$U^k(x) = U(x \wedge k), \quad x \geq 0. \quad (2)$$

and let $i^k(y)$ be the inverse of $(U^k)'$.

The following result is a counterpart of [Theorem 2.39](#) for the utility function $U^{W(\omega)}$, $\omega \in \Omega$.

Theorem 2.42. *Assume that the random variable $\frac{dQ^e}{d\mathbb{P}}$ is continuous and that the random variable W is discrete.*

Consider the function $i^W(\omega, y) := i^{W(\omega)}(y)$, for all $\omega \in \Omega$.

Let also

$$h^W(\lambda) = \mathbb{E}_{Q^e} \left[i^{W(\omega)} \left(\lambda \cdot \frac{dQ^e}{d\mathbb{P}}(\omega) \right) \right] < \infty, \quad \text{for all } \lambda > 0. \quad (3)$$

Then for any initial capital $x > 0$, there exists a unique optimal solution to the problem (2.40), which is given by $X^ = i^W(\lambda^* \cdot \frac{dQ^e}{d\mathbb{P}})$ for some $\lambda^* > 0$ such that $\mathbb{E}_{Q^e}[i^W(\lambda^* \cdot \frac{dQ^e}{d\mathbb{P}})] = x$.*

Furthermore, in [Section 2.5.2](#) we presented examples which show that the conclusion of [Theorem 2.42](#) may fail in the case where W is a continuous random variable.

To achieve this we considered the sample space $\Omega = [\frac{1}{2}, 1]$ with the probability \mathbb{P} being the normalized Lebesgue measure, i.e. $\mathbb{P}([a, b]) = 2(b - a)$ for $\frac{1}{2} \leq a < b \leq 1$.

We set $W(\omega) = 2\omega$, for $\omega \in \Omega$ and considered the utility function

$$U(x) = \begin{cases} \frac{3\sqrt{2}}{4} \cdot x^{\frac{1}{2}}, & x \in [0, \frac{1}{2}); \\ \frac{x+1}{2}, & x \in [\frac{1}{2}, 1); \\ x^{\frac{3}{4}}, & x \in [1, \infty). \end{cases}$$

[Section 2.6](#) deals with the optimal investment strategies for robust utility functional.

The main assumption here is equality

$$\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)] = \sup_{g \in C(x)} E_{\widehat{Q}}[U(g)]. \quad (4)$$

Theorem 2.44. Suppose that (2.46) holds and that $\frac{dQ^e}{d\widehat{Q}}$ is continuous. Additionally let

$$h(\lambda) = E_{Q^e} \left[i \left(\lambda \cdot \frac{dQ^e}{d\widehat{Q}} \right) \right] < \infty, \text{ for all } \lambda > 0, \quad (5)$$

be continuous, where i is the inverse to U'_c . Then there exists a unique optimal solution to the problem (2.48) given by $g^\star = i(\lambda^\star \cdot \frac{dQ^e}{d\widehat{Q}})$, where $\lambda^\star > 0$ and such that $E_{Q^e} [i(\lambda^\star \cdot \frac{dQ^e}{d\widehat{Q}})] = x$, for initial capital x .

Remark 2.45. Solution g^\star is also a unique optimal solution for the problem of maximizing concavification of the utility function, i.e.

$$\sup_{g \in C(x)} E_{\widehat{Q}} [U_c(g)] = \max_{g \in C(x)} E_{\widehat{Q}} [U_c(g)] = E_{\widehat{Q}} [U_c(g^\star)].$$

Theorem 2.46. Consider the model setting and utility function as in Section 2.3.2. Then under the assumptions of Theorem 2.44, Theorem 2.13, and Lemma 2.21, the unique optimal solution g^\star for the problem (2.48) is also an optimal solution for (2.47), in the sense that

$$\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q [U(g)] = \inf_{Q \in \mathcal{Q}} E_Q [U(g^\star)].$$

In the **third chapter** we study problems similar to those in the third, however, now we work in incomplete market model, which makes the considered model setup more general. The key technique to generalize the previous results to this extended model setup is duality theory. As before, we investigate the minimax identity for non-concave utility functions and learned how to relate the corresponding functionals of an initial non-concave utility function and its concavifications. Moreover, we present how to derive optimal investment strategies for standard and robust non-concave utility maximization problems.

Section 3.1 is devoted to the minimax identity and relations between the robust functionals of the initial and concavified utility functions.

Assumption 3.1.

For all $x > 0$ exists some $Q \in \mathcal{Q}_e$ such that $u_Q(x) < \infty$.

Assumption 3.2.

$u_{Q_0}^c(x) < \infty$ for some $x > 0$ and some $Q_0 \in \mathcal{Q}_e$.

In the [Section 3.2](#) we studied the optimal investment strategies for the maximization problems for the concavified utility functions. The [Section 3.2.1](#) dealt with the standard utility maximization problem for the concavified utility function.

We introduced the convex conjugate function V of U ,

$$V(y) = \sup_{x>0} (U(x) - xy), \quad y > 0.$$

The dual value function v_P for u_P is given by

$$v_P(y) := \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}[V(Y_T)], \quad y > 0,$$

where the space $\mathcal{Y}(y)$ is defined as

$$\mathcal{Y}(y) = \{Y \geq 0 | Y_0 = y \text{ and } XY \text{ is a } \mathbb{P} - \text{supermartingale for all } X \in \mathfrak{X}(1)\}.$$

As in [\[25, 26\]](#) consider the following sets:

$$\begin{aligned} C(x) &= \{g \in L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}) | 0 \leq g \leq X_T \text{ for some } X \in \mathfrak{X}(x)\}; \\ D(y) &= \{h \in L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}) | 0 \leq h \leq Y_T \text{ for some } Y \in \mathcal{Y}(y)\}. \end{aligned}$$

Theorem 3.16. *Assume $v_P(y) < \infty$, for all $y > 0$ (we need it for the (iii) and (iv)), [\(3.3\)](#) and [\(3.6\)](#) hold.*

For any $x_0 > 0$, there exist some $\widehat{g} \in C(x_0)$, $\widehat{y} > 0$ and $\widehat{h} \in D(\widehat{y})$ such that

- (i) $u_P^c(x_0) = v_P(\widehat{y}) + x_0\widehat{y}$,*
- (ii) $u_P^c(x_0) = \inf_{y>0} (v_P(y) + x_0y)$,*
- (iii) $v_P(\widehat{y}) = \mathbb{E}[V(\widehat{h})]$,*
- (iv) $u_P^c(x_0) = \mathbb{E}[U_c(\widehat{g})]$.*

Moreover, $v_P(\cdot)$ is convex.

Theorem 3.17. *Assume $v_P(y) < \infty$, for all $y > 0$, [\(3.3\)](#) and [\(3.6\)](#) hold.*

Then, for any $x > 0$, there exists an optimal strategy $\widehat{X} \in \mathfrak{X}(x)$ such that

$$u_P^c(x) = \mathbb{E}[U_c(\widehat{X}_T)].$$

There also exist some $\widehat{y} > 0$ and $\widehat{Y} \in \mathcal{Y}(\widehat{y})$ such that

$$v_P(\widehat{y}) = \mathbb{E}[V(\widehat{Y}_T)] \text{ and } u_P^c(x) = v_P(\widehat{y}) + x\widehat{y}.$$

Moreover, $\widehat{X}_T \in -\partial V(\widehat{Y}_T)$. Also, $\widehat{X}\widehat{Y}$ is a martingale.

Subsection 3.2.2 deals with the robust utility maximization problem for the concavified utility function.

Theorem 3.18. Fix arbitrary $Q \in \mathcal{Q}_e$. Suppose $v_Q(y) < \infty$, for all $y > 0$ (we need it for the (iii) and (iv)), (3.13) and (3.6) hold.

For any $x_0 > 0$, there exist some $\widehat{g} \in C(x_0)$, $\widehat{y} > 0$ and $\widehat{h} \in D_Q(\widehat{y})$ such that

- (i) $u_Q^c(x_0) = v_Q(\widehat{y}) + x_0\widehat{y}$,
- (ii) $u_Q^c(x_0) = \inf_{y>0} (v_Q(y) + x_0y)$,
- (iii) $v_Q(\widehat{y}) = E_Q[V(\widehat{h})]$,
- (iv) $u_Q^c(x_0) = E_Q[U_c(\widehat{g})]$.

Moreover, $v_Q(\cdot)$ is convex.

Theorem 3.19. Suppose that all assumptions of Lemma 2.21, (3.6) and (3.14) hold.

For any $x_0 > 0$, there exist some $\widehat{Q} \in \mathcal{Q}_e$, $\widehat{y} > 0$ and $\widehat{h} \in D_{\widehat{Q}}(\widehat{y})$ such that

- (i) $u^c(x_0) = u_{\widehat{Q}}^c(x_0)$,
- (ii) $u^c(x_0) = \inf_{y>0} (v(y) + x_0y)$,
- (iii) $u^c(x_0) = v(\widehat{y}) + x_0\widehat{y}$,
- (iv) $v(\widehat{y}) = v_{\widehat{Q}}(\widehat{y}) = E_{\widehat{Q}}[V(\widehat{h})]$.

Theorem 3.20. Suppose that all assumptions of Theorem 3.19 hold.

Then, for any $x > 0$, there exist some $\widehat{X} \in \mathfrak{X}(x)$ and $\widehat{Q} \in \mathcal{Q}_e$ such that

$$u^c(x) = \inf_{Q \in \mathcal{Q}} E_Q[U_c(\widehat{X}_T)] = E_{\widehat{Q}}[U_c(\widehat{X}_T)] = u_{\widehat{Q}}^c(x). \quad (6)$$

There also exist some $\widehat{y} > 0$ and $\widehat{Y} \in \mathcal{Y}_{\widehat{Q}}(\widehat{y})$ such that

$$v(\widehat{y}) = E_{\widehat{Q}}[V(\widehat{Y}_T)] \text{ and } u^c(x) = v(\widehat{y}) + x\widehat{y}. \quad (7)$$

Moreover, $\widehat{X}_T \in -\partial V(\widehat{Y}_T)$, and $\widehat{X}\widehat{Y}$ is a \widehat{Q} -martingale.

In Section 3.3 we extend these findings to the initial non-concave utility function.

Theorem 3.21. Suppose that Assumption 2.2 and Assumption 3.1 hold.

Additionally, assume that the optimal solution g^* for concavified problem (either standard: $\sup_{g \in C(x)} \mathbb{E}[U_c(g)]$ or robust: $\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U_c(g)]$) such that $g^* \in -\partial V(\lambda Z)$, where $Z \in \mathcal{Z}$ has a continuous distribution.

Then, g^* is also an optimal solution for the corresponding non-concave problem.

Conclusions section summarizes the main results of this thesis.

In **the Appendix A**, we present auxiliary statements and supplementary proofs to make reading of the main part easier for readers.

The author expresses her sincere gratitude to the scientific supervisor, Doctor of Physical and Mathematical Sciences, Professor Georgiy Shevchenko, for proposing the problems, guiding on the directions of the research, being constantly attentive and supportive, motivating and inspiring. She is also deeply indebted to the head of the Department of Probability Theory, Statistics and Actuarial Mathematics – Professor Mishura Yulia Stepanivna – for advice, motivation and inspiration; all staff of Department of Probability Theory, Statistics and Actuarial Mathematics for their help and support during her work on this thesis.

Chapter 1

Review of literature on the topic of the dissertation

The basic problem of mathematical finance is concerned with an economic agent's investment decisions in a financial market, where the agent seeks to maximize the expected utility of their terminal wealth. This can be expressed mathematically as the so-called standard utility maximization problem:

$$\mathbb{E}[U(X_T)] \rightarrow \max, X \in \mathfrak{X}(x),$$

where X represents the investor's portfolio, $\mathfrak{X}(x)$ is the set of all admissible portfolios with initial wealth $X_0 \leq x$, U is the utility function that measures the investor's satisfaction, and T is the investment horizon.

In the past, many studies have considered this problem under the assumption that there is a probability measure that accurately describes the future stock price evolution. However, in reality, there is always some level of uncertainty and investors may be averse to both risk and Knightian uncertainty, the latter refers to the uncertainty associated with choosing an appropriate model, which is often not reflected in the probabilities of known events.

Therefore, it is essential to consider both risk and Knightian uncertainty when seeking to maximize the expected utility of terminal wealth. This can be achieved by using appropriate techniques such as robust optimization, which seeks to identify the optimal investment strategy that performs well under all possible scenarios, or by incorporating multiple priors in the decision-making process, which was the idea initially taken by Savage [35]. In the spirit of it, Yaari [41], Gilboa and Schmeidler

[17] and Schmeidler [18] worked with the robust utility functional, i.e.

$$X \rightarrow \inf_{Q \in \mathcal{Q}} E_Q[U(X)],$$

where \mathcal{Q} is a set of probability measures (priors).

Note that when the set of all considered probability measures \mathcal{Q} is a singleton, the robust utility maximization problem

$$\sup_{X \in \mathfrak{X}(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(X_T)]$$

coincides with the standard utility maximization problem

$$\sup_{X \in \mathfrak{X}(x)} \mathbb{E}[U(X_T)].$$

The literature on utility maximization is rich and varied, with many papers exploring different problems and model frameworks, but there is a lack of connection between some of the contributions to field due to the diversity of problems and model frameworks they consider. For example, some papers focus on concave and non-concave utility functions, while others consider complete and incomplete market models, and still others address standard and robust utility maximization problems.

Some of these papers have made significant contributions to the field, such as the work of Schied and Wu [37], who proved the existence of optimal investment strategies for the problem of maximizing the robust utility function in a general incomplete market model framework. They assumed a strictly concave, strictly increasing, and continuously differentiable utility function that satisfies the Inada conditions. Using the minimax identity for the concave functions, they showed the existence of optimal probability measure \widehat{Q} , in the sense that $\sup_{X \in \mathfrak{X}(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(X_T)] = \sup_{X \in \mathfrak{X}(x)} E_{\widehat{Q}}[U(X_T)]$, which, together with the results of the Kramkov and Schachermayer [25, 26], established the existence of optimal investment strategy.

While considering the complete market model denote the unique equivalent local martingale measure by Q^e . In the spirit of [34], consider a utility function U on \mathbb{R}_+ which is non-decreasing upper-semicontinuous and satisfying a mild growth condition. In this thesis we also look into the standard utility maximization problem, i.e.

$$\sup_{X \in \mathfrak{X}(x)} \mathbb{E}[U(X)], \tag{1.1}$$

while considering two possibilities for the set $\mathfrak{X}(x)$ of admissible final endowments:

- the standard budget constraint:

$$\mathfrak{X}(x) = \{X \in L_+^1(Q^e) \mid E_{Q^e}[X] \leq x\}, x > 0,$$

- an additional upper bound:

$$\mathfrak{X}(x) = \{X \in L^1(Q^e) \mid 0 \leq X \leq W, E_{Q^e}[X] \leq x\}, \quad (1.2)$$

with some random variable $W : \Omega \rightarrow [0, +\infty)$.

The aforementioned Kramkov and Schachermayer's articles [25, 26] provided important insights into the problem of maximizing a strictly concave utility functional in the context of a general incomplete semimartingale model of financial markets. Their work addressed several key questions:

1. Does the optimal solution/strategy exist?
2. How may we calculate the optimal solution, provided this solution exists?
3. Does the value function satisfy the usual properties of the utility function?
4. How may we calculate the value function as well as the dual value function?

Their approach represented a significant departure from previous research in this area, which had typically focused on the assumption of a complete market and known probability measures for stock price evolution.

The paper [25] made a significant contribution to the understanding of the utility maximization problem by identifying a key market-independent assumption for answering important questions about the existence and properties of optimal solutions. Specifically, Kramkov and Schachermayer demonstrated that the answers to questions 1 and 3 are positive if and only if the asymptotic elasticity of the utility function is strictly less than one. This result provides a minimal assumption that can be made solely on the utility function to address these key questions.

In their subsequent paper, [26], Kramkov and Schachermayer further advanced the understanding of the utility maximization problem and thus clarified the conditions under which optimal solutions can be found by identifying a necessary and sufficient condition for on both the utility function and the model itself: the finiteness of the value function of the dual problem.

Several other approaches were used for robust portfolio optimization such as reducing the robust case to the standard one through proving the existence of

the “worst-case scenario measure” or “the least favourable measure”, e.g. [32, 36], a stochastic control approach, see [20], an approach using BSDEs, see [11] and references therein.

Alexander Schied [36] introduced the notion of a *least-favorable measure* in a set \mathcal{Q} of probability measures absolutely continuous with respect to \mathbb{P}^* : $Q_0 \in \mathcal{Q}$ is least-favorable with respect to \mathbb{P}^* if the density $\pi = d\mathbb{P}^*/dQ_0$ (taken in the sense of the Lebesgue decomposition) satisfies

$$Q_0[\pi \leq t] = \inf_{Q \in \mathcal{Q}} Q[\pi \leq t] \text{ for all } t > 0.$$

In a complete market model, he presented a solution to the problem of maximizing the robust utility of the terminal wealth, under a condition that set \mathcal{Q} of all measures that we are considering admits a least-favorable measure. The main result was that under condition above the robust problem is equivalent to the standard utility maximization problem with respect to the least-favorable measure. Moreover, he showed the existence of least-favorable measure in various situations.

Backhoff Veraguas and Fontbona [3] extended these results by implementing the assumption on the densities of the uncertainty set instead of the usual compactness assumption. Moreover, they have done this without relying on the existence of the worst-case measure or on any assumption implying this. Neufeld and Šikić [29, 30] studied a robust stochastic optimization problem in the quasi-sure setting in discrete-time. Their paper [29] deals with the study of the general case of concave utility functions, showing the existence of the the maximizer in different market models under the linearity-type condition, which is caused by the no-arbitrage condition. In [30], they consider the non-concave utility maximization problem and outline the conditions of maximizer’s existence.

For more results concerning the robust utility maximization problem we refer to Bartl, Kupper and Neufeld [9] and references therein.

The majority of articles on utility maximization assume that the investor’s utility function is strictly concave, strictly increasing, continuously differentiable, and satisfies the Inada conditions. While the assumption of monotonicity is natural, since an agent prefers more wealth to less, other assumptions can be omitted or relaxed. Most notably, Tversky and Kahneman in their seminal paper [21] showed, using empirical evidence, that people tend to contribute to risk aversion in choices

involving sure gains and to risk seeking in choices involving sure losses. Therefore, one has to consider utility functions which are not necessarily concave.

Jennifer Carpenter [12] and Christian Reichlin [33, 34] are two authors who have contributed to the study of utility maximization with non-concave utility functions. Carpenter's work considered the manager who operates in a complete, arbitrage-free, continuous-time financial market and who is risk-averse with utility function U which is strictly increasing, strictly concave, at least twice continuously differentiable. She studied the dynamic investment problem of choosing an admissible trading strategy π which maximizes the expected utility of the payoff of a call option on the assets plus a constant:

$$\max_{\pi} E[U(\alpha(X_T - B_T)^+ + K)],$$

where X_T is the payoff of the assets at time T , $K > 0$ represents fixed compensation and personal wealth, $\alpha > 0$ represent the percentage of positive profits, and B_T represent the option strike price. One of her main results was constructing the unique optimal solution using the concavified utility function.

Reichlin's doctoral thesis [33] provided a comprehensive analysis of the general framework of non-concave utility functions in both complete and incomplete market models. He investigated the properties of non-concave utility functions and their concavifications, and showed several useful relations between the two. One of the main results was equality between the maximization functionals for non-concave utility function U and its concavification U_c , i.e. $\sup_{X \in \mathfrak{X}(x)} \mathbb{E}[U(X_T)] =$

$\sup_{X \in \mathfrak{X}(x)} \mathbb{E}[U_c(X_T)]$. He also presented several important findings about optimal solutions for the both problems.

While considering two cases of admissible final endowments: the standard budget constraint and additional upper bound (which has not been considered before in such model setup) we extend Reichlin's results by proving new connections in the form of equalities and inequalities of the robust utility maximization functionals of initial non-concave utility functions and its concavification. Furthermore, we proceed in proving the minimax identity for the general case of non-concave utility functions. The crucial step for obtaining the mentioned results with implementing an additional upper bound is the use of the regular conditional distribution which sheds new light on the possible approaches for solving the optimization problem.

Moreover, we construct the explicit form of the optimal solution in the standard utility maximization problem including the case with implementation of the budget constraints represented by a discrete random variable. And showed that theorem may fail if budget constraints were represented by a continuous random variable.

The most of the literature on non-concave utility maximization, including this thesis, rely on the concavification principle, i.e. replacing the non-concave utility function by its concave envelope, there are also some articles which provide a general framework for solving non-concave utility maximization problems without concavification. For example, Dai et al [31] considered cases where the concavification principle may not be applied and the utility functions can be discontinuous; consult the references therein for further relevant research on the topic.

Chapter 2

Optimal investment strategies for the non-concave utility maximization problems in complete markets

In this chapter, we will delve into the minimax identity for non-concave utility functions using the concavification method. We will first introduce the concept of concavification and show how it relates to utility maximization problems. The concavification method is particularly useful for non-concave utility functions since it helps to establish conditions for the existence and uniqueness of optimal investment strategies.

Next, we will present cases in which the minimax identity holds for non-concave utility functions. The minimax identity provides a useful tool for characterizing optimal investment strategies when dealing with robust utility maximization problems. In such cases, the minimax identity helps to establish a relationship between the minimization of worst-case scenarios and the maximization of expected utility.

We will then present the optimal solution for the maximization of non-concave utility functions. This involves finding investment strategies that maximize the expected value of the non-concave utility function subject to certain constraints. Finally, we will present the optimal solution for the robust utility maximization problem under the assumption that the minimax identity holds.

2.1 Model setup

Throughout this chapter, we assume the uniqueness of equivalent local martingale measure, and hence the market model completeness.

Assumption 2.1. *There is a unique equivalent local martingale measure, which we denote as Q^e .*

Throughout the dissertation we consider standard and robust optimization problems with and without budget constraints in complete and incomplete market models; let us describe here the general model setup.

By $U(x)$ we denote the investor's utility function; assumptions on $U(x)$ will be stated later.

For the standard optimization problem we aim to identify

$$\sup_{X \in \mathfrak{X}(x)} \mathbb{E}[U(X)], \quad (2.1)$$

and for the robust optimization problem we look at

$$\sup_{X \in \mathfrak{X}(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(X)].$$

Throughout this chapter we consider two possibilities for the set $\mathfrak{X}(x)$ of admissible final endowments:

- the standard budget constraint:

$$\mathfrak{X}(x) = \{X \in L_+^1(Q^e) \mid E_{Q^e}[X] \leq x\}, x > 0,$$

- an additional upper bound:

$$\mathfrak{X}(x) = \{X \in L^1(Q^e) \mid 0 \leq X \leq W, E_{Q^e}[X] \leq x\}, \quad (2.2)$$

with some random variable $W : \Omega \rightarrow [0, +\infty)$.

As in [37], we impose the following assumptions on the set of probability measures \mathcal{Q} on (Ω, \mathcal{F}) . (Note that \mathcal{Q} is *not* the set of all measures on the measurable space (Ω, \mathcal{F}) , but just some subset satisfying the below assumptions.)

Assumption 2.2. (i) \mathcal{Q} is convex;

(ii) $\mathbb{P}[A] = 0$ if and only if $Q[A] = 0$ for all $Q \in \mathcal{Q}$;

(iii) The set $\mathcal{Z} := \{dQ/d\mathbb{P} \mid Q \in \mathcal{Q}\}$ is closed in $L^0(\mathbb{P})$ (with respect to pointwise convergence).

We also need the following notation $\mathcal{Z}_e := \{dQ/d\mathbb{P} \mid Q \in \mathcal{Q}_e\}$, where \mathcal{Q}_e denotes the set of measures in \mathcal{Q} that are equivalent to \mathbb{P} .

Proposition 2.3. *It follows from the [Assumption 2.2](#) (i) and (iii) that the set \mathcal{Q} is closed under countable convex combinations.*

Proof. See [Appendix A.1](#). □

Moreover, it follows from the Halmos–Savage theorem that given (i) and (iii), assumption (ii) is equivalent to the following:

$$Q \ll \mathbb{P} \text{ for all } Q \in \mathcal{Q} \text{ and } \mathcal{Q}_e \neq \emptyset,$$

where \mathcal{Q}_e denotes the set of measures in \mathcal{Q} that are equivalent to \mathbb{P} , for the proof see [Lemma A.1](#).

The main aim is to maximize the robust utility functional $\inf_{Q \in \mathcal{Q}} E_Q[U(X)]$ among all $X \in \mathfrak{X}(x)$.

Hence, we consider the next value function of the robust utility problem

$$u(x) := \sup_{X \in \mathfrak{X}(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(X)].$$

Denote

$$u_Q(x) := \sup_{X \in \mathfrak{X}(x)} E_Q[U(X)],$$

the value function of the optimal investment problem.

To relate the static optimization problem (2.1) to a dynamic one, one can consider the model setup as in [37] with the additional assumption that the discounted price process is locally bounded, that is a discounted price process with d assets, which is modelled by a stochastic process $S = (S_t)_{0 \leq t \leq T}$. Assume that S is a d -dimensional locally bounded semimartingale on $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to a filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$. A self-financing trading strategy is given by its initial wealth $X_0 \in \mathbb{R}$ and a d -dimensional predictable and S -integrable process $\xi = (\xi_t)_{0 \leq t \leq T}$. The corresponding

value process X satisfies

$$X_t = X_0 + \int_0^t \xi_r dS_r, \quad 0 \leq t \leq T.$$

Let $\mathcal{X}(x)$ be the set of all such processes X with $X_0 \leq x$ which are also admissible in the sense that $X_t \geq 0$, for $0 \leq t \leq T$. In this setting, the standard utility maximization problem is to maximize the expected utility of the terminal wealth for the initial capital x , i.e. to find $\sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X(T))]$. Thanks to the completeness assumption, we have that $\mathfrak{X}(x) = \{X(T) \mid X \in \mathcal{X}(x)\}$, so that dynamic utility maximization problem is equivalent to (2.1).

Example 2.4 (Cox–Ross–Rubinstein-model, also known as binomial model). Consider a financial model consisting of two assets, a bond with a fixed interest rate $r > -1$ and a stock whose price at each time step depends on the state of the economy. The economy can be in either a good or a bad state. We denote the maturity time by T . The bank account, also known as the bond, evolves according to $S_t^0 = (1 + r)^t$ for $t = 0, \dots, T$ and $r > -1$.

The sample space is given by

$$\Omega = (\omega_1, \dots, \omega_T) : \omega_i \in 0, 1, \text{ for } i = 1, \dots, T,$$

where ω_i represents the state of the economy at time i . We consider a strictly positive probability measure \mathbb{P} such that $\mathbb{P}(\omega) > 0$ for all $\omega \in \Omega$.

Let y_g be the growth rate during good times and y_b be the growth rate during bad times, where $y_g > y_b > -1$. The stock price evolves according to

$$S_t^1(\omega) = S_0^1 \cdot (1 + y_g)^{D_t(\omega)} \cdot (1 + y_b)^{t - D_t(\omega)},$$

where $S_0^1 > 0$ is the initial price of the stock, and $D_t(\omega)$ is the number of good states of the economy until time t , that is, $D_t(\omega) = \sum_{s=1}^t \omega_s$.

This model is arbitrage-free if and only if $y_g > r > y_b > -1$, and in this case, the model is also complete. The unique equivalent (local) martingale measure is given by

$$Q(\omega) = \left(\frac{r - y_b}{y_g - y_b} \right)^{D_T(\omega)} \cdot \left(1 - \frac{r - y_b}{y_g - y_b} \right)^{T - D_T(\omega)}.$$

Example 2.5 (Black–Scholes model). For a fixed time horizon, consider a continuous-time market on the time interval $[0, T]$. Let $(\Omega, \mathcal{F}, \mathbb{F} = \mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ be a filtered probability space. The market consists of a bond $S^0 = (S_t^0)_{t \in [0, T]}$ with a fixed interest rate $r \in \mathbb{R}$, defined by $S_t^0 = e^{rt}$ for all $t \in [0, T]$, and a stock $S^1 = (S_t^1)_{t \in [0, T]}$, whose price dynamics is governed by the stochastic differential equation $dS_t^1 = S_t^1(\mu dt + \sigma dB_t)$ for $t \in [0, T]$, where B_t is a standard Brownian motion w.r.t. \mathbb{F} . In other words, the stock price follows a geometric Brownian motion with volatility $\sigma > 0$ and drift $\mu - \sigma^2/2$:

$$S_t^1 = S_0^1 \cdot e^{\sigma B_t + (\mu - \sigma^2/2)t}, t \in [0, T],$$

where $S_0^1 > 0$ is the initial stock price.

The unique equivalent local martingale measure Q in this market has the Radon–Nikodym derivative $\frac{dQ}{d\mathbb{P}} = \exp\left(\frac{r-\mu}{\sigma}B_T - \left(\frac{r-\mu}{\sigma}\right)^2 \frac{T}{2}\right)$.

For brevity, we introduce the following sets, as in [25, 26]:

$$C(x) = \{g \in L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}) \mid 0 \leq g \leq X \text{ for some } X \in \mathfrak{X}(x)\}.$$

It is easy to see that

$$\begin{aligned} u(x) &= \sup_{X \in \mathfrak{X}(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(X)] = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(g)]; \\ u_Q(x) &= \sup_{X \in \mathfrak{X}(x)} E_Q[U(X)] = \sup_{g \in C(x)} E_Q[U(g)]. \end{aligned}$$

Remark 2.6. It is known from Delbaen and Schachermayer (see [13] for the case of a locally bounded semimartingale S , [14] for the general case and [15] for a more specific version) that for $g \geq 0$, it holds that

$$g \in C(x) \iff \sup_{Q \in \mathcal{M}_e} E_Q(g) \leq x \iff \sup_{Q \in \mathcal{M}_a} E_Q(g) \leq x,$$

where \mathcal{M}_e is the set of equivalent local martingale measures and \mathcal{M}_a is the set of absolutely continuous local martingale measures.

In our model setup $\mathcal{M}_e = \{Q^e\}$, and hence

$$g \in C(x) \iff E_{Q^e}(g) \leq x.$$

Consider investor's utility function $U : [0, \infty) \rightarrow \mathbb{R}$. We assume that U is non-constant, non-decreasing, upper semi-continuous and satisfies the mild growth condition:

$$\lim_{x \rightarrow \infty} \frac{U(x)}{x} = 0,$$

and that $U(\infty) > 0$. We set $U(x) = -\infty$ for $x < 0$. It follows from [2, Proposition 3.1] that $U(x)$ has a non-decreasing and continuous on $(0, \infty)$ concavification (concave envelope) $U_c(x)$, or the smallest concave function such that $U_c(x) \geq U(x)$, for all $x \in \mathbb{R}$. Concavification can be used to deal with non-concave utility function and it is crucial for this thesis. In view of this, we consider a particular utility function in the next section in order to provide more intuition about the concavification.

2.2 Special case of non-concave utility

Throughout the section we assume that utility function U is strictly increasing, strictly concave, at least twice continuously differentiable, and is defined on a domain containing $[0, \infty)$.

Moreover, suppose that $U'(x)$ approaches zero as x approaches infinity.

Define $\widehat{U} : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\widehat{U}(x) = \begin{cases} U((x - K)^+), & x \geq 0 \\ -\infty, & \text{otherwise.} \end{cases}$$

Throughout this section we will assume that investor's utility function is \widehat{U} ; we will sometimes call it *original utility function* to differentiate it from its concavification.

Function $\widehat{U}(x)$ is non-concave, hence we are going to consider its concavification, which we denote by $\widetilde{U}(x)$.

The concavified objective function \widetilde{U} is linear in the interval $[0, \widehat{x}]$, where the point $\widehat{x} > K$ is such that the slope of a line through $(0, U(0))$ and $(\widehat{x}, \widehat{U}(\widehat{x}))$ is $\widehat{U}'(\widehat{x})$. For $x \geq \widehat{x}$, the concavified objective function coincides with \widehat{U} . The existence of such concavification \widetilde{U} follows from [Lemma 2.7](#) and [Lemma 2.9](#) below.

Lemma 2.7. *There exists a unique point $\widehat{x} > K$ such that*

$$\frac{\widehat{U}(\widehat{x}) - \widehat{U}(0)}{\widehat{x}} = \widehat{U}'(\widehat{x}). \quad (2.3)$$

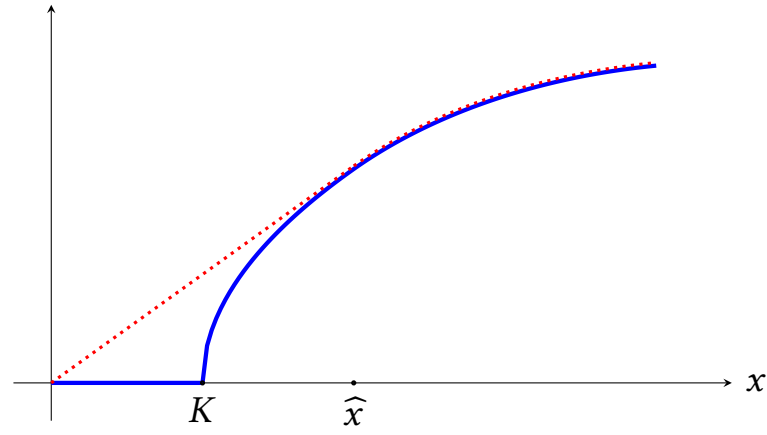


Figure 2.1. **Investor's original and concavified functions.** The solid line represents the function $\widehat{U}(x) = U((x - K)^+)$; the dotted line represents the concavification of $\widehat{U}(x)$, function $\widetilde{U}(x)$; for simplicity, in this and the following picture $U(0) = 0$.

Remark. For all $\widehat{x} \in (0, K]$ the equality (2.3) is obvious.

Proof. Noting that $\widehat{U}(0) = U(0)$, consider the linear function $y(x)$, such that

$$y(x) = kx + U(0), \text{ where } 0 < k < \frac{\widehat{U}(K+1) - U(0)}{K+1}.$$

It follows that

$$y(K+1) = k \cdot (K+1) + U(0) < \widehat{U}(K+1). \quad (2.4)$$

Since $y(x) > \widehat{U}(x)$, for $x \in (0, K)$ it follows that exists such a point $x_1 \in (K, K+1)$ for which $y(x_1) = \widehat{U}(x_1)$. By the assumptions on the utility function U , we know that $U'(\infty-) = \widehat{U}'(\infty-) = 0$. It follows that

$$\exists x_0 > x_1 : \forall x > x_0 \quad 0 < \widehat{U}'(x) < \frac{k}{2}.$$

Noting that $y'(x) = k > 0$, we obtain

$$\exists \widetilde{x} > x_1 : y(\widetilde{x}) > \widehat{U}(\widetilde{x}).$$

Noting (2.4), it follows from the intermediate value theorem that there exists a point $x_2 > K+1 > x_1$ such that $y(x_2) = \widehat{U}(x_2)$. (See Figure 2.2.)

We have

$$\widehat{U}(x_1) = y(x_1) \quad \text{and} \quad \widehat{U}(x_2) = y(x_2).$$

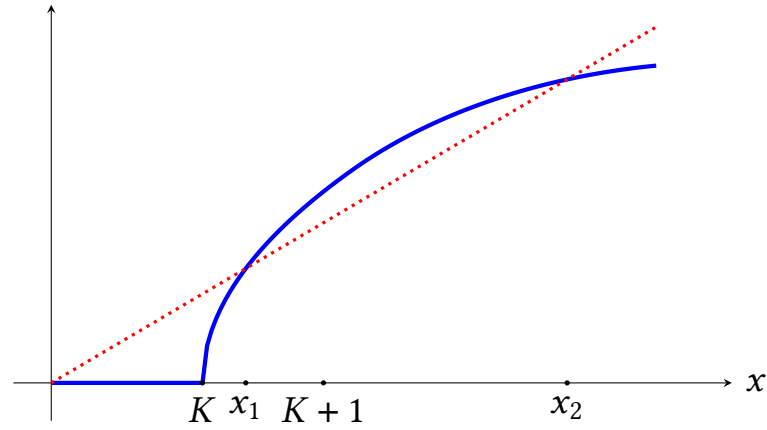


Figure 2.2. **Investor original function $\widehat{U}(x)$ and linear function $y(x)$.** The solid line represents the function $\widehat{U}(x) = U((x - K)^+)$; the dotted line represents the linear function $y(x) = kx + U(0)$; the points x_1, x_2 are such that $\widehat{U}(x_i) = y(x_i)$, for $i \in \{1, 2\}$.

Since $x_1 > K$, $\widehat{U}(\cdot)$ is continuously differentiable around the point x_1 .

Moreover,

$$\begin{aligned} \widehat{U}'(x_1) &= \lim_{x \rightarrow x_1^+} \frac{\widehat{U}(x) - \widehat{U}(x_1)}{x - x_1} = \lim_{x \rightarrow x_1^+} \frac{\widehat{U}(x) - y(x_1)}{x - x_1} \\ &\geq \lim_{x \rightarrow x_1^+} \frac{y(x) - y(x_1)}{x - x_1} = k. \end{aligned} \quad (2.5)$$

Inequality in (2.5) follows from the fact that $\widehat{U}(x)$ is a strictly concave function for $x > K$. Similarly,

$$\begin{aligned} \widehat{U}'(x_2) &= \lim_{x \rightarrow x_2^-} \frac{\widehat{U}(x_2) - \widehat{U}(x)}{x_2 - x} = \lim_{x \rightarrow x_2^-} \frac{y(x_2) - \widehat{U}(x)}{x_2 - x} \\ &\leq \lim_{x \rightarrow x_2^-} \frac{y(x_2) - y(x)}{x_2 - x} = k. \end{aligned} \quad (2.6)$$

Noting that

$$\frac{\widehat{U}(x_1) - U(0)}{x_1} = \frac{y(x_1) - U(0)}{x_1} = \frac{kx_1}{x_1} = k \stackrel{(2.5)}{\leq} \widehat{U}'(x_1)$$

and

$$\frac{\widehat{U}(x_2) - U(0)}{x_2} = \frac{y(x_2) - U(0)}{x_2} = \frac{kx_2}{x_2} = k \stackrel{(2.6)}{\geq} \widehat{U}'(x_2),$$

we obtain that the function $g(x) := \frac{\widehat{U}(x) - U(0)}{x} - \widehat{U}'(x)$ is continuous and $g(x_1) \cdot g(x_2) \leq 0$. By the intermediate value theorem function $g(x)$ has a zero in the interval $[x_1, x_2]$.

Hence,

$$\exists \hat{x} > K : \frac{\widehat{U}(\hat{x}) - U(0)}{\hat{x}} = \widehat{U}'(\hat{x}).$$

The uniqueness of such point \hat{x} follows from the fact that a tangent to strictly concave function lies above that function. \square

From [Lemma 2.7](#) it follows that the concavification $\widetilde{U} : \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$\widetilde{U}(x) = \begin{cases} \widehat{U}(x) & \text{for } x \geq \hat{x}, \\ U(0) + \widehat{U}'(\hat{x}) \cdot x & \text{for } 0 \leq x < \hat{x}, \\ -\infty & \text{for } x < 0. \end{cases} \quad (2.7)$$

Lemma 2.8. *The function $f(x) := \frac{\widehat{U}(x) - U(0)}{x}$ has following properties:*

- (i) $f(x) = 0$ for $x \in (0, K]$;
- (ii) f is increasing for $x \in (K, \hat{x})$;
- (iii) $f'(\hat{x}) = 0$;
- (iv) $f(x)$ is decreasing for $x \in (\hat{x}, \infty)$.

Proof. See [Appendix A.2](#). \square

Lemma 2.9. *Function*

$$\widetilde{U}(x) = \begin{cases} \widehat{U}(x), & x \geq \hat{x}; \\ U(0) + \widehat{U}'(\hat{x}) \cdot x, & 0 \leq x < \hat{x}; \\ -\infty, & x < 0. \end{cases}$$

is strictly increasing, concave and continuously differentiable on the $(0, \infty)$.

Proof. The proof is straightforward. \square

To summarize, we obtain the concavified objective function \widetilde{U} , which satisfies $\widetilde{U} \geq \widehat{U}$ and denoted by

$$\widetilde{U}(x) = \begin{cases} \widehat{U}(x), & x \geq \hat{x}; \\ U(0) + \widehat{U}'(\hat{x}) \cdot x, & 0 \leq x < \hat{x}; \\ -\infty, & x < 0. \end{cases}$$

Moreover, the function \tilde{U} is concave, strictly increasing, continuously differentiable on $(0, +\infty)$, satisfies the Inada condition in the ∞ , however the Inada condition in 0 does not hold.

Example 2.10. Consider the function

$$U(x) = (x - K)_+^j, \text{ for } K > 0, \text{ and } 0 < j < 1.$$

The concavification of this function is

$$U_c(x) = \begin{cases} \left(\frac{K}{1-j} - K\right)^j + j \left(\frac{K}{1-j} - K\right)^{j-1} \left(x - \frac{K}{1-j}\right), & x < \frac{K}{1-j}; \\ (x - K)^j, & x \geq \frac{K}{1-j}. \end{cases}$$

Then, the point \hat{x} , starting from which objective function U and it's concavification U_c coincides, is equal to $\frac{K}{1-j}$.

2.3 Minimax identity for non-concave utility function in a complete market

This section is devoted to study of the minimax identity for the robust non-concave utility functional in a complete market model, i.e.

$$u(x) := \sup_{X \in \mathfrak{X}(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(X)] = \inf_{Q \in \mathcal{Q}} \sup_{X \in \mathfrak{X}(x)} E_Q[U(X)].$$

We keep all the assumptions from [Section 2.1](#) on the set of all probability measures \mathcal{Q} and the utility function U intact.

We assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless and suppose that [Assumption 2.2](#) holds.

Recall that

$$\begin{aligned} u(x) &= \sup_{X \in \mathfrak{X}(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(X)] = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(g)]; \\ u_Q(x) &= \sup_{X \in \mathfrak{X}(x)} E_Q[U(X)] = \sup_{g \in C(x)} E_Q[U(g)]. \end{aligned}$$

Define

$$u^c(x) := \sup_{X \in \mathfrak{X}(x)} \inf_{Q \in \mathcal{Q}} E_Q[U_c(X)] = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U_c(g)];$$

In our case, the function $U_c(\cdot)$ is not strictly concave and does not satisfy the Inada conditions at 0, so we cannot simply apply the previous results without modifications.

To address this issue, we need the following lemma. While its proof is similar to that of [37, Lemma 3.4], we give it for completeness.

Lemma 2.14. *Under Assumptions 2.2 and 2.12, we have*

$$\begin{aligned} u^c(x) &= \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U_c(g)] = \inf_{Q \in \mathcal{Q}} \sup_{g \in C(x)} E_Q[U_c(g)] \\ &= \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)] = \inf_{Q \in \mathcal{Q}_e} \sup_{g \in C(x)} E_Q[U_c(g)]. \end{aligned}$$

Remark. *The lemma above also holds for a utility function defined on the domain $(0, \infty)$.*

Proof. Take arbitrary $\varepsilon > 0$. It follows from the definition of $C(x)$ that $g + \varepsilon \in C(x + \varepsilon)$ whenever $g \in C(x)$. Therefore, using the properties of conditional expectations, we get

$$\begin{aligned} u^c(x + \varepsilon) &\geq \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U_c(\varepsilon + g)] \\ &= \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} \left[E_Q \left(\frac{dP}{dQ} \right) \cdot E_P \left[U_c(\varepsilon + g) \cdot \frac{dQ}{dP} \right] \right] \\ &= \sup_{g \in C(x)} \inf_{Z \in \mathcal{Z}} \mathbb{E}[Z U_c(\varepsilon + g)]. \end{aligned}$$

Next, we show that $Z \rightarrow \mathbb{E}[Z U_c(\varepsilon + g)]$ is, for each $g \in C(x)$, a weakly lower semi-continuous affine functional defined on the weakly compact convex set \mathcal{Z} .

Consider the sequence $Z_n \xrightarrow{d} Z$. We need to show that

$$\liminf_{n \rightarrow \infty} \mathbb{E}[Z_n U_c(\varepsilon + g)] \geq \mathbb{E}[Z U_c(\varepsilon + g)]. \quad (2.8)$$

By the Skorokhod representation theorem, there exist a sequence Y_n , $n \geq 1$, and a random variable Y such that $Y_n \rightarrow Y$, $n \rightarrow \infty$, a.s., Y has the same distribution as Z , and for each $n \geq 1$, Y_n has the same distribution as Z_n .

Since $U_c(\varepsilon + g)$ is bounded from below, there is a $c > 0$ such that $U_c(\varepsilon + g) + c > 0$. Moreover, $Y_n \cdot (U_c(\varepsilon + g) + c)$ is nonnegative and converges to $Y \cdot (U_c(\varepsilon + g) + c)$ a.s.. From Fatou's lemma and definition of Y_n we obtain

$$\begin{aligned} \liminf_{n \rightarrow \infty} \mathbb{E}[Z_n \cdot (U_c(\varepsilon + g) + c)] &= \liminf_{n \rightarrow \infty} \mathbb{E}[Y_n \cdot (U_c(\varepsilon + g) + c)] \\ &\geq \mathbb{E}[Y \cdot (U_c(\varepsilon + g) + c)] = \mathbb{E}[Z \cdot (U_c(\varepsilon + g) + c)]. \end{aligned}$$

Noting that $\mathbb{E}[Z_n \cdot (U_c(\varepsilon + g) + c)] = \mathbb{E}[Z_n U_c(\varepsilon + g)] + c$, we derive equation (2.8).

From [37, Lemma 3.2] it follows that \mathcal{Z} is a weakly compact convex set. Also for any $g_1, g_2 \in C(x)$ and $0 < \lambda < 1$,

$$\begin{aligned} \mathbb{E}[Z U_c(\varepsilon + \lambda g_1 + (1 - \lambda)g_2)] &= \mathbb{E}[Z U_c(\lambda(\varepsilon + g_1) + (1 - \lambda)(\varepsilon + g_2))] \\ &\geq \mathbb{E}[Z(\lambda U_c(\varepsilon + g_1) + (1 - \lambda)U_c(\varepsilon + g_2))] \\ &= \lambda \mathbb{E}[Z U_c(\varepsilon + g_1)] + (1 - \lambda) \mathbb{E}[Z U_c(\varepsilon + g_2)]. \end{aligned}$$

Hence, for each $Z \in \mathcal{Z}$, $g \rightarrow \mathbb{E}[Z U_c(\varepsilon + g)]$ is a concave functional defined on the convex set $C(x)$, see [25, Proposition 3.1]. Since almost sure convergence implies weak convergence, the conditions of the lop sided minimax theorem [1, Chapter 6, p. 295] are satisfied, and thus

$$\sup_{g \in C(x)} \min_{Z \in \mathcal{Z}} \mathbb{E}[Z U_c(\varepsilon + g)] = \min_{Z \in \mathcal{Z}} \sup_{g \in C(x)} \mathbb{E}[Z U_c(\varepsilon + g)].$$

Therefore, we obtain

$$\begin{aligned} u^c(x + \varepsilon) &\geq \min_{Q \in \mathcal{Q}} \sup_{g \in C(x)} E_Q[U_c(\varepsilon + g)] \geq \inf_{Q \in \mathcal{Q}} \sup_{g \in C(x)} E_Q[U_c(g)] \\ &\geq \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U_c(g)] = u^c(x). \end{aligned}$$

The last inequality follows from the fact that for all $Q \in \mathcal{Q}$ and $g \in C(x)$

$$\sup_{g \in C(x)} E_Q[U_c(g)] \geq \inf_{Q \in \mathcal{Q}} E_Q[U_c(g)].$$

Sending $\varepsilon \downarrow 0$ and using the continuity of u^c , we obtain the first part of lemma.

Now Assumption 2.12 combined with [37, Lemma 3.3] imply that $u^c(x) = \inf_{Q \in \mathcal{Q}_e} u_Q^c(x)$. Thus,

$$\begin{aligned} u^c(x) &= \inf_{Q \in \mathcal{Q}_e} u_Q^c(x) = \inf_{Q \in \mathcal{Q}_e} \sup_{g \in C(x)} E_Q[U_c(g)] \\ &\geq \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)] \geq \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U_c(g)] = u^c(x), \end{aligned}$$

as required. \square

2.3.2 Minimax identity for the objective function $U(x)$

In this section, we prove lemmas that complete the proof of [Theorem 2.13](#).

Lemma 2.15. *If [Assumption 2.2](#) and [Assumption 2.11](#) hold, then for all $g \in C(x)$*

$$\inf_{Q \in \mathcal{Q}} E_Q[U(g)] = \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)]. \quad (2.9)$$

Remark 2.16. *The statement holds for U_c verbatim.*

Proof. Fix $g \in C(x)$. Note that [Assumption 4.9](#) implies the existence of a measure $Q_0 \in \mathcal{Q}_e$ such that $E_{Q_0}[U(g)] < \infty$.

Take $Q_1 \in \mathcal{Q}$ and define $0 \leq t < 1$ the measure $Q_t = tQ_1 + (1-t)Q_0$. Evidently, $Q_t \in \mathcal{Q}_e$.

Let $(Q_1^{(n)}) \in \mathcal{Q}$ be such that $E_{Q_1^{(n)}}[U(g)] \rightarrow \inf_{Q \in \mathcal{Q}} E_Q[U(g)] < \infty$.

Thanks to continuity of function $f(t) = E_{Q_t}[U(g)] = tE_{Q_1}[U(g)] + (1-t)E_{Q_0}[U(g)]$ on $[0, 1]$, one has

$$\inf_{Q \in \mathcal{Q}} E_Q[U(g)] = \lim_{n \rightarrow \infty} E_{Q_1^{(n)}}[U(g)] = \lim_{n \rightarrow \infty} \lim_{t \rightarrow 1} E_{Q_t^{(n)}}[U(g)], \quad (2.10)$$

where $Q_t^{(n)} := tQ_1^{(n)} + (1-t)Q_0$.

Since $Q_t^{(n)} \in \mathcal{Q}_e$ for $n \geq 1$, there exists a subsequence $(Q^{(n_k)})_{k \in \mathbb{N}} \in \mathcal{Q}_e$ such that

$$\lim_{n \rightarrow \infty} \lim_{t \rightarrow 1} E_{Q_t^{(n)}}[U(g)] = \lim_{k \rightarrow \infty} E_{Q^{(n_k)}}[U(g)] \geq \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)]. \quad (2.11)$$

From [\(2.10\)](#) and [\(2.11\)](#) we obtain

$$\inf_{Q \in \mathcal{Q}} E_Q[U(g)] \geq \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)] \geq \inf_{Q \in \mathcal{Q}} E_Q[U(g)],$$

yielding the proof. □

To prove equality [\(5★\)](#), we refer to the paper by Christian Reichlin [\[34\]](#). He considered a non-constant, increasing and upper semi-continuous function $U : (0, \infty) \rightarrow \mathbb{R}$ with $U(\infty) > 0$ and that satisfies a mild growth condition:

$$\lim_{x \rightarrow \infty} \frac{U(x)}{x} = 0.$$

Reichlin studied

$$u(x, U) = \sup\{\mathbb{E}[U(f)] \mid f \in L_+^0(\Omega, \mathcal{F}, \mathbb{P}) \text{ with } E_Q(f) \leq x\},$$

for a (pricing) measure $Q \approx P$.

For convenience, we introduce one more notation.

$$C'(x) := \{f \in L_+^0(\Omega, \mathcal{F}, \mathbb{P}) \text{ with } E_Q(f) \leq x\}. \quad (2.12)$$

Note that $u(x, U) = \sup_{f \in C'(x)} \mathbb{E}[U(f)]$.

One of the main results of the Reichlin's paper was the [34, Theorem 5.1] which is

Theorem 2.17 ([34, Theorem 5.1]). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be atomless. Then it holds that*

$$u(x, U) = u(x, U_c) \text{ for all } x > 0.$$

The non-concave problem $u(x, U)$ admits a maximizer if and only if the concavified problem $u(x, U_c)$ admits a maximizer. Every maximizer for the non-concave problem $u(x, U)$ also maximizes the concavified problem $u(x, U_c)$.

In view of Remark 2.6, we can use the Q^e as a pricing measure in [34] to derive

$$f \in C(x) \iff f \in C'(x).$$

Then it follows from the Theorem 2.17 that

$$\sup_{g \in C(x)} \mathbb{E}[U(g)] = \sup_{g \in C(x)} \mathbb{E}[U_c(g)], \text{ for all } x > 0.$$

Lemma 2.18. *Suppose Assumption 2.1 holds and assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless.*

Then it holds that

$$\inf_{Q \in \mathcal{Q}^e} \sup_{g \in C(x)} E_Q[U(g)] = \inf_{Q \in \mathcal{Q}^e} \sup_{g \in C(x)} E_Q[U_c(g)].$$

Proof. First, note that in Theorem 2.17 only the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the pricing measure $Q^p \approx \mathbb{P}$ were used. Moreover, the condition $E_{Q^p}(f) \leq x$ does not depend on \mathbb{P} . If $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, then (Ω, \mathcal{F}, Q) is atomless for all $Q \approx \mathbb{P}$. Moreover, the set $L_+^0(\Omega, \mathcal{F}, Q)$ is the same for all $Q \approx \mathbb{P}$. Hence, everything remains true if we substitute \mathbb{P} with another measure $\tilde{Q} \approx \mathbb{P}$. Furthermore, equivalent local martingale measure Q^e remains the same and unique. Thus, we use Q^e as a pricing measure in Reichlin's paper.

Hence,

$$\sup_{g \in C(x)} E_Q[U(g)] = \sup_{g \in C(x)} E_Q[U_c(g)], \text{ for all } Q \in \mathcal{Q}_e. \quad (2.13)$$

Consequently,

$$\inf_{Q \in \mathcal{Q}_e} \sup_{g \in C(x)} E_Q[U(g)] = \inf_{Q \in \mathcal{Q}_e} \sup_{g \in C(x)} E_Q[U_c(g)].$$

□

Proof of the Theorem 2.13.

- (1★) - (3★) follows from the Lemma 2.14;
- (4★) follows from the fact that $U_c \geq U$;
- (5★) follows from the Lemma 2.18;
- To obtain (6★) we need to take the $\sup_{g \in C(x)}$ of the both sides in the equality (2.9);
- The inequality (7★) follows from the fact that for all $\hat{Q} \in \mathcal{Q}$ and all $h \in C(x)$ holds

$$\inf_{Q \in \mathcal{Q}} E_Q[U(h)] \leq \sup_{g \in C(x)} E_{\hat{Q}}[U(g)];$$

- Since $\mathcal{Q}_e \subseteq \mathcal{Q}$, the inequality (8★) is clear.

□

2.3.3 A deeper dive into the minimax identity

In this subsection, we consider a utility function $U: (0, \infty) \rightarrow \mathbb{R}$, which is strictly increasing, continuously differentiable and satisfies the mild growth condition:

$$\lim_{x \rightarrow \infty} \frac{U(x)}{x} = 0.$$

As was stated in the Section 2.1, $U(x)$ has a non-decreasing and continuous on $(0, +\infty)$ concave envelope $U_c(\cdot)$.

Under assumptions from [Section 2.1](#) the concavified objective function has useful properties, as demonstrated in the following lemma from [[34](#), Lemma 2.8]:

Lemma 2.19. *The concave envelope U_c of U is finite, continuous on $(0, \infty)$, and satisfies a mild growth condition. The set $\{U < U_c\} := \{x \in \mathbb{R}_+ \mid U(x) < U_c(x)\}$ is open, and U is affine on each of the open intervals contained in it.*

Proof. See [[34](#), Lemma 2.8]. □

By applying [Lemma 2.19](#), we can express the set where $U(x)$ is strictly less than its concavification $U_c(x)$ as the union of bounded (open) intervals: $x \in \mathbb{R}_+ \mid U(x) < U_c(x) = \bigcup_i (a_i, b_i)$, where $0 \leq a_1 < b_1 < \infty$ and $0 < a_i < b_i < \infty$ for $i > 1$. Additionally, we will assume that U is strictly concave on $\{x \geq 0 \mid U(x) = U_c(x)\}$. Let us denote the intervals where $U(x)$ is strictly less than its concavification by $(a_i, b_i), i \geq 1$. Hence, $\{x \in \mathbb{R}_+ : U(x) < U_c(x)\} = \bigcup_i (a_i, b_i)$, where for all $i \geq 1, 0 \leq a_i < b_i < \infty$.

Moreover, under our assumptions on utility function $U(x)$, it's concave envelope has the following properties.

Lemma 2.20. *The function $U_c(\cdot)$ is strictly increasing and continuously differentiable on $(0, \infty)$.*

Proof. See [Appendix A.7](#). □

Lemma 2.21. *Suppose [Assumption 2.2](#) holds and that for all $Q \in \mathcal{Q}_e : u_Q^c(x) < \infty$ for some $x > 0$.*

Additionally, assume that $\lim_{x \rightarrow \infty} \frac{u_Q^c(x)}{x} = 0$, for each $Q \in \mathcal{Q}_e$. Then, for any $x > 0$, there exist some $\widehat{g} \in C(x)$ and $\widehat{Q} \in \mathcal{Q}$ such that

$$u^c(x) = \inf_{Q \in \mathcal{Q}} E_Q[U_c(\widehat{g})] = E_{\widehat{Q}}[U_c(\widehat{g})] = u_{\widehat{Q}}^c(x).$$

Remark 2.22. *Since the authors of [[37](#)] consider the general framework of the incomplete market, this lemma holds for the framework that we consider in [Chapter 3](#) without any changes in proof.*

Proof. The proof follows from the first part of the proof of [37, Theorem 2.6 and Lemma 4.1 (a)]. Note that we have $U : [0, \infty) \rightarrow \mathbb{R}$. Hence, we do not have to consider $U(g + \varepsilon)$ which simplifies the proof of [37, Lemma 4.1 (a)] in our case.

The full proof of the part $u^c(x) = u_{\widehat{Q}}^c(x)$ can be done similarly to the proof of [Theorem 2.23](#) part (ii) \Rightarrow (i), noting that minimax identity for concavified objective function U_c holds. \square

The main argument in the proof of minimax identity for the robust utility maximization problem is the lop sided minimax theorem by Aubin and Ekeland, see [1, Chapter 6, p. 295], which holds if the functional $g \rightarrow \mathbb{E}[ZU(g)]$ is concave. Since we consider the non-concave utility function U , we can not prove the minimax identity in this case similarly. A more general case of the minimax identity was proved by Maurice Sion, see [40]. However, to use Sion's minimax theorem, we still need functional $g \rightarrow \mathbb{E}[ZU(g)]$ to be quasi-concave, which is not true in the general case, even in the simplest case of an indicator functions multiplied by the constants.

While we do not prove the minimax identity for a general non-concave functions, we show that it holds under quite mild assumptions.

Theorem 2.23. *Suppose that all assumptions from [Theorem 2.13](#) hold.*

Then, the next two equalities are equivalent

$$(i) \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)] = \sup_{g \in C(x)} E_{\widehat{Q}}[U(g)], \text{ for } \widehat{Q} \in \mathcal{Q}_e;$$

$$(ii) \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)] = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)].$$

Proof. Suppose (i) holds, then from (2.13) it follows that

$$\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)] = \sup_{g \in C(x)} E_{\widehat{Q}}[U(g)] = \sup_{g \in C(x)} E_{\widehat{Q}}[U_c(g)] \geq \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)].$$

Noting that for all $x \geq 0 : U_c(x) \geq U(x)$, we have

$$\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)] \leq \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)].$$

Which gives us (ii).

Assume (ii) holds, then from the [Theorem 2.13](#) we obtain that

$$\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)] = \inf_{Q \in \mathcal{Q}_e} \sup_{g \in C(x)} E_Q[U(g)] = \inf_{Z \in \mathcal{Z}_e} \sup_{g \in C(x)} \mathbb{E}[ZU(g)].$$

Take a sequence $(Z_n)_{n \in \mathbb{N}} \in \mathcal{Z}_e$ such that

$$\lim_{n \rightarrow \infty} \sup_{g \in C(x)} \mathbb{E}[Z_n U(g)] = \inf_{Z \in \mathcal{Z}_e} \sup_{g \in C(x)} \mathbb{E}[Z U(g)].$$

Using the Kolmos-type argument [13, Lemma A1.1] there exists a sequence $(\widehat{Z}_n)_{n \in \mathbb{N}} \in \text{conv}(Z_n, Z_{n+1}, \dots)$ such that \widehat{Z}_n converges a.s. to some \widehat{Z} .

Moreover, $(\widehat{Z}_n)_{n \in \mathbb{N}} \subset \mathcal{Z}_e$, because $(Z_n)_{n \in \mathbb{N}} \subset \mathcal{Z}_e \subset \mathcal{Z}$. Proposition 2.3 implies that $(\widehat{Z}_n)_{n \in \mathbb{N}} \subset \mathcal{Z}$ and also for all $n \in \mathbb{N}$, $\widehat{Z}_n \approx \mathbb{P}$. It follows from Assumption 2.2 (iv) that $\widehat{Z} \in \mathcal{Z}_e$.

For each $n \in \mathbb{N}$, there exists a non-negative sequence $(\alpha_i^n)_{i \in \mathbb{N}}$ such that $\sum_{i \in \mathbb{N}} \alpha_i^n = 1$ and $\widehat{Z}_n = \sum_{i \in \mathbb{N}} \alpha_i^n Z_{n+i-1}$.

Hence,

$$\sup_{g \in C(x)} \mathbb{E}[\widehat{Z}_n U(g)] = \sup_{g \in C(x)} \mathbb{E} \left[U(g) \sum_{i \in \mathbb{N}} \alpha_i^n Z_{n+i-1} \right] \leq \sum_{i \in \mathbb{N}} \alpha_i^n \sup_{g \in C(x)} \mathbb{E}[Z_{n+i-1} U(g)].$$

As a result,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \sup_{g \in C(x)} \mathbb{E}[\widehat{Z}_n U(g)] &\leq \limsup_{n \rightarrow \infty} \sum_{i \in \mathbb{N}} \alpha_i^n \sup_{g \in C(x)} \mathbb{E}[Z_{n+i-1} U(g)] \\ &= \inf_{Z \in \mathcal{Z}_e} \sup_{g \in C(x)} \mathbb{E}[Z U(g)], \end{aligned}$$

where the last equality holds due to the Proposition A.2, for $h_n = \sup_{g \in C(x)} \mathbb{E}[Z_n U(g)]$ and Assumption 2.11.

Noting that

$$\liminf_{n \rightarrow \infty} \sup_{g \in C(x)} \mathbb{E}[\widehat{Z}_n U(g)] \geq \inf_{Z \in \mathcal{Z}_e} \sup_{g \in C(x)} \mathbb{E}[Z U(g)],$$

we have

$$\inf_{Z \in \mathcal{Z}_e} \sup_{g \in C(x)} \mathbb{E}[Z U(g)] = \lim_{n \rightarrow \infty} \sup_{g \in C(x)} \mathbb{E}[\widehat{Z}_n U(g)]. \quad (2.14)$$

Since $U(g)$ is bounded from below, it follows that exists $c > 0$ such that $U(g) + c > 0$. Fatou's lemma yields that

$$\mathbb{E} \left[\liminf_{n \rightarrow \infty} \widehat{Z}_n \cdot (U(g) + c) \right] \leq \liminf_{n \rightarrow \infty} \mathbb{E} \left[\widehat{Z}_n \cdot (U(g) + c) \right].$$

Moreover,

$$\begin{aligned}\mathbb{E} \left[\liminf_{n \rightarrow \infty} \widehat{Z}_n \cdot c \right] &= c, \\ \liminf_{n \rightarrow \infty} \mathbb{E} \left[\widehat{Z}_n \cdot c \right] &= c.\end{aligned}$$

Hence, for all $g \in C(x)$ one has

$$\mathbb{E} \left[\liminf_{n \rightarrow \infty} \widehat{Z}_n U(g) \right] \leq \liminf_{n \rightarrow \infty} \mathbb{E} \left[\widehat{Z}_n U(g) \right].$$

This immediately implies

$$\sup_{g \in C(x)} \mathbb{E} \left[\liminf_{n \rightarrow \infty} \widehat{Z}_n U(g) \right] \leq \sup_{g \in C(x)} \liminf_{n \rightarrow \infty} \mathbb{E} \left[\widehat{Z}_n U(g) \right]. \quad (2.15)$$

Noting that for all $Z_n \in \mathcal{Z}_e$ and all $g \in C(x)$ holds $\mathbb{E} \left[\widehat{Z}_n U(g) \right] \leq \sup_{g \in C(x)} \mathbb{E} \left[\widehat{Z}_n U(g) \right]$, we have

$$\liminf_{n \rightarrow \infty} \mathbb{E} \left[\widehat{Z}_n U(g) \right] \leq \liminf_{n \rightarrow \infty} \sup_{g \in C(x)} \mathbb{E} \left[\widehat{Z}_n U(g) \right].$$

Since g is arbitrary, it follows that

$$\sup_{g \in C(x)} \liminf_{n \rightarrow \infty} \mathbb{E} \left[\widehat{Z}_n U(g) \right] \leq \liminf_{n \rightarrow \infty} \sup_{g \in C(x)} \mathbb{E} \left[\widehat{Z}_n U(g) \right]. \quad (2.16)$$

Combining (2.14), (2.15) and (2.16), we arrive at

$$\begin{aligned}\sup_{g \in C(x)} \mathbb{E}[\widehat{Z}U(g)] &= \sup_{g \in C(x)} \mathbb{E} \left[\liminf_{n \rightarrow \infty} \widehat{Z}_n U(g) \right] \stackrel{(2.15)}{\leq} \sup_{g \in C(x)} \liminf_{n \rightarrow \infty} \mathbb{E} \left[\widehat{Z}_n U(g) \right] \\ &\stackrel{(2.16)}{\leq} \liminf_{n \rightarrow \infty} \sup_{g \in C(x)} \mathbb{E} \left[\widehat{Z}_n U(g) \right] \stackrel{(2.14)}{=} \inf_{Z \in \mathcal{Z}_e} \sup_{g \in C(x)} \mathbb{E} [ZU(g)].\end{aligned}$$

The proof is concluded by observing that

$$\inf_{Z \in \mathcal{Z}_e} \sup_{g \in C(x)} \mathbb{E} [ZU(g)] \leq \sup_{g \in C(x)} \mathbb{E}[\widehat{Z}U(g)] = \sup_{g \in C(x)} E_{\widehat{Q}}[U(g)].$$

□

Lemma 2.24. *Suppose that Assumptions 2.2, 2.11 and 2.12 hold.*

Then, for all $g \in C(x)$,

- (i) $\inf_{Q \in \mathcal{Q}_e} E_Q[U(g)] = E_{\widehat{Q}}[U(g)]$, where $\widehat{Q} \in \mathcal{Q}_e$;

(ii) $\inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)] = E_{\tilde{Q}}[U_c(g)]$, where $\tilde{Q} \in \mathcal{Q}_e$.

Proof. See [Appendix A.4](#). □

[Lemma 2.24](#) shows that for all $g \in C(x)$ infimum attained in some measure $Q_g \in \mathcal{Q}_e$. However, this measure Q_g depends on the choice of the random variable $g \in C(x)$.

The next theorem shows that if Q_g was the same for all $g \in C(x)$, then the minimax identity was true. Moreover, it shows that this assumption can be simplified.

Assumption 2.25. Assume that the set \mathcal{Q} in [Lemma 2.21](#) can be replaced by \mathcal{Q}_e , i.e. for any $x > 0$, there exists some $\hat{g} \in C(x)$ and $\hat{Q}_0 \in \mathcal{Q}_e$ such that

$$u^c(x) = \inf_{Q \in \mathcal{Q}} E_Q[U_c(\hat{g})] = E_{\hat{Q}_0}[U_c(\hat{g})] = u_{\hat{Q}_0}^c(x).$$

Further on, $\hat{Q}_0 \in \mathcal{Q}_e$ denotes a measure such that

$$u^c(x) = u_{\hat{Q}_0}^c(x).$$

Theorem 2.26. Suppose that all assumptions from [Lemma 2.21](#) and [Theorem 2.13](#) are fulfilled and that [Assumption 2.25](#) holds. Additionally, let at least one of the conditions below hold:

- (i) There exists a measure $\hat{Q} \in \mathcal{Q}_e$ such that for all $g \in C(x)$: $\inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)] = E_{\hat{Q}}[U_c(g)]$;
- (ii) For any sequence $g_n \in C(x)$ such that $\lim_{n \rightarrow \infty} E_{\hat{Q}_0}[U_c(g_n)] = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)]$, it holds that $\lim_{n \rightarrow \infty} E_{\hat{Q}_0}[U_c(g_n)] = \lim_{n \rightarrow \infty} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g_n)]$.

Then, we have

$$\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)] = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)],$$

and all inequalities in the [Theorem 2.13](#) are equalities.

Proof. Let us prove the first part, i.e:

$$(i) \implies \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)] = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)].$$

The proof of [Lemma 2.18](#) together with [[34](#), Proposition 5.3] imply that for all $g \in C(x)$ there exists $g^\star \in C(x)$ such that $\{g^\star \in \{U < U_c\}\} = \emptyset$ and

$$E_{\widehat{Q}}[U_c(g)] = E_{\widehat{Q}}[U_c(g^\star)] = E_{\widehat{Q}}[U(g^\star)].$$

Since $\{g^\star \in \{U < U_c\}\} = \emptyset$, we have

$$\inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g^\star)] = \inf_{Q \in \mathcal{Q}_e} E_Q[U(g^\star)].$$

As a result, (i) yields

$$\begin{aligned} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)] &= E_{\widehat{Q}}[U_c(g)] = E_{\widehat{Q}}[U_c(g^\star)] \stackrel{(i)}{=} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g^\star)] \\ &= \inf_{Q \in \mathcal{Q}_e} E_Q[U(g^\star)] \leq \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)]. \end{aligned}$$

Since $g \in C(x)$ was arbitrary, one has

$$\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)] \leq \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)].$$

Noting that $U_c \geq U$, this implies

$$\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)] = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)].$$

Now, we prove second part, i.e.

$$(ii) \implies \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)] = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)].$$

From the definition of \widehat{Q}_0 , see [Assumption 2.25](#), it follows that

$$u^c(x) = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U_c(g)] = \sup_{g \in C(x)} E_{\widehat{Q}_0}[U_c(g)] = u_{\widehat{Q}_0}^c(x). \quad (2.17)$$

Owing to [Lemma 2.14](#), it holds that

$$\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U_c(g)] = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)].$$

Now take a sequence $g_n \in C(x)$ such that

$$\liminf_{n \rightarrow \infty} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g_n)] = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)]. \quad (2.18)$$

Noting that $\inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g_n)] \leq E_{\widehat{Q}_0}[U_c(g_n)] \leq \sup_{g \in C(x)} E_{\widehat{Q}_0}[U_c(g)]$, we have

$$\lim_{n \rightarrow \infty} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g_n)] \leq \lim_{n \rightarrow \infty} E_{\widehat{Q}_0}[U_c(g_n)] \leq \sup_{g \in C(x)} E_{\widehat{Q}_0}[U_c(g)].$$

From (2.17) and (2.18) we obtain

$$\lim_{n \rightarrow \infty} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g_n)] = \lim_{n \rightarrow \infty} E_{\widehat{Q}_0}[U_c(g_n)] = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)]. \quad (2.19)$$

As in the first part of the proof, for all $g_n \in C(x)$ exists $g_n^* \in C(x)$ satisfying $\{g_n^* \in \{U < U_c\}\} = \emptyset$ and

$$E_{\widehat{Q}_0}[U_c(g_n)] = E_{\widehat{Q}_0}[U_c(g_n^*)] = E_{\widehat{Q}_0}[U(g_n^*)]. \quad (2.20)$$

By taking $\lim_{n \rightarrow \infty}$ to the first equality of (2.20) from (ii) and definition of the sequence g_n^* one has

$$\begin{aligned} \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)] &= \lim_{n \rightarrow \infty} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g_n^*)] = \lim_{n \rightarrow \infty} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g_n^*)] \\ &\leq \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)]. \end{aligned}$$

The proof is concluded, since $U_c \geq U$. □

Consider the conjugate of U :

$$V(y) := \sup_{x > 0} \{U(x) - xy\}.$$

In what follows ∂V denotes the subdifferential of V .

Theorem 2.27. *Suppose that all assumptions from Theorem 2.13 hold. Additionally, assume that $\frac{dQ^e}{dP}$ has a continuous distribution and that for some $\lambda (\geq 0)$ the maximizer for $u^c(x)$ satisfies $g^* \in -\partial V(\lambda \cdot \frac{dQ^e}{dP})$.*

Then g^ is also a maximizer for $u(x)$. Moreover, we have*

$$\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)] = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)].$$

Hence, all inequalities in the Theorem 2.13 are equalities.

Remark. *This theorem also holds verbatim for a utility function defined on $(0, \infty)$.*

Proof. Since g^\star is a maximizer for $u^c(x)$, we have

$$\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)] = \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g^\star)].$$

It follows from the proof of [34, Lemma 5.7] that $\mathbb{P}(g^\star \in \{U < U_c\}) = 0$. Hence,

$$\inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g^\star)] = \inf_{Q \in \mathcal{Q}_e} E_Q[U(g^\star)].$$

Thus, we obtain

$$\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)] = \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g^\star)] = \inf_{Q \in \mathcal{Q}_e} E_Q[U(g^\star)] \leq \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)].$$

Since $U_c \geq U$, we have

$$\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)] = \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g^\star)] = \inf_{Q \in \mathcal{Q}_e} E_Q[U(g^\star)] = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)].$$

Now, from Lemma 2.15 it follows that g^\star is a maximizer for $u(x)$. \square

Example 2.28. Consider a two-period binomial model, see Example 2.4. Each probability measure is defined by $Q(\{\omega\}) = q^{D_2(\omega)} \cdot (1-q)^{2-D_2(\omega)}$, for some $q \in (0, 1)$, where $D_2(\omega)$ the number of good periods.

We wish to consider a subset \mathcal{Q} of all measures, so we fix $0 < \underline{q} < \bar{q} < 1$ and consider the set which determined by q such that $\underline{q} \leq q \leq \bar{q}$. Note that \mathcal{Q} coincides with the set \mathcal{Q}_e of all equivalent measures in the set \mathcal{Q} .

The set \mathcal{Q} is not convex, but it is closed with respect to the almost sure convergence.

Denote $\omega_1 = (0, 0)$, $\omega_2 = (1, 0)$, $\omega_3 = (0, 1)$, $\omega_4 = (1, 1)$, where 0 corresponds to a bad state of economy and 1 to a good one.

Thus, we have

$$\begin{aligned} X_1(\omega_1) &= (x_0 - x_0^0)(1+r) + (1+y_b)x_0^0, \\ X_1(\omega_2) &= (x_0 - x_0^0)(1+r) + (1+y_g)x_0^0, \\ X_1(\omega_3) &= (x_0 - x_0^0)(1+r) + (1+y_b)x_0^0, \\ X_1(\omega_4) &= (x_0 - x_0^0)(1+r) + (1+y_g)x_0^0, \end{aligned}$$

where $X_0 = x_0$ is the initial capital, x_0^0 is the capital invested in stock at time $t = 0$, X_1 is the payoff we obtain at time $t = 1$.

At maturity $T = 2$, one has

$$\begin{aligned} X_2(\omega_1) &= [(x_0 - x_0^0)(1+r) + (1+y_b)x_0^0 - x_1^0](1+r) + (1+y_b)x_1^0, \\ X_2(\omega_2) &= [(x_0 - x_0^0)(1+r) + (1+y_g)x_0^0 - x_1^1](1+r) + (1+y_b)x_1^1, \\ X_2(\omega_3) &= [(x_0 - x_0^0)(1+r) + (1+y_b)x_0^0 - x_1^0](1+r) + (1+y_g)x_1^0, \\ X_2(\omega_4) &= [(x_0 - x_0^0)(1+r) + (1+y_g)x_0^0 - x_1^1](1+r) + (1+y_g)x_1^1, \end{aligned}$$

where x_1^0 and x_1^1 denote respectively the capital invested in stock at time $t = 1$, if at time $t = 1$ economy was in a bad, respectively good, state; X_2 is the payoff that obtain at maturity.

We are interested in verifying the equality below:

$$\sup_{x_0^0, x_1^0, x_1^1} \inf_{Q \in \mathcal{Q}} E_Q[U(X_2)] \stackrel{?}{=} \inf_{Q \in \mathcal{Q}} \sup_{x_0^0, x_1^0, x_1^1} E_Q[U(X_2)], \quad (2.21)$$

for all admissible X , i.e. $X_t \geq 0$, for all $t \in \{0, 1, 2\}$. A python script were written to this end (see [Appendix A.6](#)), and we have considered a number of different non-concave utility functions, most being of the form $U[(x - K)_+]$, where function U is concave, e.g. $(x - K)_+^j$, $-\exp[-(x - K)_+]$, $\log[(x - K)_+]$. The absolute value of the difference between the right-hand and left-hand sides of (2.21) was between 10^{-5} and 10^{-11} , which illustrates the fact that the minimax identity holds for non-concave utility functions.

2.3.4 Constrained case

This section examines the minimax identity for a robust non-concave utility functional in a complete market model. The utility function is defined as follows:

$$u(x) := \sup_{X \in \mathfrak{X}(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(X)] = \inf_{Q \in \mathcal{Q}} \sup_{X \in \mathfrak{X}(x)} E_Q[U(X)].$$

Here, $\mathfrak{X}(x)$ is the set of admissible final endowments, and we consider an additional upper bound for this set, given by:

$$\mathfrak{X}^W(x) = \{X \in L^1(Q^e) \mid 0 \leq X \leq W, E_{Q^e}[X] \leq x\}, \quad (2.22)$$

where $W : \Omega \rightarrow [0, +\infty)$ is a random variable. Throughout this section, we will use the Borel sense to understand the measurability of real-valued functions.

As noted in [16, Chapter 3], this formulation has various applications. For example, we can consider an agent who aims to reach a certain random final wealth W , which may represent the future price of a particular asset or the claim the agent has to pay. In this scenario, any utility for an amount above W is zero, while the agent still applies their utility function for endowments below W . Because we assume completeness, we can safely replace endowments X that exceed W with positive probability with $X \wedge W$.

We maintain all the assumptions from Section 2.3 on the set of all probability measures Q and the utility function U . For technical reasons, we also assume that (Ω, \mathcal{F}) is a standard Borel space, implying the existence of a regular conditional distribution given W . We require that this conditional distribution is atomless, which means that the constraint W leaves enough randomness.

Formally, we make the following assumptions:

- Assumption 2.29.**
1. (Ω, \mathcal{F}) is a standard Borel space.
 2. There exists a regular conditional distribution given W , which is atomless. That is, there exists a function $P : \mathcal{F} \times (0, \infty) \rightarrow [0, \infty)$ such that for all $v > 0$, $P(\cdot, v)$ is an atomless probability measure, and for all $A \in \mathcal{F}$, $P(A, \cdot)$ is a measurable function satisfying $P(A, W) = \mathbb{P}(A \mid W)$ almost surely.

Remark 2.30. A simple sufficient condition for (Ω, \mathcal{F}) to be a standard Borel space is that \mathcal{F} is generated by a countable collection of real-valued random variables, which is satisfied in the majority of applications. Since we assume market completeness, this assumption is automatically fulfilled unless one considers completion of \mathcal{F} .

The second assumption is also relatively mild. For example, it is satisfied if there exists a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ that is independent of W and has a continuous distribution. We also emphasize that this assumption does not contradict market completeness. A simple example is a market with two independent continuously distributed assets S^1 and S^2 , where $W = f(S^1)$.

As in [7] for each $k > 0$ denote

$$U^k(x) = U(x \wedge k), \quad x \geq 0. \quad (2.23)$$

Note that the function U^k and its concavification U_c^k satisfy all of the assumptions on the utility function U and its concavification U_c . Moreover, $U_c^k(x) = U^k(x)$, for all $x \geq k$.

Our goal is to prove some equalities and inequalities, related to the minimax identity for the robust non-concave utility functionals:

$$\sup_{X \in \mathfrak{X}_x^W} \inf_{Q \in \mathcal{Q}} E_Q[U^{W(\omega)}(X)] = \inf_{Q \in \mathcal{Q}} \sup_{X \in \mathfrak{X}_x^W} E_Q[U^{W(\omega)}(X)],$$

where the budget set \mathfrak{X}_x^W is defined by

$$\mathfrak{X}_x^W := \{X \in L^1(Q^e) \mid 0 \leq X \leq W, E_{Q^e}[X] \leq x\},$$

where $x > 0$ is the initial wealth and Q^e is the unique equivalent local martingale measure.

Introduce the following notation:

$$\begin{aligned} u_c^W(x) &:= \sup_{X \in \mathfrak{X}_x^W} \inf_{Q \in \mathcal{Q}} E_Q[U_c^{W(\omega)}(X)]; \\ u_Q^W(x) &:= \sup_{X \in \mathfrak{X}_x^W} E_Q[U^{W(\omega)}(X)]; \\ u_{c,Q}^W(x) &:= \sup_{X \in \mathfrak{X}_x^W} E_Q[U_c^{W(\omega)}(X)]. \end{aligned}$$

Remark. Since $U^k(x) \leq U(x)$, $x \geq 0$ [Assumption 2.11](#) and [2.12](#) provide the finiteness of the value function above.

It is natural to consider only the case where

$$E_{Q^e}[W] > x, \tag{2.24}$$

as otherwise, thanks to the monotonicity of U , the optimization problem has a trivial solution $X^* = W$.

The formulation of the next theorems and lemmas are the same as in [Section 2.3](#). However, because of the boundness assumption on the endowments the proof of the corresponding statements will be different.

Theorem 2.31. *Under Assumptions [2.2](#), [2.11](#), [2.12](#), [2.29](#), we have the following:*

Lemma 2.33. *Suppose that Assumption 2.2 and Assumption 2.12 hold.*

Then, we have

$$\begin{aligned} u_c^W(x) &= \sup_{X \in \mathfrak{X}_x^W} \inf_{Q \in \mathcal{Q}} E_Q[U_c^{W(\omega)}(X)] = \inf_{Q \in \mathcal{Q}} \sup_{X \in \mathfrak{X}_x^W} E_Q[U_c^{W(\omega)}(X)] \\ &= \sup_{X \in \mathfrak{X}_x^W} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c^{W(\omega)}(X)] = \inf_{Q \in \mathcal{Q}_e} \sup_{X \in \mathfrak{X}_x^W} E_Q[U_c^{W(\omega)}(X)]. \end{aligned}$$

Proof. Take $\varepsilon > 0$. Consider $Y = (X + \varepsilon) \wedge W$, where $X \in \mathfrak{X}_x^W$. Then $Y \in \mathfrak{X}_{x+\varepsilon}^W$, since $0 \leq Y \leq W$ and $E_{Q^e}[Y] = E_{Q^e}[(X + \varepsilon) \wedge W] \leq E_{Q^e}[X + \varepsilon] \leq x + \varepsilon$.

Define $Y_{X,\varepsilon}^W := \{Y \in L^1(Q^e) \mid Y = (X + \varepsilon) \wedge W, X \in \mathfrak{X}_x^W\}$. Then $Y_{X,\varepsilon}^W \subset \mathfrak{X}_{x+\varepsilon}^W$. Thus, it holds that

$$\begin{aligned} u_c^W(x + \varepsilon) &= \sup_{\bar{X} \in \mathfrak{X}_{x+\varepsilon}^W} \inf_{Q \in \mathcal{Q}} E_Q[U_c^{W(\omega)}(\bar{X})] \\ &\geq \sup_{Y \in Y_{X,\varepsilon}^W} \inf_{Q \in \mathcal{Q}} E_Q[U_c^{W(\omega)}(Y)] = \sup_{Y \in Y_{X,\varepsilon}^W} \inf_{Q \in \mathcal{Q}} \mathbb{E} \left[U_c^{W(\omega)}(Y) \cdot \frac{dQ}{dP} \right] \\ &= \sup_{Y \in Y_{X,\varepsilon}^W} \inf_{Z \in \mathcal{Z}} \mathbb{E}[ZU_c^{W(\omega)}(Y)]. \end{aligned}$$

The proof of [22, Lemma 1] shows that for each $X \in \mathfrak{X}(x)$, the map $Z \mapsto \mathbb{E}[ZU_c^{W(\omega)}(Y)]$ is a weakly lower semicontinuous affine functional defined on the weakly compact convex set \mathcal{Z} . Moreover, it shows that for each $Z \in \mathcal{Z}$, $X \mapsto \mathbb{E}[ZU_c^{W(\omega)}(X + \varepsilon)]$ is a concave functional. Therefore, $X \mapsto \mathbb{E}[ZU_c^{W(\omega)}(Y)]$ is a concave functional defined on the convex set \mathfrak{X}_x^W , for each $Z \in \mathcal{Z}$.

Noting that almost sure convergence implies weak convergence, the conditions of the lop-sided minimax theorem [1, Chapter 6, p. 295] are satisfied, so

$$\sup_{Y \in Y_{X,\varepsilon}^W} \min_{Z \in \mathcal{Z}} \mathbb{E}[ZU_c^{W(\omega)}(Y)] = \min_{Z \in \mathcal{Z}} \sup_{Y \in Y_{X,\varepsilon}^W} \mathbb{E}[ZU_c^{W(\omega)}(Y)].$$

Hence, we arrive at

$$\begin{aligned} u_c^W(x + \varepsilon) &\geq \sup_{Y \in Y_{X,\varepsilon}^W} \inf_{Z \in \mathcal{Z}} \mathbb{E}[ZU_c^{W(\omega)}(Y)] = \min_{Z \in \mathcal{Z}} \sup_{Y \in Y_{X,\varepsilon}^W} \mathbb{E}[ZU_c^{W(\omega)}(Y)] \\ &\geq \inf_{Q \in \mathcal{Q}} \sup_{Y \in Y_{X,\varepsilon}^W} E_Q[U_c^{W(\omega)}(Y)] \\ &\stackrel{Y \geq X}{\geq} \inf_{Q \in \mathcal{Q}} \sup_{Y \in Y_{X,\varepsilon}^W} E_Q[U_c^{W(\omega)}(X)] \geq \sup_{X \in \mathfrak{X}_x^W} \inf_{Q \in \mathcal{Q}} E_Q[U_c^{W(\omega)}(X)] = u_c^W(x). \end{aligned}$$

The last inequality follows from the fact that for all $Q \in \mathcal{Q}$ and $X \in \mathfrak{X}_x^W$

$$\sup_{X \in \mathfrak{X}_x^W} E_Q[U_c^{W(\omega)}(X)] \geq \inf_{Q \in \mathcal{Q}} E_Q[U_c^{W(\omega)}(X)].$$

By sending $\varepsilon \downarrow 0$ and using the continuity of the concave function u_c^W on the set $(0, +\infty)$, we obtain the first part of the lemma.

From [Assumption 2.12](#) and [[37](#), Lemma 3.3], we have $u_c^W(x) = \inf_{Q \in \mathcal{Q}_e} u_{c,Q}^W(x)$; the proof is similar to that of [[22](#), Lemma 2].

Hence,

$$\begin{aligned} u_c^W(x) &= \inf_{Q \in \mathcal{Q}_e} u_{c,Q}^W(x) = \inf_{Q \in \mathcal{Q}_e} \sup_{X \in \mathfrak{X}_x^W} E_Q[U_c^{W(\omega)}(X)] \\ &\geq \sup_{X \in \mathfrak{X}_x^W} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c^{W(\omega)}(X)] \geq \sup_{X \in \mathfrak{X}_x^W} \inf_{Q \in \mathcal{Q}} E_Q[U_c^{W(\omega)}(X)] = u_c^W(x). \end{aligned}$$

This concludes the proof. \square

2.3.6 Minimax identity for the objective function $U(x)$

In this section, we present auxiliary results that will aid us in proving [Theorem 2.31](#). We begin with the following lemma.

Lemma 2.34. *Under the assumptions of [Assumption 2.2](#) and [Assumption 2.11](#), for all $X \in \mathfrak{X}_x^W$, we have*

$$\inf_{Q \in \mathcal{Q}} E_Q[U^W(X)] = \inf_{Q \in \mathcal{Q}_e} E_Q[U^W(X)]. \quad (2.25)$$

Proof. The proof of [Theorem 2.34](#) follows the same approach as in the unconstrained case. For details, see [[22](#), Lemma 2]. \square

Lemma 2.35. *Let $\{P_v, v \in (0, \infty)\}$ be a family of atomless probability measures on a standard Borel space (Ω, \mathcal{F}) , such that for any $A \in \mathcal{F}$, $P_v(A)$ is measurable. Then, for all $Q \in \mathcal{Q}_e$, for all $X \in \mathfrak{X}_x^W$ there exists $X^* \in \mathfrak{X}_x^W$ such that*

$$E_Q[U^W(X^*)] = E_Q[U_c^W(X^*)] = E_Q[U_c^W(X)] \geq E_Q[U^W(X)]. \quad (2.26)$$

Proof. The main idea of the proof is to utilize the ideas of [[34](#), Proposition 5.3] in our conditional setting.

For a fixed $Q \in \mathcal{Q}_e$, define the Radon–Nikodym derivatives of Q^e with respect to \mathbb{P} and Q as $\psi = dQ^e/d\mathbb{P}$, $\varphi = dQ^e/dQ$. Since ψ is positive, we have that $M(v) := \int_{\Omega} \psi P(d\omega, v)$ is also positive. Thus, the measure

$$P_Q(A, v) = \frac{\int_A \psi P(d\omega, v)}{M(v)}, A \in \mathcal{F},$$

is a probability measure. It is easy to see that it is measurable in v and $P_Q(A, W) = Q(A | W)$ Q -a.s. We can check that P_Q is measurable in v and that $P_Q(A, W) = Q(A | W)$ Q -almost surely, in other words, that P_Q is a regular conditional probability $P(\cdot | W)$.

Next, we apply [Lemma A.6](#) to the functions $Y(x, \omega) = X(\omega)$, $\phi(v, \omega) = \varphi(\omega)$ and $P_v(A) = P_Q(A, v)$. This yields a jointly measurable function $Y^*(v, \omega)$ such that, for all $v > 0$,

$$\begin{aligned} E_{P_Q(\cdot, v)}[Y^*(v, \omega)\varphi(\omega)] &\leq E_{P_Q(\cdot, v)}[X(\omega)\varphi(\omega)], \text{ and} \\ E_{P_Q(\cdot, v)}[U^v(Y^*(v, \omega))] &= E_{P_Q(\cdot, v)}[U_c^v(Y^*(v, \omega))] = E_{P_Q(\cdot, v)}[U_c^v(X(\omega))]. \end{aligned}$$

Set $X^*(\omega) = Y^*(W(\omega), \omega)$. Then, we have

$$\begin{aligned} E_{Q^e}[X^*] &= E_Q[Y^*(W(\omega), \omega)\varphi] = E_Q[E_Q[Y^*(W(\omega), \omega)\varphi | W]] \\ &= E_Q[E_{P_Q(\cdot, v)}[Y^*(v, \omega)\varphi(\omega)]|_{v=W}] \leq E_Q[E_{P_Q(\cdot, v)}[X(\omega)\varphi(\omega)]|_{v=W}] \\ &= E_Q[E_Q[X(\omega)\varphi | W]] = E_Q[X(\omega)\varphi] = E_{Q^e}[X] \leq x, \end{aligned}$$

so $X^* \in \mathfrak{X}_x^W$. Further,

$$\begin{aligned} E_Q[U_c^W(X^*)] &= E_Q[U_c^{W(\omega)}(Y^*(W(\omega), \omega))] \\ &= E_Q[E_Q[U_c^{W(\omega)}(Y^*(W(\omega), \omega)) | W]] \\ &= E_Q[E_{P_Q(\cdot, v)}[U_c^v(Y^*(v, \omega))]|_{v=W}] = E_Q[E_{P_Q(\cdot, v)}[U_c^v(X(\omega))]|_{v=W}] \\ &= E_Q[E_Q[U_c^v(X(\omega)) | W]] = E_Q[U_c^v(X(\omega))]. \end{aligned}$$

The equality $E_Q[U^W(X^*)] = E_Q[U_c^W(X^*)]$ is proved similarly. Finally, the inequality $E_Q[U_c^W(X)] \geq E_Q[U^W(X)]$ is obvious, since $U_c^W \geq U^W$. \square

Lemma 2.36. *If [Assumption 2.29](#) holds, then for all $Q \in \mathcal{Q}_e$ it holds that*

$$\sup_{X \in \mathfrak{X}_x^W} E_Q[U^W(X)] = \sup_{X \in \mathfrak{X}_x^W} E_Q[U_c^{W(\omega)}(X)], \text{ for all } x > 0.$$

Proof. Apply $\sup_{X \in \mathfrak{X}_x^W}$ to both sides of (2.26). This yields

$$\begin{aligned} \sup_{X \in \mathfrak{X}_x^W} E_Q[U^{W(\omega)}(X^*)] &= \sup_{X \in \mathfrak{X}_x^W} E_Q[U_c^{W(\omega)}(X^*)] \\ &= \sup_{X \in \mathfrak{X}_x^W} E_Q[U_c^{W(\omega)}(X)] \geq \sup_{X \in \mathfrak{X}_x^W} E_Q[U^{W(\omega)}(X)]. \end{aligned} \quad (2.27)$$

Since $Q \in \mathcal{Q}_e$, $X \in \mathfrak{X}_x^W$ are arbitrary and $X^* \in \mathfrak{X}_x^W$, the inequality in (2.27) is an equality, as claimed. \square

Lemma 2.37. *Under the Assumption 2.29,*

$$\inf_{Q \in \mathcal{Q}_e} \sup_{X \in \mathfrak{X}_x^W} E_Q[U^{W(\omega)}(X)] = \inf_{Q \in \mathcal{Q}_e} \sup_{X \in \mathfrak{X}_x^W} E_Q[U_c^{W(\omega)}(X)].$$

Proof. Follows immediately from Lemma 2.36. \square

Proof of Theorem 2.31.

- (1★) - (3★) follows from Lemma 2.33;
- (4★) follows from the fact that $U_c^{W(\omega)} \geq U^{W(\omega)}$;
- (5★) follows from Lemma 2.37;
- To obtain (6★) we need to apply $\sup_{X \in \mathfrak{X}_x^W}$ to both sides of the equality (2.25);
- The inequality (7★) follows from the fact that for all $\hat{Q} \in \mathcal{Q}$ and all $\hat{X} \in \mathfrak{X}_x^W$, we have

$$\inf_{Q \in \mathcal{Q}} E_Q[U^{W(\omega)}(\hat{X})] \leq \sup_{X \in \mathfrak{X}_x^W} E_{\hat{Q}}[U^{W(\omega)}(X)];$$

- Since $\mathcal{Q}_e \subseteq \mathcal{Q}$, the inequality (8★) is clear. \square

2.4 Optimal investments for non-concave utility maximization problems in complete market

The aim of this section is to develop optimal investment strategies for a general framework of non-concave utility functions, by considering two cases of budget

constraints. We assume that the market model is arbitrage-free and complete, which implies the existence of a unique equivalent local martingale measure denoted as Q^e .

For any initial capital $x > 0$, let $\mathfrak{X}(x)$ be the set of all possible random endowments corresponding to x , i.e., all non-negative random variables X such that $E_{Q^e}[X] \leq x$. We are interested in two optimization problems: the standard maximization problem of the utility functional for initial capital $x > 0$,

$$\sup_{X \in \mathfrak{X}(x)} \mathbb{E}[U(X)], \quad (2.28)$$

and the maximization of the robust utility functional,

$$\sup_{X \in \mathfrak{X}(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(X_T)]. \quad (2.29)$$

Merton [27] and [28] proved that the optimal solution to the standard optimization problem can be written using the inverse to the derivative of the utility function and the “state price density”. However, in our case, the utility function and its concavified version will not be strictly concave, and it is not clear how the solution will look like in this case.

In this section, we slightly change the assumptions on the utility function $U(x)$, namely we consider the utility function $U: (0, \infty) \rightarrow \mathbb{R}$, while keeping all other assumptions and the model setup as in Chapter 2. Hence, we consider a utility function U which is non-decreasing, upper-semicontinuous, defined on a domain $(0, \infty)$ and satisfying the mild growth condition:

$$\lim_{x \rightarrow \infty} \frac{U(x)}{x} = 0.$$

By $U_c(x)$ we denote the concave envelope of utility function $U(x)$.

Additionally, we will assume that U is strictly concave on $\{x \geq 0 \mid U(x) = U_c(x)\}$. Recall from Section 2.3.2 that $\{x \in \mathbb{R}_+ : U(x) < U_c(x)\} = \bigcup_i (a_i, b_i)$, where for all $i \geq 1$, $0 \leq a_i < b_i < \infty$.

We will need the following well-known result.

Lemma 2.38. *The concavified objective function U_c has both one-sided derivatives at any point $x > 0$, and the right derivative is less or equal than the left derivative. The derivatives coincide (and hence U_c is differentiable) everywhere except at most countable set.*

Moreover, for all $x, y \in \mathbb{R}$, we have:

$$U_c(x) - U_c(y) \leq U'_c(y^+)(x - y), \quad (2.30)$$

$$U_c(x) - U_c(y) \leq U'_c(y^-)(x - y). \quad (2.31)$$

Based on the assumptions regarding the utility function U , we can conclude that the derivative U'_c of the concavified objective function U_c is decreasing, and since U_c satisfies a mild growth condition, it follows that $\lim_{x \rightarrow \infty} U'_c(x) = 0$. However, U'_c is not strictly decreasing on the intervals (a_i, b_i) , as it is constant on each of them.

2.4.1 Optimal investments for the standard utility maximization problem in complete market

The purpose of this section is to develop optimal investment strategies within the framework of non-concave utility functions in a complete market. We consider the standard maximization problem of the utility functional $\mathbb{E}[U(X)]$.

Recall from [Section 2.3.2](#) that the set $x \in \mathbb{R}_+ : U(x) < U_c(x)$ can be expressed as a countable union of finite, disjoint open intervals, specifically $x \in \mathbb{R}_+ : U(x) < U_c(x) = \bigcup_i (a_i, b_i)$.

Let $c_i, i \geq 1$ denote the set of all non-differentiability points of the concavified objective function U_c , excluding the points a_i and b_i . We use \mathcal{B} to represent the set of all points a_i, b_i, c_i , where $i \geq 1$.

From the [Lemma 2.38](#) it follows

$$U'_c(b_i^+) \leq U'_c(b_i^-) = U'_c(a_i^+) \leq U'_c(a_i^-), \quad (2.32)$$

$$U'_c(c_i^+) \leq U'_c(c_i^-). \quad (2.33)$$

Define the (one-sided) inverse of $U'_c(\cdot)$:

$$i(y) = \begin{cases} (U'_c)^{-1}(y) & \\ \cdot \mathbb{I}\{y \in \mathbb{R}^+ \setminus \{\cup_{i \in \mathbb{N}} [U'_c(b_i^+), U'_c(a_i^-)] \cup [U'_c(c_i^+), U'_c(c_i^-)]\}\} & \\ + \sum_i a_i \cdot \mathbb{I}\{y \in [U'_c(a_i^+), U'_c(a_i^-)]\} & \\ + \sum_i b_i \cdot \mathbb{I}\{y \in [U'_c(b_i^+), U'_c(b_i^-)]\} & \\ + \sum_i c_i \cdot \mathbb{I}\{y \in [U'_c(c_i^+), U'_c(c_i^-)]\}, & y < U'_c(0^+); \\ 0, & y \geq U'_c(0^+). \end{cases}$$

For clarity, we present below a sketch of plots for U'_c and $i(y)$ in [Figure 2.3](#) and [Figure 2.4](#) respectively.

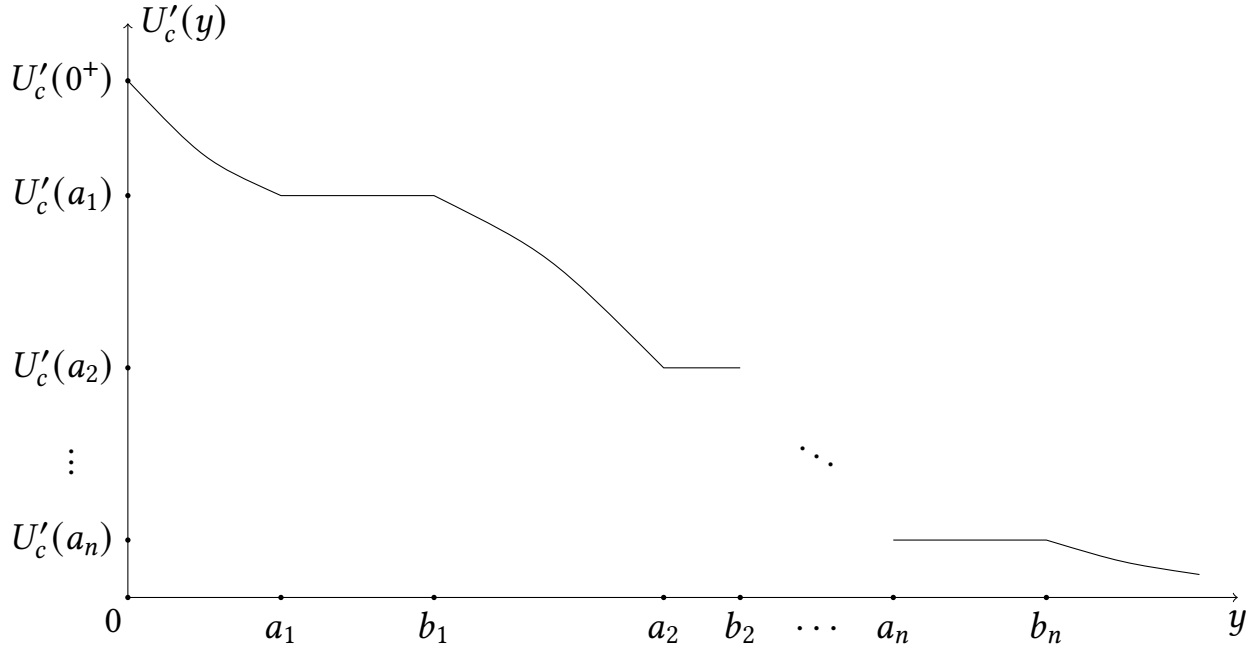


Figure 2.3. The derivative U'_c of a continuously differentiable concavified objective function U_c .

Theorem 2.39. Assume that $\frac{dQ^e}{d\mathbb{P}}$ is continuous and

$$h(\lambda) = \mathbb{E}_{Q^e} \left[i \left(\lambda \cdot \frac{dQ^e}{d\mathbb{P}} \right) \right] < \infty, \text{ for all } \lambda > 0. \quad (2.34)$$

Then, for any initial capital $x > 0$, the unique optimal solution to the problem (2.28) is given by $X^* = i(\lambda^* \cdot \frac{dQ^e}{d\mathbb{P}})$, where $\lambda^* > 0$ is such that $\mathbb{E}_{Q^e} [i(\lambda^* \cdot \frac{dQ^e}{d\mathbb{P}})] = x$.

Proof. The function i is inverse to U'_c in the sense $y \in [U'_c(i(y)^-), U'_c(i(y)^+)]$, which, thanks to (2.30) and (2.31) amounts to

$$\forall z > 0 \quad U_c(z) - U_c(i(y)) \leq y(z - i(y)). \quad (2.35)$$

Denote $\mathcal{I} = \bigcup_{i \geq 1} \{U'_c(a_i^-)\} \cup \{U'_c(a_i^+)\} \cup \{U'_c(b_i^+)\} \cup \{U'_c(c_i^-)\} \cup \{U'_c(c_i^+)\} \cup \{U'_c(0+)\}$. Note that if $y \in \mathbb{R}^+ \setminus \mathcal{I}$, then U_c is differentiable in y and the inequality in (2.35) is strict:

$$\forall z > 0, z \neq i(y) \quad U_c(z) - U_c(i(y)) < y(z - i(y)). \quad (2.36)$$

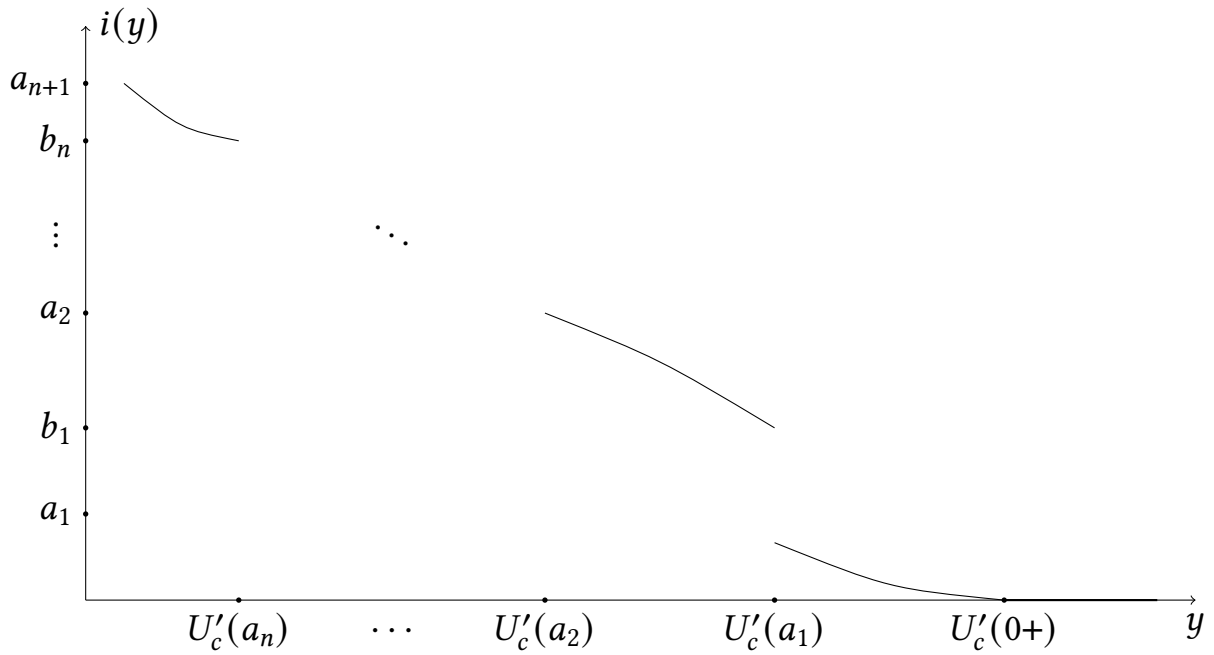


Figure 2.4. The inverse function $i(y)$ for the derivative of the concavified objective function.

The statement above is obvious for

$y \in \bigcup_{i \geq 1} (U'_c(a_i^+), U'_c(a_i^-)) \cup (U'_c(b_i^+), U'_c(b_i^-)) \cup (U'_c(c_i^+), U'_c(c_i^-))$. For other $y \in \mathbb{R}^+ \setminus \mathcal{I}$, the statement follows from assumption that U , and hence U_c , is strictly concave on $\{x \geq 0 \mid U(x) = U_c(x)\}$.

Since $\frac{dQ^e}{d\mathbb{P}} > 0$ a.s., the function

$$h(\lambda) = \mathbb{E}_{Q^e} \left[i \left(\lambda \cdot \frac{dQ^e}{d\mathbb{P}} \right) \right] = \mathbb{E} \left[\frac{dQ^e}{d\mathbb{P}} \cdot i \left(\lambda \cdot \frac{dQ^e}{d\mathbb{P}} \right) \right]$$

is well-defined for all $\lambda > 0$. Since the function i is decreasing, the function h is decreasing as well. Let us show its continuity. The set A of discontinuity points of the function i is at most countable. Let λ_n be a decreasing sequence such that $\lim_{n \rightarrow \infty} \lambda_n = \lambda$. If $\omega \in \Omega$ is such that $\lim_{n \rightarrow \infty} i \left(\lambda_n \cdot \frac{dQ^e}{d\mathbb{P}}(\omega) \right) \neq i \left(\lambda \cdot \frac{dQ^e}{d\mathbb{P}}(\omega) \right)$, then $\lambda \cdot \frac{dQ^e}{d\mathbb{P}}(\omega) \in A$. But $\mathbb{P}(\lambda \cdot \frac{dQ^e}{d\mathbb{P}} \in A) = 0$, since $\frac{dQ^e}{d\mathbb{P}}$ has continuous distribution. Therefore,

$$\mathbb{P} \left[\lim_{n \rightarrow \infty} i \left(\lambda_n \cdot \frac{dQ^e}{d\mathbb{P}} \right) \neq i \left(\lambda \cdot \frac{dQ^e}{d\mathbb{P}} \right) \right] = 0.$$

Since i is non-increasing and right-continuous, it follows from the monotone

convergence theorem and (2.34) that

$$\lim_{n \rightarrow \infty} \mathbb{E}_{Q^e} \left[i \left(\lambda_n \cdot \frac{dQ^e}{d\mathbb{P}} \right) \right] = \mathbb{E}_{Q^e} \left[i \left(\lambda \cdot \frac{dQ^e}{d\mathbb{P}} \right) \right],$$

as required.

Note in passing that

$$\lim_{\lambda \rightarrow \infty} h(\lambda) = 0, \quad \lim_{\lambda \rightarrow 0} h(\lambda) = \infty.$$

Hence, there exists $\lambda^* > 0$ such that $h(\lambda^*) = \mathbb{E}_{Q^e} [i(\lambda^* \cdot \frac{dQ^e}{d\mathbb{P}})] = x$.

Denote $X^* = i(\lambda^* \cdot \frac{dQ^e}{d\mathbb{P}})$. It remains to show that X^* is optimal. Take another feasible random variable $X \in \mathfrak{X}(x)$, i.e. such that $\mathbb{E}_{Q^e}[X] = x$ and $X \geq 0$, which is not almost surely equal to X^* . Note that $i(\cdot)$ and hence X^* never takes on the values where $U \neq U_c$. Moreover, thanks to the continuity of distribution $\frac{dQ^e}{d\mathbb{P}}$ one has

$$\mathbb{P} \left[\lambda^* \cdot \frac{dQ^e}{d\mathbb{P}} \in \mathbb{R}^+ \setminus \mathcal{I} \right] = 1.$$

Therefore,

$$\begin{aligned} \mathbb{P} \left[U_c(X) - U_c(X^*) \stackrel{(2.35)}{\leq} \lambda^* \cdot \frac{dQ^e}{d\mathbb{P}} (X - X^*) \right] &= 1, \\ \mathbb{P} \left[U_c(X) - U_c(X^*) \stackrel{(2.36)}{<} \lambda^* \cdot \frac{dQ^e}{d\mathbb{P}} (X - X^*) \right] &> 0. \end{aligned} \quad (2.37)$$

Consequently,

$$\begin{aligned} \mathbb{E}[U(X) - U(X^*)] &= \mathbb{E}[U(X) - U_c(X^*)] \leq \mathbb{E}[U_c(X) - U_c(X^*)] \\ &\stackrel{(2.37)}{<} \mathbb{E} \left[\lambda^* \cdot \frac{dQ^e}{d\mathbb{P}} (X - X^*) \right] = \lambda^* \cdot \mathbb{E} \left[\frac{dQ^e}{d\mathbb{P}} X - x \right] \leq 0. \end{aligned} \quad (2.38)$$

Hence, for all $X \in \mathfrak{X}(x)$ which do not coincide X^* we have

$$\mathbb{E}[U(X)] < \mathbb{E}[U(X^*)].$$

As a result,

$$\sup_{X \in \mathfrak{X}(x)} \mathbb{E}[U(X)] \leq \mathbb{E}[U(X^*)].$$

Noting that $X^* \geq 0$ and $\mathbb{E}_{Q^e}(X^*) = x$, we obtain

$$\sup_{X \in \mathfrak{X}(x)} \mathbb{E}[U(X)] = \max_{X \in \mathfrak{X}(x)} \mathbb{E}[U(X)] = \mathbb{E}[U(X^*)].$$

□

Remark 2.40. Solution X^* is also a unique optimal solution to the maximization problem for concavified utility function, i.e.

$$\sup_{X \in \mathfrak{X}(x)} \mathbb{E}[U_c(X)] = \max_{X \in \mathfrak{X}(x)} \mathbb{E}[U_c(X)] = \mathbb{E}[U_c(X^*)],$$

which follows from the inequality (2.38) using the same arguments as for U .

Example 2.41. Consider the function

$$U(x) = (x - K)_+^j, \text{ for } K > 0, \text{ and } 0 < j < 1.$$

Its concavification is

$$U_c(x) = \begin{cases} \left(\frac{K}{1-j} - K\right)^j + j \left(\frac{K}{1-j} - K\right)^{j-1} \left(x - \frac{K}{1-j}\right), & x < \frac{K}{1-j}; \\ (x - K)^j, & x \geq \frac{K}{1-j}. \end{cases}$$

We have that $a_1 = 0, b_1 = \frac{K}{1-j}, a_2 = \infty$. The derivative of $U_c(x)$ is

$$U'_c(x) = \begin{cases} j \left(\frac{K}{1-j} - K\right)^{j-1}, & x < \frac{K}{1-j}; \\ j (x - K)^{j-1}, & x \geq \frac{K}{1-j}. \end{cases}$$

The function i is given by

$$i(y) = \begin{cases} (U'_c)^{-1}(y) \cdot \mathbb{I}\{y \in (0, U'_c(\frac{K}{1-j}))\}, & y < U'_c(0^+); \\ 0, & y \geq U'_c(0^+). \end{cases}$$

This can be simplified as

$$i(y) = \begin{cases} (U'_c)^{-1}(y), & y < j \left(\frac{K}{1-j} - K\right)^{j-1}; \\ 0, & y \geq j \left(\frac{K}{1-j} - K\right)^{j-1}. \end{cases}$$

Hence, we obtain that

$$i(y) = \begin{cases} \left(\frac{y}{j}\right)^{\frac{1}{j-1}} + K, & y < j \left(\frac{K}{1-j} - K\right)^{j-1}; \\ 0, & y \geq j \left(\frac{K}{1-j} - K\right)^{j-1}. \end{cases}$$

Consider the standard Black–Scholes model. Then,

$$\frac{dQ^e}{d\mathbb{P}} = \exp \left\{ \frac{r - \mu}{\sigma} B_T - \left(\frac{r - \mu}{\sigma} \right)^2 \frac{T}{2} \right\}.$$

According to Theorem 2.39, we need to find λ^* such that

$$\mathbb{E} \left[\frac{dQ^e}{d\mathbb{P}} i \left(\lambda^* \cdot \frac{dQ^e}{d\mathbb{P}} \right) \right] = x.$$

Equivalently,

$$\int_{\Omega} e^{\left(\frac{r-\mu}{\sigma} B_T - \left(\frac{r-\mu}{\sigma}\right)^2 \frac{T}{2}\right)} \cdot \left(\left(\frac{\lambda^* e^{\left(\frac{r-\mu}{\sigma} B_T - \left(\frac{r-\mu}{\sigma}\right)^2 \frac{T}{2}\right)}}{j} \right)^{\frac{1}{j-1}} + K \right) \\ \times \mathbb{I}_{\left\{ \lambda^* e^{\left(\frac{r-\mu}{\sigma} B_T - \left(\frac{r-\mu}{\sigma}\right)^2 \frac{T}{2}\right)} < j \left(\frac{K}{1-j} - K \right)^{j-1} \right\}} d\mathbb{P} = x.$$

Denote by $X_T := \exp\left(\frac{r-\mu}{\sigma} B_T - \left(\frac{r-\mu}{\sigma}\right)^2 \frac{T}{2}\right)$, thus $X_T \sim LN\left(-\left(\frac{r-\mu}{\sigma}\right)^2 \frac{T}{2}, \left(\frac{r-\mu}{\sigma}\right)^2 T\right)$. Thus,

$$\int_{\Omega} X_T \cdot \left(\left(\frac{\lambda^* X_T}{j} \right)^{\frac{1}{j-1}} + K \right) \cdot \mathbb{I}_{\left\{ \lambda^* X_T < j \left(\frac{K}{1-j} - K \right)^{j-1} \right\}} d\mathbb{P} = x.$$

We can write the last expression as a sum of two integrals:

$$\int_{\Omega} X_T \cdot \left(\left(\frac{\lambda^* X_T}{j} \right)^{\frac{1}{j-1}} + K \right) \cdot \mathbb{I}_{\left\{ \lambda^* X_T < j \left(\frac{K}{1-j} - K \right)^{j-1} \right\}} d\mathbb{P} \\ = K \int_{\Omega} X_T \cdot \mathbb{I}_{\left\{ \lambda^* X_T < j \left(\frac{K}{1-j} - K \right)^{j-1} \right\}} d\mathbb{P} + \left(\frac{\lambda^*}{j} \right)^{\frac{1}{j-1}} \int_{\Omega} X_T^{\frac{j}{j-1}} \cdot \mathbb{I}_{\left\{ \lambda^* X_T < j \left(\frac{K}{1-j} - K \right)^{j-1} \right\}} d\mathbb{P}.$$

Denote $c(\lambda^*) = \frac{j}{\lambda^*} \left(\frac{K}{1-j} - K \right)^{j-1}$. Hence, we need to find λ^* such that

$$K \int_{\Omega} X_T \cdot \mathbb{I}_{\{X_T < c(\lambda^*)\}} d\mathbb{P} + \left(\frac{\lambda^*}{j} \right)^{\frac{1}{j-1}} \int_{\Omega} X_T^{\frac{j}{j-1}} \cdot \mathbb{I}_{\{X_T^{\frac{j}{j-1}} > c(\lambda^*)^{\frac{j}{j-1}}\}} d\mathbb{P} = x.$$

Noting that $X_T^{\frac{j}{j-1}}$ has a lognormal distribution, each of these integrals can be calculated using the Black–Scholes formula, allowing to compute λ^* numerically.

The unique optimal solution to the problem (2.28) is then given by $X^* = i(\lambda^* \cdot \frac{dQ^e}{d\mathbb{P}})$.

2.5 Constrained optimization

This section is similar in nature to Section 2.4, as we are concerned with finding the optimal solution to the standard utility maximization problem. Specifically, we

aim to identify

$$\sup_{X \in \mathfrak{X}(x)} \mathbb{E}[U(X)], \quad (2.39)$$

which is the unconstrained optimization problem. However, we follow the approach of [16, Chapter 3] and consider a modified version of the optimization problem (2.28), subject to a budget constraint. Specifically, we assume that there is an upper bound on the endowment, given by a random variable $W : \Omega \rightarrow [0, +\infty)$, and define the set of admissible payoffs as

$$\mathfrak{X}^W := \{X \in L^0(\mathbb{P}) \mid 0 \leq X \leq W \text{ } \mathbb{P}\text{-a.s.}\}.$$

Our goal is to maximize $\mathbb{E}[U(X)]$ for $X \in \mathfrak{X}_x^W$, where the budget set \mathfrak{X}_x^W is defined by

$$\mathfrak{X}_x^W := \{X \in L^1(Q^e) \mid 0 \leq X \leq W, \mathbb{E}_{Q^e}[X] \leq x\},$$

with $x > 0$ being the initial wealth and Q^e being the unique equivalent local martingale measure. Hence, we aim to find the optimal solution to the following constrained optimization problem:

$$\sup_{X \in L^1(Q^e)} \mathbb{E}[U(X)] \text{ subject to } 0 \leq X \leq W \text{ and } \mathbb{E}_{Q^e}(X) \leq x. \quad (2.40)$$

We keep all of the assumptions from Section 2.4 on the utility function U and its concave envelope U_c . As before, we consider only the case where

$$\mathbb{E}_{Q^e}[W] > x,$$

as otherwise the set of admissible endowments has at most one element, making the optimization problem meaningless.

2.5.1 Optimal investments for the constrained utility maximization problem in complete market

The goal of this section is to identify the optimal investment strategies for the standard constrained utility maximization problem (2.40).

Recall notation and basic properties of

$$U^k(x) = U(x \wedge k), \quad x \geq 0,$$

for each $k > 0$, see (2.23):

- the function U^k and its concavification U_c^k satisfy all of the assumptions on the utility function U and its concavification U_c .
- $U_c^k(x) = U^k(x)$, for all $x \geq k$.

Similarly to the [Section 2.4](#) define the set $\{a_{i,k}, b_{i,k}\}_{i \in \mathbb{N}}$ such that $\{x \in \mathbb{R}_+ : U^k(x) < U_c^k(x)\} = \bigcup_i (a_{i,k}, b_{i,k})$.

Define the set $\{c_{i,k}\}_{i \in \mathbb{N}}$ as set of all points where U_c^k is non-differentiable, with the exception of the points $a_{i,k}$ and $b_{i,k}$. Consider the set $\mathcal{B}^k = \bigcup_{i \in \mathbb{N}} \{a_{i,k} \cup b_{i,k} \cup c_{i,k} \cup k\}$.

On the set $\{x \in (0, k) : U_c^k(x) = U^k(x)\}$, U^k is strictly concave, therefore, with the exception of set \mathcal{B}^k , the derivative of U^k is strictly decreasing.

Define the inverse of the $(U_c^k)'$:

$$i^k(y) = \begin{cases} k, & y = 0; \\ ((U_c^k)')^{-1}(y) \\ \cdot \mathbf{I}\{y \in \mathbb{R}^+ \setminus \{\bigcup_{i \in \mathbb{N}} [(U_c^k(b_{i,k}^+))', (U_c^k(a_{i,k}^-))'] \cup [(U_c^k(c_{i,k}^+))', (U_c^k(c_{i,k}^-))']]\}\} \\ + \sum_i a_{i,k} \cdot \mathbf{I}\{y \in [(U_c^k(a_{i,k}^+))', (U_c^k(a_{i,k}^-))']\} \\ + \sum_i b_{i,k} \cdot \mathbf{I}\{y \in [(U_c^k(b_{i,k}^+))', (U_c^k(b_{i,k}^-))']\} \\ + \sum_i c_{i,k} \cdot \mathbf{I}\{y \in [(U_c^k(c_{i,k}^+))', (U_c^k(c_{i,k}^-))']\}, & 0 < y < (U_c^k(0^+))'; \\ 0, & y \geq (U_c^k(0^+))'. \end{cases}$$

Let us now establish an analogue of [Theorem 2.39](#) for the utility function $U^{W(\omega)}$, $\omega \in \Omega$.

Theorem 2.42. *Assume that the random variable $\frac{dQ^e}{dP}$ is continuous and that the random variable W is discrete.*

Consider the function $i^W(\omega, y) := i^{W(\omega)}(y)$, for all $\omega \in \Omega$.

Let also

$$h^W(\lambda) = E_{Q^e} \left[i^{W(\omega)} \left(\lambda \cdot \frac{dQ^e}{dP}(\omega) \right) \right] < \infty, \text{ for all } \lambda > 0. \quad (2.41)$$

Then for any initial capital $x > 0$, the unique optimal solution to the problem [\(2.40\)](#) is given by $X^* = i^W(\lambda^* \cdot \frac{dQ^e}{dP})$ for some $\lambda^* > 0$ such that $E_{Q^e}[i^W(\lambda^* \cdot \frac{dQ^e}{dP})] = x$.

Remark 2.43. *Thanks to W being discrete, the function $i^{W(\omega)}(\lambda \cdot \frac{dQ^e}{dP}(\omega))$ is clearly measurable. The measurability actually holds for any random variable W thanks to monotonicity of $i(\cdot)$ in both arguments.*

Proof. From the definition of $i^{W(\omega)}$ it follows that for any $z > 0$

$$U_c^{W(\omega)}(z) - U_c^{W(\omega)}(i^{W(\omega)}(y)) \leq y(z - i^{W(\omega)}(y)). \quad (2.42)$$

The proof of (2.42) is similar to the corresponding part of proof of [Theorem 2.39](#), with the exception of the case $y = 0$, which we will address immediately. For $y = 0$, inequality (2.42) is equivalent to the requirement that for any $z > 0$

$$U_c^{W(\omega)}(z) - U_c^{W(\omega)}(W(\omega)) \leq 0,$$

which does hold true since $U^{W(\omega)}$ is decreasing.

Let us set $\mathcal{I}^{W(\omega)} := \bigcup_{i \in \mathbb{N}} \{(U_c^{W(\omega)}(a_{i,\omega}^-))' \} \cup \{(U_c^{W(\omega)}(a_{i,\omega}^+))' \} \cup \{(U_c^{W(\omega)}(b_{i,\omega}^+))' \} \cup \{(U_c^{W(\omega)}(c_{i,\omega}^-))' \} \cup \{(U_c^{W(\omega)}(c_{i,\omega}^+))' \} \cup \{(U_c^{W(\omega)}(0+))' \} \cup \{0\}$. Similarly to the corresponding part of proof of [Theorem 2.39](#), for all $y \in \mathbb{R}^+ \setminus \mathcal{I}^{W(\omega)}$ and for any $z > 0, z \neq i^{W(\omega)}(y)$, we have

$$U_c^{W(\omega)}(z) - U_c^{W(\omega)}(i^{W(\omega)}(y)) < y(z - i^{W(\omega)}(y)). \quad (2.43)$$

Since $\frac{dQ^e}{d\mathbb{P}} > 0$ a.s., the function

$$h^W(\lambda) = \mathbb{E}_{Q^e} \left[i^W \left(\lambda \cdot \frac{dQ^e}{d\mathbb{P}} \right) \right] = \mathbb{E} \left[\frac{dQ^e}{d\mathbb{P}} \cdot i^W \left(\lambda \cdot \frac{dQ^e}{d\mathbb{P}} \right) \right],$$

is well defined for all $\lambda > 0$. Since $\frac{dQ^e}{d\mathbb{P}} > 0$ a.s., $h^W(\lambda)$ is continuous and decreasing due to the continuity of $i^W(\cdot)$ and the fact that i^W is decreasing (see [Theorem 2.39](#)).

Additionally, we have that

$$\lim_{\lambda \rightarrow \infty} h^W(\lambda) = 0, \quad h^W(0) = \mathbb{E}_{Q^e} [W].$$

Since $\mathbb{E}_{Q^e} [W] > x$, there exists a positive λ^* such that $h(\lambda^*) = \mathbb{E}_{Q^e} [i^W(\lambda^* \cdot \frac{dQ^e}{d\mathbb{P}})] = x$. Let $X^* = i^W(\lambda^* \cdot \frac{dQ^e}{d\mathbb{P}})$. It is evident that $X^* \in \mathcal{X}_x^W$. The proof of the optimality of X^* is similar to the corresponding part of the proof of [Theorem 2.39](#). Essentially, it follows from the fact that

$$\mathbb{P} \left[\lambda^* \cdot \frac{dQ^e}{d\mathbb{P}} \in \mathbb{R}^+ \setminus \mathcal{I}^W \right] = 1,$$

which implies

$$\mathbb{P} \left[U_c^W(X) - U_c^W(X^*) \stackrel{(2.42)}{\leq} \lambda^* \cdot \frac{dQ^e}{d\mathbb{P}} (X - X^*) \right] = 1, \quad (2.44)$$

$$\mathbb{P} \left[U_c^W(X) - U_c^W(X^*) \stackrel{(2.43)}{<} \lambda^* \cdot \frac{dQ^e}{d\mathbb{P}} (X - X^*) \right] > 0. \quad (2.45)$$

The proof concludes as in [Theorem 2.39](#). \square

2.5.2 Examples

In this section, we provide examples that demonstrate that the conclusion of [Theorem 2.42](#) may not hold when W is a continuous random variable.

We begin by presenting the general setup for the examples. The sample space is $\Omega = [\frac{1}{2}, 1]$ and the probability measure \mathbb{P} is the normalized Lebesgue measure. Specifically, for $\frac{1}{2} \leq a < b \leq 1$, we have $\mathbb{P}([a, b]) = 2(b - a)$.

Let $W(\omega) = 2\omega$ for $\omega \in \Omega$, and consider the utility function

$$U(x) = \begin{cases} \frac{3\sqrt{2}}{4} \cdot x^{\frac{1}{2}}, & x \in [0, \frac{1}{2}); \\ \frac{x+1}{2}, & x \in [\frac{1}{2}, 1); \\ x^{\frac{3}{4}}, & x \in [1, \infty). \end{cases}$$

Observe that the function $U(x)$ is increasing, continuous on $(0, +\infty)$ and satisfies the mild growth condition.

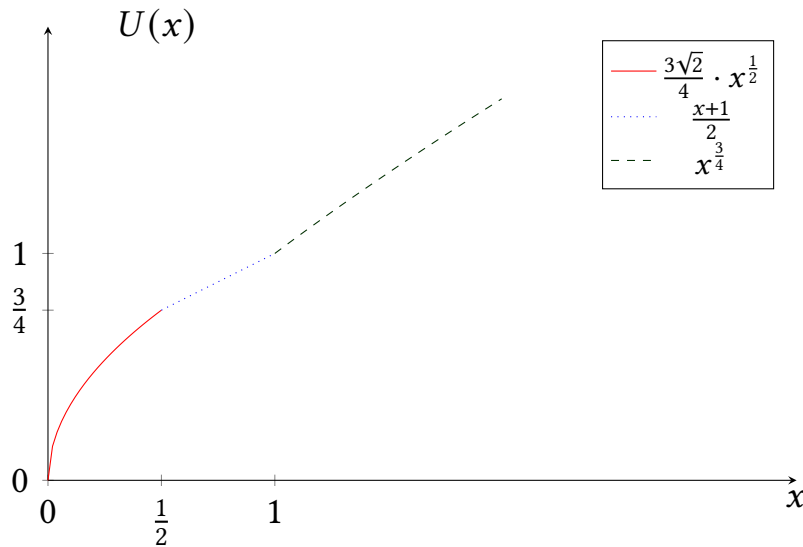


Figure 2.5. The utility function U .

As before, for each $\omega \in [\frac{1}{2}, 1]$ denote

$$U^{W(\omega)} = \begin{cases} U(x), & x \in [0, W(\omega)]; \\ U(W(\omega)), & x \in (W(\omega), +\infty). \end{cases}$$

To construct the concavification, one needs to build the straight line passing through the points $(\frac{1}{2}, \frac{3}{4})$ and $(2\omega, (2\omega)^{\frac{3}{4}})$. Therefore, the concavification $U_c^{W(\omega)}$ of

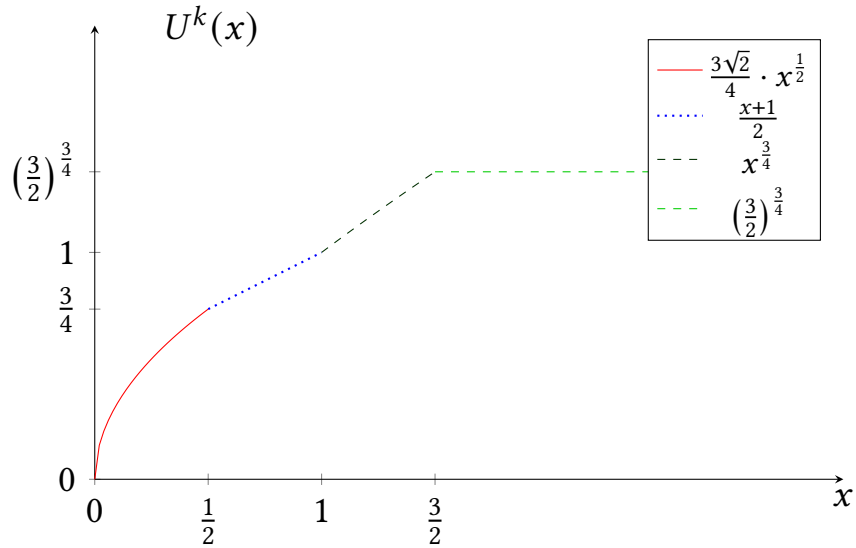


Figure 2.6. The utility function U^k , for $k = \frac{3}{2}$.

$U^{W(\omega)}$ is given by:

$$U_c^{W(\omega)}(x) = \begin{cases} \frac{3\sqrt{2}}{4} \cdot x^{\frac{1}{2}}, & x \in [0, \frac{1}{2}); \\ \frac{4(2\omega)^{\frac{3}{4}}-3}{2(4\omega-1)} \cdot x - \frac{4(2\omega)^{\frac{3}{4}}-3}{4(4\omega-1)} + \frac{3}{4}, & x \in [\frac{1}{2}, 2\omega); \\ (2\omega)^{\frac{3}{4}}, & x \in [2\omega, \infty). \end{cases}$$

Note that the concavification $U_c^{W(\omega)}$ is still increasing and continuous on $(0, +\infty)$, as it preserves the original properties of $U^{W(\omega)}$.

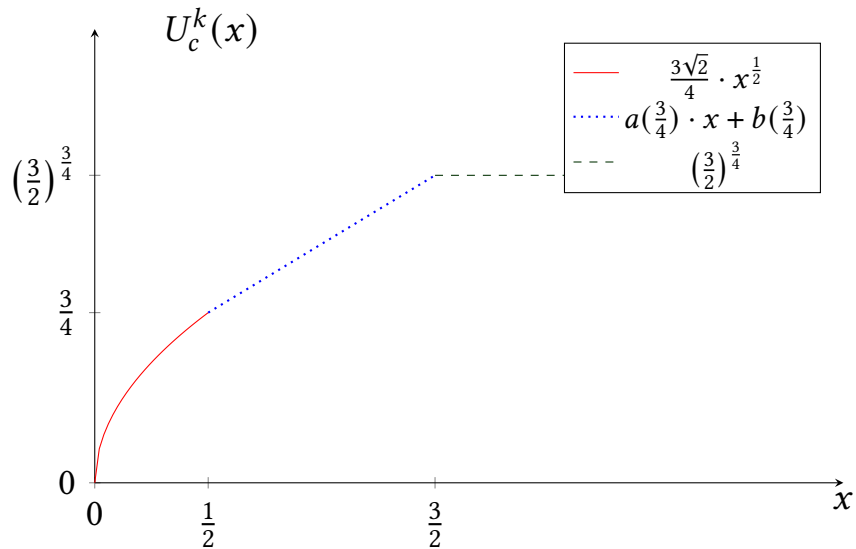


Figure 2.7. The concavified utility function U_c^k , for $k = \frac{3}{2}$.

The tangent line to the function $x^{\frac{3}{4}}, x \geq 1$ at the point $(\frac{1}{2}, \frac{3}{4})$ touches the function $x^{\frac{3}{4}}, x \geq 1$ approximately at (2.12, 1.76). Since $x \in [\frac{1}{2}, 2\omega)$ and $2\omega \leq 2$, it follows that $U_c^{W(\omega)} \geq U^{W(\omega)}$. Hence, $U_c^{W(\omega)}$ is indeed the concavification for $U^{W(\omega)}$.

For convenience let us abbreviate $a(\omega) := \frac{4(2\omega)^{\frac{3}{4}} - 3}{2(4\omega - 1)}$ and $b(\omega) := -\frac{4(2\omega)^{\frac{3}{4}} - 3}{4(4\omega - 1)} + \frac{3}{4}$. Observing that

$$(U_c^{W(\omega)})' \left(\frac{1}{2}^- \right) = \frac{3}{4} > \frac{4(2\omega)^{\frac{3}{4}} - 3}{2(4\omega - 1)} = (U_c^{W(\omega)})' \left(\frac{1}{2}^+ \right), \text{ for } \omega > \frac{1}{4},$$

we conclude that $U_c^{W(\omega)}$ is also concave and is the smallest concave function which is not less than the utility function $U^{W(\omega)}$. Note also that $U^{W(\omega)}$ is strictly concave on the set where $U_c^{W(\omega)}(x) = U^{W(\omega)}(x), x < W(\omega)$.

The derivative of $U_c^{W(\omega)}(x)$ is

$$(U_c^{W(\omega)}(x))' = \begin{cases} \frac{3\sqrt{2}}{8} \cdot x^{-\frac{1}{2}} & x \in (0, \frac{1}{2}); \\ \frac{4(2\omega)^{\frac{3}{4}} - 3}{2(4\omega - 1)}, & x \in [\frac{1}{2}, 2\omega); \\ 0, & x \in [2\omega, \infty); \end{cases}$$

and the inverse function $i^{W(\omega)}(y)$ is

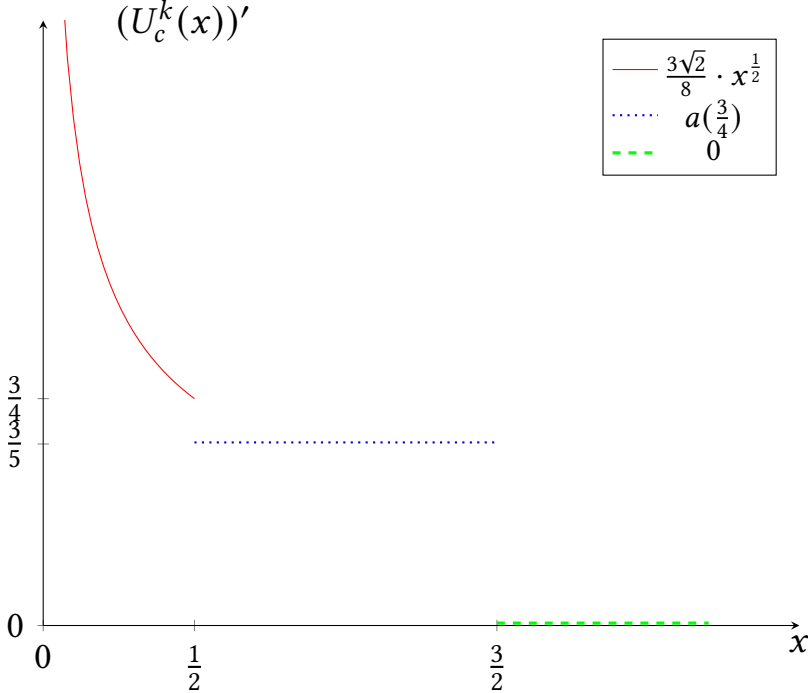


Figure 2.8. The derivative of concavified utility function $(U_c^k)'$, for $k = \frac{3}{2}$.

$$i^{W(\omega)}(y) = \begin{cases} 2\omega, & y \in [0, c(\omega)); \\ \frac{1}{2}, & y \in [c(\omega), \frac{3}{4}); \\ \frac{9}{32} \cdot y^{-2}, & y \in [\frac{3}{4}, \infty), \end{cases}$$

where $c(\omega) := \frac{4(2\omega)^{\frac{3}{4}} - 3}{2(4\omega - 1)}$.

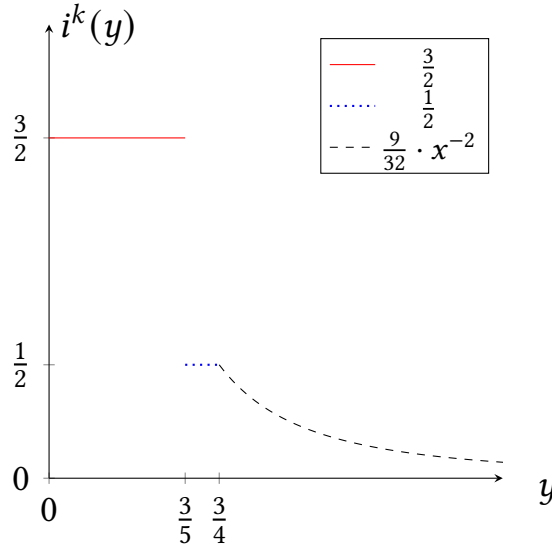


Figure 2.9. The inverse function i^k of $(U_c^k)'$, for $k = \frac{3}{2}$.

2.5.2.1 The function $h^W(\lambda)$ may be discontinuous

In this example, the function $h^W(\lambda)$ is not continuous and misses some values of admissible initial capitals.

Consider $\frac{dQ^e}{dP}(\omega) = \frac{1}{\lambda^*}c(\omega)$, for some λ^* . In this case we have

$$\int_{\Omega} \frac{dQ^e}{dP}(\omega) \mathbf{P}(d\omega) = 1.$$

Thus,

$$\int_{\Omega} \frac{1}{\lambda^*} c(\omega) \mathbf{P}(d\omega) = 1 \iff$$

$$\int_{\frac{1}{2}}^1 \frac{1}{\lambda^*} \cdot \frac{4(2\omega)^{\frac{3}{4}} - 3}{2(4\omega - 1)} \mathbf{P}(d\omega) = 1 \iff$$

$$\frac{1}{\lambda^\star} \int_{\frac{1}{2}}^1 \frac{4(2\omega)^{\frac{3}{4}} - 3}{2(4\omega - 1)} \cdot 2d\omega = 1.$$

Let us compute the integral

$$\begin{aligned} & \int_{\frac{1}{2}}^1 \frac{4(2\omega)^{\frac{3}{4}} - 3}{2(4\omega - 1)} \cdot 2d\omega \\ &= -\frac{4}{3} + \frac{4 \cdot 2^{\frac{3}{4}}}{3} - 2^{\frac{1}{4}} \cdot \arctan(2^{\frac{1}{4}}) + 2^{\frac{1}{4}} \cdot \arctan(\sqrt{2}) \\ &+ 2^{\frac{1}{4}} \cdot \operatorname{arctanh}(2^{\frac{1}{4}}) - 2^{\frac{1}{4}} \cdot \operatorname{arctanh}(\sqrt{2}) - \frac{3 \ln(3)}{4} \approx 0.59. \end{aligned}$$

Hence, $\lambda^\star \approx 0.59$.

Let us choose an initial capital $x \in \left(\frac{1}{2}, \frac{3}{2}\right)$ and let us prove that $E_{Q^e}(W) > x$:

$$\begin{aligned} E_{Q^e}(W) &= \mathbb{E} \left[\frac{dQ^e}{d\mathbb{P}} \cdot 2\omega \right] = \int_{\frac{1}{2}}^1 \frac{4(2\omega)^{\frac{3}{4}} - 3}{2(4\omega - 1)} \cdot \frac{1}{\lambda^\star} \cdot 2\omega d\omega \\ &= \frac{1}{\lambda^\star} \cdot \left(-\frac{167}{84} + \frac{38 \cdot 2^{\frac{3}{4}}}{21} - \frac{\arctan(2^{\frac{1}{4}})}{2^{\frac{3}{4}}} + \frac{\arctan(\sqrt{2})}{2^{\frac{3}{4}}} \right. \\ &\quad \left. + \frac{\operatorname{arctanh}(2^{\frac{1}{4}})}{2^{\frac{3}{4}}} - \frac{\operatorname{arctanh}(\sqrt{2})}{2^{\frac{3}{4}}} - \frac{3 \ln(3)}{8} \right) \approx 1.51. \end{aligned}$$

Now we show that function $h^W(\lambda) := E_{Q^e} \left[i^W \left(\lambda \frac{dQ^e}{d\mathbb{P}} \right) \right]$ is not continuous at λ^\star .

Take an increasing sequence $\lambda_n \uparrow \lambda^\star$. Then

$$h^W(\lambda^\star) = E_{Q^e} \left[i^W \left(\lambda^\star \frac{dQ^e}{d\mathbb{P}} \right) \right] = E_{Q^e} \left[i^W(c(\omega)) \right] = E_{Q^e} \left[\frac{1}{2} \right] = \frac{1}{2}.$$

Since $\lambda_n \uparrow \lambda^\star$, it follows that $\lambda_n \frac{dQ^e}{d\mathbb{P}} \uparrow \lambda^\star \frac{dQ^e}{d\mathbb{P}}$. Moreover, $\lambda_n \frac{dQ^e}{d\mathbb{P}} < \lambda^\star \frac{dQ^e}{d\mathbb{P}} =: c(\omega)$.

Thus

$$\begin{aligned} h^W(\lambda_n) &= E_{Q^e} \left[i^W \left(\lambda_n \frac{dQ^e}{d\mathbb{P}} \right) \right] = E_{Q^e} \left[2\omega \right] \\ &= \mathbb{E} \left[\frac{dQ^e}{d\mathbb{P}} \cdot 2\omega \right] = \int_{\frac{1}{2}}^1 \frac{4(2\omega)^{\frac{3}{4}} - 3}{2(4\omega - 1)} \cdot \frac{1}{\lambda^\star} \cdot 2\omega d\omega \approx 1.51. \end{aligned}$$

Since $h^W(\lambda)$ is decreasing, it does not attain the values between $\frac{1}{2}$ and $\frac{3}{2}$.

2.5.2.2 The inequality (2.45) may fail

In this example we consider $x = \frac{1}{2}$ as initial capital. As mentioned in the previous example, we know that there exists a value of $\lambda^* \approx 0.59$ such that $h^W(\lambda^*) = \frac{1}{2}$.

Denoting $v(x) = (U_c^W)'(x)$, consider the probability

$$\begin{aligned} & \mathbb{P}\left(\lambda^* \frac{dQ^e}{d\mathbb{P}} \notin \mathbb{R} \setminus \mathcal{I}^W\right) = \mathbb{P}\left(\lambda^* \frac{dQ^e}{d\mathbb{P}} \in \mathcal{I}^W\right) \\ &= \mathbb{P}\left(\lambda^* \frac{dQ^e}{d\mathbb{P}} \in \left\{v\left(\frac{1^-}{2}\right), v\left(\frac{1^+}{2}\right), v((2\omega)^+), v(0^+), 0\right\}\right) \\ &= \mathbb{P}\left(\lambda^* \frac{dQ^e}{d\mathbb{P}} \in \left\{\frac{3}{4}, c(\omega), v(0^+), 0\right\}\right) \geq \mathbb{P}\left(\lambda^* \frac{dQ^e}{d\mathbb{P}} = c(\omega)\right) = 1. \end{aligned}$$

Hence, $\mathbb{P}\left(\lambda^* \frac{dQ^e}{d\mathbb{P}} \in \mathbb{R} \setminus \mathcal{I}^W\right) = 0$.

2.6 Optimal investments for the robust utility functional maximization problem in complete market

This section presents optimal investment strategies for the problem of maximizing the robust utility function $\sup_{X \in \mathfrak{X}(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(X_T)]$ under certain assumptions. We will use the results from [Section 2.4.1](#). Hence, we assume that the equality

$$\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)] = \sup_{g \in C(x)} E_{\widehat{Q}}[U(g)], \quad (2.46)$$

holds for some $\widehat{Q} \in \mathcal{Q}_e$.

Under the assumptions of [Theorem 2.13](#), the assumption above is equivalent to the minimax identity, as shown in [Theorem 2.23](#). Using [Remark 2.6](#), we can reformulate our problem as follows:

$$\text{Find the optimal solution for } \sup_{g \in L_+^0(\Omega, \mathcal{F}, \mathbb{P})} \inf_{Q \in \mathcal{Q}} E_Q[U(g)] \text{ subject to } E_{Q^e}(g) \leq x, \quad (2.47)$$

where x is the initial capital.

Assuming that (2.46) holds, we consider the problem:

$$\text{Find the optimal solution for } \sup_{g \in L_+^0(\Omega, \mathcal{F}, \mathbb{P})} E_{\widehat{Q}}[U(g)] \text{ subject to } E_{Q^e}(g) \leq x, \quad (2.48)$$

where x is the initial capital.

Theorem 2.44. *Suppose that (2.46) holds. Additionally, assume that $\frac{dQ^e}{d\widehat{Q}}$ is continuous. Denote*

$$i(y) = \begin{cases} (U'_c)^{-1}(y) \\ \cdot \mathbb{I}\{y \in \mathbb{R}^+ \setminus \{\cup_{i \in \mathbb{N}} \{[U'_c(b_i^+), U'_c(a_i^-)] \cup [U'_c(c_i^+), U'_c(c_i^-)]\}\}\} \\ + \sum_i a_i \cdot \mathbb{I}\{y \in [U'_c(a_i^+), U'_c(a_i^-)]\} \\ + \sum_i b_i \cdot \mathbb{I}\{y \in [U'_c(b_i^+), U'_c(b_i^-)]\} \\ + \sum_i c_i \cdot \mathbb{I}\{y \in [U'_c(c_i^+), U'_c(c_i^-)]\}, & y < U'_c(0^+); \\ 0, & y \geq U'_c(0^+). \end{cases}$$

Assume that

$$h(\lambda) = E_{Q^e} \left[i \left(\lambda \cdot \frac{dQ^e}{d\widehat{Q}} \right) \right] < \infty, \text{ for all } \lambda > 0. \quad (2.49)$$

Then the unique optimal solution to the problem (2.48) is given by $g^\star = i(\lambda^\star \cdot \frac{dQ^e}{d\widehat{Q}})$, where $\lambda^\star > 0$ and such that $E_{Q^e} [i(\lambda^\star \cdot \frac{dQ^e}{d\widehat{Q}})] = x$, for initial capital x .

Remark. If $a_1 = 0$, then we need to consider $U'_c(a_1^+)$.

Proof. We note that the proof is similar to the proof of [Theorem 2.39](#), but instead of using \mathbb{P} , we consider the measure \widehat{Q} , which is equivalent to \mathbb{P} . Thus, the set $L_+^0(\Omega, \mathcal{F}, \mathbb{P})$ coincides with $L_+^0(\Omega, \mathcal{F}, \widehat{Q})$, and the unique equivalent local martingale measure Q^e remains the same. \square

Remark 2.45. *The solution g^\star is also a unique optimal solution for the problem of maximizing the concavification of the utility function, i.e.,*

$$\sup_{g \in C(x)} E_{\widehat{Q}}[U_c(g)] = \max_{g \in C(x)} E_{\widehat{Q}}[U_c(g)] = E_{\widehat{Q}}[U_c(g^\star)],$$

which follows from [Remark 2.40](#) and the proof of the theorem above.

Theorem 2.46. *Consider the model setting and utility function as in [Section 2.3.2](#). Then under the assumptions of [Theorem 2.44](#), [Theorem 2.13](#), and [Lemma 2.21](#), the*

unique optimal solution g^* for the problem (2.48) is also an optimal solution for (2.47), in the sense that

$$\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(g)] = \inf_{Q \in \mathcal{Q}} E_Q[U(g^*)].$$

Proof. It follows from Lemma 2.21 that there exists $\widehat{g} \in C(x)$ such that

$$u^c(x) = \inf_{Q \in \mathcal{Q}} E_Q[U_c(\widehat{g})]. \quad (2.50)$$

Using (2.46), we also have

$$u^c(x) = u_{\widehat{Q}}^c(x) = E_{\widehat{Q}}[U_c(g^*)]. \quad (2.51)$$

Furthermore,

$$\inf_{Q \in \mathcal{Q}} E_Q[U_c(\widehat{g})] \leq E_{\widehat{Q}}[U_c(\widehat{g})] \leq u_{\widehat{Q}}^c(x).$$

Therefore, combining (2.50) and (2.51), we obtain

$$E_{\widehat{Q}}[U_c(\widehat{g})] = E_{\widehat{Q}}[U_c(g^*)].$$

By Remark 2.45, since \widehat{g} is also an optimal solution of the problem (2.48), we have $\widehat{g} = g^*$ almost surely.

Using Remark 2.16 and (2.50), we get

$$u^c(x) = \inf_{Q \in \mathcal{Q}} E_Q[U_c(\widehat{g})] = \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(\widehat{g})] = \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g^*)] = \inf_{Q \in \mathcal{Q}} E_Q[U_c(g^*)]. \quad (2.52)$$

From Theorem 2.23, we know that assumption (2.46) implies

$$u^c(x) = u(x). \quad (2.53)$$

Note that g^* never takes on the values where $U < U_c$. Therefore,

$$\inf_{Q \in \mathcal{Q}_e} E_Q[U(g^*)] = \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g^*)]. \quad (2.54)$$

Combining (2.52), (2.53) and (2.54) we obtain

$$u^c(x) = \inf_{Q \in \mathcal{Q}} E_Q[U_c(g^*)] = \inf_{Q \in \mathcal{Q}_e} E_Q[U(g^*)] \stackrel{\text{Lemma 2.15}}{=} \inf_{Q \in \mathcal{Q}} E_Q[U(g^*)] = u(x).$$

□

Conclusions for Chapter 2

In this chapter, we explored the properties of concavified utility functions and examined the special case of concavification. We established the minimax identity for non-concave utility functions by deriving equalities and inequalities to relate the robust utility functional of the initial utility function and its concavification. We presented cases in which all the inequalities become equalities, and hence, the minimax identity for the initial non-concave utility function holds.

We also demonstrated the existence of the "worst-case scenario" measure in two cases of the infimum of the expected value of the initial utility function and its concavification.

Furthermore, we derived the existence and uniqueness of the optimal solution to the standard non-concave utility maximization problem and constructed its explicit form under the assumption of standard budget constraints. We showed that this solution is also a unique optimal solution for the maximization problem of the concavified utility function. In the case of implementing an additional upper bound given by some random variable, we proved a similar theorem if the given random variable is discrete and presented examples that show the previous conclusions may fail in the case of a continuous random variable that represents an upper bound.

Finally, we demonstrated the existence and uniqueness of the optimal solution to the robust non-concave utility maximization problem and provided a way to construct it.

Chapter 3

Optimal investments for the non-concave utility maximization problems in incomplete market

In this chapter we will deal with incomplete markets. The results we will show are similar to those from [Chapter 2](#). However, the main difference is that we no longer assume that the equivalent local martingale measure is unique. We only assume the no-arbitrage type condition that the set of all equivalent local martingale measures \mathcal{M}_e is not empty. We keep the same notation as in [Chapter 2](#) while extending its framework.

The aim of this chapter is finding the optimal investment strategies for standard as well as robust non-concave utility maximization problems.

For model details see [Section 2.1](#).

We assume that for the set \mathcal{Q} of all measures [Assumption 2.2](#) holds.

3.1 The minimax identity for the non-concave utility in incomplete market

As before, we consider a utility function $U : (0, \infty) \rightarrow \mathbb{R}$, which is non-constant, non-decreasing, upper semi-continuous and satisfies the mild growth condition:

$$\lim_{x \rightarrow \infty} \frac{U(x)}{x} = 0,$$

and such that $U(\infty) > 0$. Set $U(x) = -\infty$ for $x < 0$, and denote $U(0) := \lim_{x \downarrow 0} U(x)$ and $U(\infty) = \lim_{x \uparrow \infty} U(x)$.

It follows from the [2, Proposition 3.1] that $U(x)$ has a non-decreasing and continuous on $(0, +\infty)$ concave envelope, which is denoted by $U_c(x)$.

Recall that

$$\begin{aligned} u(x) &= \sup_{X \in \mathfrak{X}(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(X_T)] = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(g)]; \\ u_Q(x) &= \sup_{X \in \mathfrak{X}(x)} E_Q[U(X_T)] = \sup_{g \in C(x)} E_Q[U(g)]. \end{aligned}$$

To exclude the trivialities, we impose

Assumption 3.1.

For all $x > 0$ exists some $Q \in \mathcal{Q}_e$ such that $u_Q(x) < \infty$.

Denote

$$\begin{aligned} u^c(x) &:= \sup_{X \in \mathfrak{X}(x)} \inf_{Q \in \mathcal{Q}} E_Q[U_c(X_T)] = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U_c(g)]; \\ u_Q^c(x) &:= \sup_{X \in \mathfrak{X}(x)} E_Q[U_c(X_T)] = \sup_{g \in C(x)} E_Q[U_c(g)]. \end{aligned}$$

In this chapter we study the same minimax identity, but in a more general framework.

Besides [Assumption 2.2](#), assume the following.

Assumption 3.2.

$u_{Q_0}^c(x) < \infty$ for some $x > 0$ and some $Q_0 \in \mathcal{Q}_e$.

Lemma 3.3. *Suppose that [Assumption 2.2](#) and [Assumption 3.2](#) hold. Then, one has*

$$\begin{aligned} u^c(x) &= \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U_c(g)] = \inf_{Q \in \mathcal{Q}} \sup_{g \in C(x)} E_Q[U_c(g)] \\ &= \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)] = \inf_{Q \in \mathcal{Q}_e} \sup_{g \in C(x)} E_Q[U_c(g)]. \end{aligned}$$

Proof. Noting that the function $U_c(x)$ is non-decreasing, continuous on $(0, +\infty)$ and concave the proof is the same as proof of [Lemma 2.14](#). \square

Lemma 3.4. *If [Assumptions 2.2](#) and [3.1](#) hold, then for all $g \in C(x)$,*

$$\inf_{Q \in \mathcal{Q}} E_Q[U(g)] = \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)].$$

Proof. The proof is the same as proof of [Lemma 2.15](#). □

Lemma 3.5. *If [Assumption 2.2](#) and [Assumption 3.1](#) hold, then*

$$\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(g)] = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)].$$

Proof. The statement is clear from [Lemma 3.4](#). □

The proof of forthcoming results will be based on the ideas of Christian Reichlin [\[33\]](#). First, let us introduce the notation and assumption used therein. Recall, that $V(y)$ is the conjugate of U (and U_c):

$$V(y) = \sup_{x > 0} \{U(x) - xy\}, \quad y > 0,$$

the asymptotic elasticity of V was defined by

$$AE_0(V) := \limsup_{y \rightarrow 0} \sup_{q \in \partial V(y)} \frac{|q|y}{V(y)};$$

where ∂V denotes the subdifferential of V ; more detailed in the [\[33, Chapter II\]](#).

Assumption 3.6. $AE_0(V) < \infty$.

To define the utility price, that we will use in the following assumption, we first need to consider a few notations:

$$Y(y) := \{Y \geq 0 \mid \mathbb{E}[YX] \leq xy \text{ for all } X \in \mathfrak{X}(x)\};$$

$$D(y) := \{h \in L_+^0(\Omega, \mathcal{F}, \mathbb{P}) \mid 0 \leq h \leq Y_T, \text{ for some } Y \in Y(y)\}.$$

Definition. *For a distribution F on \mathbb{R}_+ , the utility price is defined as*

$$UP(F) := \sup_{h \in D(1)} \left\{ \int_0^1 q_F(s) q_h(1-s) ds \right\}.$$

The last assumption we need is

Assumption 3.7. *For any $Q \in \mathcal{Q}_e$ and any distribution F_Q on \mathbb{R}_+ with $UP(F_Q) < \infty$, there exists $f_Q \in C(UP(F_Q))$ with $f_Q \sim F_Q$. The index of distribution shows the probability measure under which we consider it.*

For more information on the utility price $UP(F)$, see [33, Chapter IV].

Remark 3.8. Reichlin considers the [Assumption 3.7](#) only with respect to the objective measure \mathbb{P} . However, since we are interested in the robust utility maximization problem, we need his assumption to work for all $Q \in \mathcal{Q}_e$.

Theorem 3.9 (Theorem 4.6 by Reichlin in [33]). Suppose that [Assumption 3.1](#) (for \mathbb{P} and some $x > 0$), [Assumption 3.6](#) and [Assumption 3.7](#) (for probability measure \mathbb{P}) are satisfied. Then the non-concave problem $u_{\mathbb{P}}(x)$ admits a maximizer and it holds that

$$u_{\mathbb{P}}(x) = u_{\mathbb{P}}^c(x), \text{ for all } x > 0. \quad (3.1)$$

Proof. See [33, Chapter IV]. □

Remark 3.10. At the end of [33, Section IV.5.1] Reichlin showed that [Assumption 3.7](#) is not necessary for equation (3.1) to hold.

Now we can introduce our lemma.

Lemma 3.11. Suppose that [Assumptions 3.1](#), [3.6](#) and [3.7](#) are satisfied. Then, for all $x > 0$, it holds that

$$\inf_{Q \in \mathcal{Q}_e} \sup_{g \in C(x)} E_Q[U(g)] = \inf_{Q \in \mathcal{Q}_e} \sup_{g \in C(x)} E_Q[U_c(g)].$$

Proof. There is a sequence $Q_n \in \mathcal{Q}_e$ such that

$$\lim_{n \rightarrow \infty} \sup_{g \in C(x)} E_{Q_n}[U(g)] = \inf_{Q \in \mathcal{Q}_e} \sup_{g \in C(x)} E_Q[U(g)].$$

It follows from [Assumption 3.1](#) that exists $N \in \mathbb{N}$ such that for all $n > N$, we have $\sup_{g \in C(x)} E_{Q_n}[U(g)] < \infty$.

As in the proof of [Lemma 2.18](#), we can apply [Theorem 3.9](#) for any $Q_n \in \mathcal{Q}_e$, $n > N$. All assumptions remains the same, and the set \mathcal{M}_e of all equivalent local martingale measures does not change. Moreover, the set $L_+^0(\Omega, \mathcal{F}, Q)$ is the same for all $Q \in \mathcal{Q}_e$, which means that everything remains true if one uses arbitrary measure $Q_n \in \mathcal{Q}_e$, $n > N$, in place of \mathbb{P} .

Hence, from the [Theorem 3.9](#) follows that

$$\sup_{g \in C(x)} E_{Q_n}[U(g)] = \sup_{g \in C(x)} E_{Q_n}[U_c(g)], \text{ for all } Q_n \in \mathcal{Q}_e \text{ with } n > N.$$

From now on, assume that $U(0)$ is finite, i.e. $U : [0, \infty) \rightarrow \mathbb{R}$. The next two theorems present the conditions under which all inequalities in the [Theorem 3.13](#) are equalities and, consequently, when the minimax identity holds for the general framework of non-concave utility functions.

Theorem 3.14. *Suppose that all assumptions from [Theorem 3.13](#) holds.*

Then, for all $x > 0$, the next two equalities are equivalent

- (i) $\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)] = \sup_{g \in C(x)} E_{\widehat{Q}}[U(g)],$ for $\widehat{Q} \in \mathcal{Q}_e$;
- (ii) $\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)] = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)].$

Proof. Suppose (i) holds. If $\sup_{g \in C(x)} E_{\widehat{Q}}[U(g)] = \infty$, then, since $U \leq U_c$, it follows that

$$\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)] = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)] = \infty.$$

However, this case is trivial. Hence, assume $\sup_{g \in C(x)} E_{\widehat{Q}}[U(g)] < \infty$.

Therefore, from [Theorem 3.9](#), due to the arguments of [Lemma 3.11](#), it follows that

$$\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)] = \sup_{g \in C(x)} E_{\widehat{Q}}[U(g)] = \sup_{g \in C(x)} E_{\widehat{Q}}[U_c(g)] \geq \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)].$$

Noting that for all $x \geq 0$ one has $U_c(x) \geq U(x)$,

$$\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)] \leq \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)].$$

Which gives us (ii).

The proof from (ii) to (i) is the same as in [Theorem 2.23](#). □

Recall [Remark 2.22](#) and \widehat{Q}_0 from [Assumption 2.25](#).

Theorem 3.15. *Suppose that all assumptions from [Lemma 2.21](#) and [Theorem 3.13](#) hold. Assume that at least one of the items below holds*

- (i) *Exists such a measure $\widehat{Q} \in \mathcal{Q}_e$ that for all $g \in C(x) : \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)] = E_{\widehat{Q}}[U_c(g)];$*
- (ii) *For any sequence $g_n \in C(x)$ such that $\lim_{n \rightarrow \infty} E_{\widehat{Q}_0}[U_c(g_n)] = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)]$ it holds that $\lim_{n \rightarrow \infty} E_{\widehat{Q}_0}[U_c(g_n)] = \lim_{n \rightarrow \infty} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g_n)].$*

Then, one has

$$\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)] = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g)].$$

Hence, all inequalities in the [Theorem 3.13](#) turn to equalities.

Proof. The proof is the same as in [Theorem 2.26](#), however, we can no longer use the [\[34, Proposition 5.3\]](#). Instead, we refer to the proofs of [Theorem 4.6](#) and [Proposition 4.7](#) [\[33\]](#).

From [\[33, Proposition 4.7\]](#) follows that for each $g \in C(x)$, there is a distribution F^\star , such that for each $h \in D(1)$, there exists $g_h^\star \in L_+^0(\Omega, \mathcal{F}, \mathbb{P})$ with $g_h^\star \sim F^\star$, moreover, $\mathbb{E}[hg_h^\star] \leq \mathbb{E}[hg]$ and $\mathbb{E}[U(g_h^\star)] = \mathbb{E}[U_c(g)]$. Besides, it follows from the first part of the proof of [\[33, Proposition 4.7\]](#) that $\mathbb{P}(g_h^\star \in \{U < U_c\}) = 0$. Next, it follows from the proof of the [\[33, Theorem 4.6\]](#) that there exists $g^\star \in C(x)$ with $g^\star \sim F^\star$. By combining all of the results above, we obtain that for each $g \in C(x)$, exists $g^\star \in C(x)$ such that $\mathbb{P}(g^\star \in \{U < U_c\}) = 0$ and

$$\mathbb{E}[U_c(g)] = \mathbb{E}[U(g^\star)] = \mathbb{E}[U_c(g^\star)]. \quad (3.2)$$

As before, due to all assumptions of the theorem, we can use $Q \sim \mathbb{P}$ in place of \mathbb{P} . Hence, [\(3.2\)](#) holds for measures \widehat{Q} and \widehat{Q}_0 . This is the result of [\[34, Proposition 5.3\]](#) used in the proof of [Theorem 2.26](#). Also, instead of [Lemma 2.14](#), as in the proof of [Theorem 2.26](#), is used [Lemma 3.3](#). \square

3.2 Optimal investments for the concavified maximization problems in incomplete market

In this section, we focus on constructing the optimal investment strategies for concavified utility functions within the context of incomplete markets. Our objective is to maximize the utility function, specifically $\sup_{X \in \mathfrak{X}(x)} \mathbb{E}[U_c(X_T)]$.

Consider a utility function $U : [0, \infty) \rightarrow \mathbb{R}$ which is strictly increasing, at least twice continuously differentiable and satisfying a mild growth condition:

$$\lim_{x \rightarrow \infty} \frac{U(x)}{x} = 0.$$

As before by, $U_c : [0, \infty) \rightarrow \mathbb{R}$ denote the concave envelope of U . The concave envelope U_c is increasing, continuously differentiable on $(0, \infty)$, and it also satisfies the mild growth condition, see [Lemma 2.19](#) and [Lemma 2.20](#).

3.2.1 Optimal investments for the standard problem of maximization the concavified utility function in incomplete market

This section focuses on presenting the optimal investment strategies for the standard problem of maximizing the concavified utility function in an incomplete market. Specifically, we aim to find $\sup_{X \in \mathfrak{X}(x)} \mathbb{E}[U_c(X_T)]$.

Recall that

$$u_p^c(x) = \sup_{X \in \mathfrak{X}(x)} \mathbb{E}[U_c(X_T)].$$

To exclude trivialities, we assume throughout this section that

$$u_p^c(x) < \infty \text{ for some } x > 0. \quad (3.3)$$

From now on we use the duality theory. Let us first introduce some relevant notation and results.

3.2.1.1 Duality theory

Define the convex conjugate function V of U :

$$V(y) = \sup_{x>0} (U(x) - xy), \quad y > 0.$$

The utility function U in our case is not concave and does not satisfy the Inada condition in zero. Moreover, the concave envelope U_c is not strictly concave and the Inada condition in zero does not hold as well. Hence, the convex conjugate V is not smooth and not strictly decreasing in general.

It follows from the [\[34, Lemma 2.9\]](#) that the function V is convex, decreasing, finite on $(0, \infty)$. The utility U and its concave envelope U_c have the same convex conjugate. Moreover, it holds that

$$U_c(x) - xy = V(y) \iff x \in -\partial V(y) \iff y \in \partial U_c(x). \quad (3.4)$$

The dual value function v_P for u_P is given by

$$v_P(y) := \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}[V(Y_T)], \quad y > 0,$$

where the space $\mathcal{Y}(y)$ is defined as

$$\mathcal{Y}(y) = \{Y \geq 0 \mid Y_0 = y \text{ and } XY \text{ is a } \mathbb{P} - \text{supermartingale for all } X \in \mathfrak{X}(1)\}.$$

As in [25, 26] consider sets:

$$\begin{aligned} C(x) &= \{g \in L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}) \mid 0 \leq g \leq X_T \text{ for some } X \in \mathfrak{X}(x)\}; \\ D(y) &= \{h \in L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}) \mid 0 \leq h \leq Y_T \text{ for some } Y \in \mathcal{Y}(y)\}. \end{aligned}$$

It is easy to see that

$$\begin{aligned} u_P(x) &= \sup_{g \in C(x)} \mathbb{E}[U(g)]; \\ u_P^c(x) &= \sup_{g \in C(x)} \mathbb{E}[U_c(g)]; \\ v_P(y) &= \inf_{h \in D(y)} \mathbb{E}[V(h)]. \end{aligned}$$

Moreover, if there is an optimal g or h , then they are clearly correspond to some $X \in \mathfrak{X}(x)$ or $Y \in \mathcal{Y}(y)$ respectively.

The subset S of $L_+^0(\Omega, \mathcal{F}_T, \mathbb{P})$ is called *solid* if from $0 \leq f \leq g$ and $g \in S$ it follows that $f \in S$.

By virtue of [25, Proposition 3.1], we have the following:

- (i) $C(x)$ and $D(y)$ are subsets of $L_+^0(\Omega, \mathcal{F}_T, \mathbb{P})$ which are convex, solid and closed in the topology of convergence in measure.
- (ii)

$$\begin{aligned} g \in C(x) &\iff g \geq 0 \text{ and } \sup_{h \in D(y)} \mathbb{E}[hg] \leq xy; \\ h \in D(y) &\iff h \geq 0 \text{ and } \sup_{g \in C(x)} \mathbb{E}[hg] \leq xy. \end{aligned} \tag{3.5}$$

- (iii) $C(x)$ is a bounded subset of $L_+^0(\Omega, \mathcal{F}_T, \mathbb{P})$ and contains the constant function x .

Assume that

$$U_c(0) \geq 0. \tag{3.6}$$

In this case,

$$V(y) = \sup_{x>0} (U_c(x) - xy) \geq U_c(0) \geq 0 \text{ for all } y > 0;$$

$$V_n(y) = \sup_{0<x\leq n} (U_c(x) - xy) \geq U_c(0) \geq 0 \text{ for all } y > 0.$$

Since $V_n(y) \geq 0$ for all $y > 0$ and in view of the assumption (3.3), it follows from the proof of the [25, Lemma 3.4] that

$$v_P(y) = \sup_{x>0} (u_P^c(x) - xy) \text{ for each } y > 0. \quad (3.7)$$

Theorem 3.16. *Let the assumptions (3.3) and (3.6) hold. Then, for any $x_0 > 0$, there exist some $\widehat{g} \in C(x_0)$, $\widehat{y} > 0$ and $\widehat{h} \in D(\widehat{y})$ such that*

- (i) $u_P^c(x_0) = v_P(\widehat{y}) + x_0\widehat{y}$,
- (ii) $u_P^c(x_0) = \inf_{y>0} (v_P(y) + x_0y)$.

If additionally $v_P(y) < \infty$ for all $y > 0$, then

- (iii) $v_P(\widehat{y}) = \mathbb{E}[V(\widehat{h})]$,
- (iv) $u_P^c(x_0) = \mathbb{E}[U_c(\widehat{g})]$.

Moreover, $v_P(\cdot)$ is convex.

Proof. Fix $x_0 > 0$. For all $y > 0$, we have

$$u_P^c(x_0) \leq \sup_{x>0} [u_P^c(x) - xy] + x_0y. \quad (3.8)$$

Let us prove that there exists $\widehat{y} > 0$ such that for all $x > 0$ the following holds:

$$u_P^c(x_0) \geq u_P^c(x) - x\widehat{y} + x_0\widehat{y}. \quad (3.9)$$

This is equivalent to

$$u_P^c(x) - u_P^c(x_0) \leq \widehat{y}(x - x_0).$$

Since u_P^c is concave (see [37, Lemma 3.1 (b)]), Lemma A.7 yields the existence of such \widehat{y} . Moreover, from $u_P^{c'}(x_0^+) \leq u_P^{c'}(x_0^-)$ it follows that $\widehat{y} \in [u_P^{c'}(x_0^+), u_P^{c'}(x_0^-)]$.

Therefore, (3.9) gives

$$u_P^c(x_0) \geq \sup_{x>0} [u_P^c(x) - x\widehat{y}] + x_0\widehat{y}.$$

Thus, in view of (3.8), we have

$$u_p^c(x_0) = \sup_{x>0} [u_p^c(x) - x\widehat{y}] + x_0\widehat{y}.$$

From (3.7) we then derive (i).

To prove (ii), note that thanks to (3.7), it holds for all $x_0, y > 0$ that

$$u_p^c(x_0) \leq v_P(y) + xy.$$

Hence,

$$u_p^c(x_0) \leq \inf_{y>0} (v_P(y) + x_0y).$$

From (i) we have that

$$u_p^c(x_0) \geq \inf_{y>0} (v_P(y) + x_0y).$$

This gives us (ii).

The definition of $v_P(\widehat{y})$ implies the existence of a sequence $(h_n) \in D(\widehat{y})$ such that

$$\lim_{n \rightarrow \infty} \mathbb{E}[V(h_n)] = \inf_{h \in D(\widehat{y})} \mathbb{E}[V(h)].$$

By utilizing the Kolmos-type argument (see [13, Lemma A1.1]), we can find a sequence $(\widehat{h}_n), n \in \mathbb{N}$ that belongs to the convex hull of (h_n, h_{n+1}, \dots) and converges almost surely to some non-negative \widehat{h} .

Since $D(\widehat{y})$ is both convex and closed with respect to convergence in measure, it follows that $D(\widehat{y})$ is closed under countable convex combinations. Therefore, $(\widehat{h}_n) \in D(\widehat{y})$. Furthermore, as $D(\widehat{y})$ is closed in the topology of convergence in measure, we have $\widehat{h} \in D(\widehat{y})$.

For each $n \in \mathbb{N}$, there exists a sequence $\alpha_i^n \in \mathbb{R}^+$ such that $\sum_{i \in \mathbb{N}} \alpha_i^n = 1$ and

$$\widehat{h}_n = \sum_{i \in \mathbb{N}} \alpha_i^n h_{n+i-1}.$$

By noting that V is a convex function and continuous on $(0, \infty)$, and considering the assumption of the theorem, we draw the following conclusion:

$$\limsup_{n \rightarrow \infty} \mathbb{E}[V(\widehat{h}_n)] = \limsup_{n \rightarrow \infty} \mathbb{E} \left[V \left(\sum_{i \in \mathbb{N}} \alpha_i^n h_{n+i-1} \right) \right]$$

$$\stackrel{\text{continuity of } V}{\leq} \limsup_{n \rightarrow \infty} \sum_{i \in \mathbb{N}} \alpha_i^n \cdot \mathbb{E}[V(h_{n+i-1})] = \inf_{h \in D(\hat{y})} \mathbb{E}[V(h)].$$

The last equality here holds due to [Proposition A.2](#).

Since $\liminf_{n \rightarrow \infty} \mathbb{E}[V(\hat{h}_n)] \leq \limsup_{n \rightarrow \infty} \mathbb{E}[V(\hat{h}_n)]$ and $\hat{h}_n \in D(\hat{y})$, we obtain

$$\lim_{n \rightarrow \infty} \mathbb{E}[V(\hat{h}_n)] = \inf_{h \in D(\hat{y})} \mathbb{E}[V(h)].$$

Noting that $V(y) \geq 0$, $y > 0$, one has

$$\mathbb{E}[V(\hat{h})] = \mathbb{E}[\liminf_{n \rightarrow \infty} V(\hat{h}_n)] \leq \liminf_{n \rightarrow \infty} \mathbb{E}[V(\hat{h}_n)] = \inf_{h \in D(\hat{y})} \mathbb{E}[V(h)].$$

This proves (iii).

Since $v_P(y) < \infty$ for $y > 0$, it follows from (ii) that

$$\lim_{x \rightarrow \infty} \frac{u_P^c(x)}{x} = 0.$$

Hence, (iv) follows from [\[26, Lemma 1\]](#).

It remains to show that $v_P(\cdot)$ is convex. For $\lambda \in (0, 1)$, observe that for $\hat{h}_1 \in D(y_1)$ and $\hat{h}_2 \in D(y_2)$, we have $(\lambda\hat{h}_1 + (1-\lambda)\hat{h}_2) \in D(\lambda y_1 + (1-\lambda)y_2)$, and therefore,

$$\begin{aligned} \lambda v_P(y_1) + (1-\lambda)v_P(y_2) &= \lambda \mathbb{E}[V(\hat{h}_1)] + (1-\lambda)\mathbb{E}[V(\hat{h}_2)] \geq \\ \mathbb{E}[V(\lambda\hat{h}_1 + (1-\lambda)\hat{h}_2)] &\geq \inf_{h \in D(\lambda y_1 + (1-\lambda)y_2)} \mathbb{E}[V(h)] = v_P(\lambda y_1 + (1-\lambda)y_2). \end{aligned}$$

□

Theorem 3.17. *Let $v_P(y) < \infty$ for all $y > 0$, and the assumptions [\(3.3\)](#) and [\(3.6\)](#) hold.*

Then, for any $x > 0$, there exists an optimal strategy $\hat{X} \in \mathfrak{X}(x)$ such that

$$u_P^c(x) = \mathbb{E}[U_c(\hat{X}_T)].$$

There also exist some $\hat{y} > 0$ and $\hat{Y} \in \mathcal{Y}(\hat{y})$ such that

$$v_P(\hat{y}) = \mathbb{E}[V(\hat{Y}_T)] \text{ and } u_P^c(x) = v_P(\hat{y}) + x\hat{y}.$$

Furthermore, $\hat{X}_T \in -\partial V(\hat{Y}_T)$, and $\hat{X}\hat{Y}$ is a martingale.

Proof. It follows from [Theorem 3.16](#) (iv) that for all $x > 0$, there exists an optimal strategy $\hat{X} \in \mathfrak{X}(x)$. Furthermore, from [Theorem 3.16](#) (i) and (iii), there exist $\hat{y} > 0$ and $\hat{Y} \in \mathcal{Y}(\hat{y})$ such that

$$u_P^c(x) = \mathbb{E}[U_c(\hat{X}_T)];$$

$$\begin{aligned} v_P(\widehat{y}) &= \mathbb{E}[V(\widehat{Y}_T)]; \\ u_P^c(x) &= v_P(\widehat{y}) + x\widehat{y}. \end{aligned}$$

Since V is a convex conjugate of U_c , we have that

$$V(\widehat{h}) + \widehat{g}\widehat{h} - U_c(\widehat{g}) \geq 0,$$

where $\widehat{h} = \widehat{Y}_T$. Hence,

$$\begin{aligned} 0 &\leq \mathbb{E}[V(\widehat{h}) + \widehat{g}\widehat{h} - U_c(\widehat{g})] \\ &= v_P(\widehat{y}) + \mathbb{E}[\widehat{g}\widehat{h}] - u_P^c(x) \stackrel{(3.5)}{\leq} v_P(\widehat{y}) + x\widehat{y} - u_P^c(x) = 0. \end{aligned} \quad (3.10)$$

Consequently,

$$V(\widehat{h}) + \widehat{g}\widehat{h} - U_c(\widehat{g}) = 0.$$

From (3.4) we have that

$$\widehat{g} \in -\partial V(\widehat{h}).$$

This is equivalent to

$$\widehat{X}_T \in -\partial V(\widehat{Y}_T).$$

Note also that (3.10) implies the equality $\mathbb{E}[\widehat{X}_T \widehat{Y}_T] = \mathbb{E}[\widehat{g}\widehat{h}] = x\widehat{y}$.

Since $\widehat{X}\widehat{Y}$ is a \mathbb{P} -supermartingale for all $X \in \mathcal{X}(x)$, it holds for all $s \leq T$ that

$$\mathbb{E}[\widehat{X}_T \widehat{Y}_T | \mathcal{F}_s] \leq \widehat{X}_s \widehat{Y}_s.$$

This leads to

$$\mathbb{E}[\widehat{X}_T \widehat{Y}_T] \leq \mathbb{E}[\widehat{X}_s \widehat{Y}_s] \leq \widehat{X}_0 \widehat{Y}_0 = x\widehat{y}.$$

Hence, all the above inequalities are in fact equalities, and for all $0 \leq s \leq T$,

$$\mathbb{E}[\widehat{X}_s \widehat{Y}_s] = \widehat{X}_0 \widehat{Y}_0 = x\widehat{y}. \quad (3.11)$$

Assume that there exists $A \in \mathcal{F}$ such that $\mathbb{P}(A) > 0$ and $\mathbb{E}[\widehat{X}_t \widehat{Y}_t | \mathcal{F}_s] < \widehat{X}_s \widehat{Y}_s$ on A for some $0 \leq s < t$. Then, $\mathbb{E}[\widehat{X}_t \widehat{Y}_t] < \mathbb{E}[\widehat{X}_s \widehat{Y}_s]$, contradicting (3.11). Hence, $\mathbb{E}[\widehat{X}_t \widehat{Y}_t | \mathcal{F}_s] = \widehat{X}_s \widehat{Y}_s$ \mathbb{P} -a.s. for all $0 \leq s < t$. \square

3.2.2 Optimal investments for the problem of maximization the robust concavified utility functional in incomplete market

In this subsection we present the optimal investment strategies for the maximization problem of the robust concavified utility function, i.e. $\sup_{X \in \mathfrak{X}(x)} \inf_{Q \in \mathcal{Q}} E_Q[U_c(X_T)]$.

This problem was already solved by authors of [37] in general incomplete market for a strictly increasing, strictly concave, continuously differentiable utility functions defined on $(0, \infty)$ and satisfying the Inada conditions. The concavified utility function U_c does not satisfy all the assumptions above. Nevertheless, U_c is strictly increasing, concave, continuously differentiable on the $(0, \infty)$, and it satisfies the Inada conditions at ∞ , see Lemma 2.19 and Lemma 2.20.

Moreover, we will assume that (3.6) holds, and hence, $V(y) \geq 0$ and $V_n(y) = \sup_{0 < x \leq n} (U_c(x) - xy) \geq 0$ for all $y > 0$.

In order to establish the results of this subsection, we will often refer to the article [37], so let us we present its notation.

First, consider the sets

$$\begin{aligned} \mathcal{Y}_Q(y) &= \{Y \geq 0 \mid Y_0 = y \text{ and } XY \text{ is a } Q\text{-supermartingale for all } X \in \mathfrak{X}(1)\}; \\ D_Q(y) &= \{h \in L_+^0(\Omega, \mathcal{F}_T, \mathbb{P}) \mid 0 \leq h \leq Y_T \text{ for some } Y \in \mathcal{Y}_Q(y)\}. \end{aligned}$$

Recall from Lemma 3.3 that

$$u^c(x) = \inf_{Q \in \mathcal{Q}_e} u_Q^c(x). \quad (3.12)$$

Denote the dual value function v_Q by

$$v_Q(y) := \inf_{Y \in \mathcal{Y}_Q(y)} E_Q[V(Y_T)].$$

It is easy to see that $v_Q(y) = \inf_{h \in D_Q(y)} E_Q[V(h)]$.

The dual value function of the robust problem is defined by

$$v(y) := \inf_{Q \in \mathcal{Q}_e} v_Q(y).$$

Assume that for all $Q \in \mathcal{Q}_e$,

$$u_Q^c(x) < \infty \text{ for some } x > 0. \quad (3.13)$$

Theorem 3.18. Fix arbitrary $Q \in \mathcal{Q}_e$. Suppose that the assumptions (3.13) and (3.6) hold.

For any $x_0 > 0$, there exist some $\widehat{g} \in C(x_0)$, $\widehat{y} > 0$ and $\widehat{h} \in D_Q(\widehat{y})$ such that

- (i) $u_Q^c(x_0) = v_Q(\widehat{y}) + x_0\widehat{y}$,
- (ii) $u_Q^c(x_0) = \inf_{y>0} (v_Q(y) + x_0y)$.

Inf additionally $v_Q(y) < \infty$ for all $y > 0$, then

- (iii) $v_Q(\widehat{y}) = E_Q[V(\widehat{h})]$,
- (iv) $u_Q^c(x_0) = E_Q[U_c(\widehat{g})]$.

Moreover, $v_Q(\cdot)$ is convex.

Proof. The proof is the same as that of [Theorem 3.16](#). □

We are now in the position to formulate a similar result for the robust problem. We want the dual value function to be finite, so we introduce the following assumption:

$$v_Q(y) < \infty \text{ for all } Q \in \mathcal{Q}_e \text{ and all } y > 0. \quad (3.14)$$

Theorem 3.19. Suppose that all assumptions of [Lemma 2.21](#) hold as well as (3.6) and (3.14).

For any $x_0 > 0$, there exist some $\widehat{Q} \in \mathcal{Q}_e$, $\widehat{y} > 0$ and $\widehat{h} \in D_{\widehat{Q}}(\widehat{y})$ such that

- (i) $u^c(x_0) = u_{\widehat{Q}}^c(x_0)$,
- (ii) $u^c(x_0) = \inf_{y>0} (v(y) + x_0y)$,
- (iii) $u^c(x_0) = v(\widehat{y}) + x_0\widehat{y}$,
- (iv) $v(\widehat{y}) = v_{\widehat{Q}}(\widehat{y}) = E_{\widehat{Q}}[V(\widehat{h})]$.

Proof. The proof of (i) follows from [Lemma 2.21](#) due to [Remark 2.22](#) and [Lemma 2.21](#).

Note that \widehat{Z} is the Radon–Nikodym density of \widehat{Q} , so it is an almost sure limit of a sequence $Z_n \in \mathcal{Z}_e$ with $u_{Q_n}^c(x_0) \rightarrow u^c(x_0)$, $n \rightarrow \infty$ (see proofs of [Theorem 2.23](#) or [[37](#), Lemma 4.1 (a)]). Furthermore, it follows from the first part of the proof of [[37](#), Lemma 4.1 (a)] that

$$u^{c'}(x_0^+) \leq \liminf_{n \rightarrow \infty} u_{Q_n}^{c'}(x_0^+) \leq \limsup_{n \rightarrow \infty} u_{Q_n}^{c'}(x_0^-) \leq u^{c'}(x_0^-). \quad (3.15)$$

From [Theorem 3.18](#) (ii) it follows that for all $Q \in \mathcal{Q}_e$ and $y > 0$,

$$u_Q^c(x_0) \leq v_Q(y) + x_0y.$$

Therefore,

$$\inf_{Q \in \mathcal{Q}_e} u_Q^c(x_0) \leq \inf_{Q \in \mathcal{Q}_e} (v_Q(y) + x_0 y).$$

Consequently, it follows from the (3.12) and the definition of dual value function for robust problem that

$$u^c(x_0) \leq v(y) + x_0 y.$$

Since $y > 0$ is arbitrary, we have that

$$u^c(x_0) \leq \inf_{y>0} (v(y) + x_0 y). \quad (3.16)$$

To prove the opposite inequality, consider the sequence Q_n such that

$$u^c(x_0) = \lim_{n \rightarrow \infty} u_{Q_n}^c(x_0).$$

From the Theorem 3.18 (i) follows that for each Q_n exists $\widehat{y}_n \in [u_{Q_n}^{c'}(x_0^+), u_{Q_n}^{c'}(x_0^-)]$ such that

$$u_{Q_n}^c(x_0) = v_{Q_n}(\widehat{y}_n) + x_0 \widehat{y}_n.$$

Taking a subsequence if necessary, we can assume that sequence (\widehat{y}_n) converges to some $\widehat{y} \in [u^{c'}(x_0^+), u^{c'}(x_0^-)]$, due to (3.15). Since u^c is strictly increasing, we have $\widehat{y} > 0$. Therefore, we have that

$$\begin{aligned} u^c(x_0) &= \lim_{n \rightarrow \infty} u_{Q_n}^c(x_0) = \lim_{n \rightarrow \infty} (v_{Q_n}(\widehat{y}_n) + x_0 \widehat{y}_n) \\ &\geq \limsup_{n \rightarrow \infty} (v(\widehat{y}_n) + x_0 \widehat{y}_n) \geq \inf_{y>0} (v(y) + x_0 y). \end{aligned} \quad (3.17)$$

The proof of (ii) is concluded by noting (3.16).

To prove (iii), we use the results of the [37]. Thanks to (3.6), it holds that $V(y) \geq 0$ as well as $V_n(y) \geq 0$. From the assumption (3.14) we obtain that $v(y) < \infty$. Hence, it follows from the [37, Lemma 3.7] that v is convex, thus it is continuous on $(0, \infty)$. Since \widehat{y}_n converges almost surely to $\widehat{y} > 0$, it follows from (3.17) that

$$u^c(x_0) = \lim_{n \rightarrow \infty} (v(\widehat{y}_n) + x_0 \widehat{y}_n) = v(\widehat{y}) + x_0 \widehat{y}. \quad (3.18)$$

Thus, it remains to show (iv). It follows from (3.17) and (3.18) that

$$\lim_{n \rightarrow \infty} v_{Q_n}(\widehat{y}_n) = v(\widehat{y}).$$

Theorem 3.18 (iii) implies the existence of $h'_n \in D_{Q_n}(\widehat{y}_n)$, such that

$$v_{Q_n}(\widehat{y}_n) = E_{Q_n}[V(h'_n)],$$

for all $n \in \mathbb{N}$. This is equivalent to the existence of $h_n \in D(\widehat{y}_n)$ such that

$$v_{Q_n}(\widehat{y}_n) = \mathbb{E} \left[Z_n V \left(\frac{h_n}{Z_n} \right) \right],$$

where $Z_n \in \mathcal{Z}_e$ are the densities corresponding to the measures Q_n .

The Kolmos–type argument (see [13, Lemma A1.1]) yields a sequence $(\widetilde{h}_n)_{n \in \mathbb{N}} \in \text{conv}(h_n, h_{n+1}, \dots)$ such that \widetilde{h}_n converges \mathbb{P} –a.s. to some \widetilde{h} .

Now we show that $\widetilde{h} \in D(\widehat{y})$. For each $n \in \mathbb{N}$, there is a sequence of $\alpha_i^n \in [0, 1]$ such that $\sum_i \alpha_i^n = 1$ and $\widetilde{h}_n = \sum_i \alpha_i^n h_{n+i-1}$. It follows from (3.5) that

$$h_n \geq 0 \text{ and } \sup_{g \in C(x)} \mathbb{E}[h_n g] \leq x \widehat{y}_n. \quad (3.19)$$

Consider the expectation $\mathbb{E}[\widetilde{h}g]$. For all $g \in C(x)$, we have that

$$\begin{aligned} \mathbb{E}[\widetilde{h}g] &= \mathbb{E} \left[\lim_{n \rightarrow \infty} \sum_i \alpha_i^n h_{n+i-1} g \right] \stackrel{\text{Fatou's lemma}}{\leq} \liminf_{n \rightarrow \infty} \sum_i \alpha_i^n \mathbb{E}[h_{n+i-1} g] \\ &\stackrel{(3.19)}{\leq} \liminf_{n \rightarrow \infty} \sum_i \alpha_i^n x \widehat{y}_{n+i-1} = x \liminf_{n \rightarrow \infty} \sum_i \alpha_i^n \widehat{y}_{n+i-1} \stackrel{\text{Proposition A.2}}{=} x \lim_{n \rightarrow \infty} \widehat{y}_n = x \widehat{y}. \end{aligned}$$

Thus, noting that $\widetilde{h} \geq 0$, we derive from (3.5) that $\widetilde{h} \in D(\widehat{y})$. Hence, the proof of [37, Lemma 3.7] yields that

$$v(\widehat{y}) = \mathbb{E} \left[\widehat{Z} V \left(\frac{\widetilde{h}}{\widehat{Z}} \right) \right].$$

Equivalently,

$$v(\widehat{y}) = \mathbb{E}[\widehat{Z} V(\widehat{h})] = E_{\widehat{Q}}[V(\widehat{h})] \quad (3.20)$$

for $\widehat{h} \in D_{\widehat{Q}}(\widehat{y})$.

Using the definitions, we obtain that

$$v(\widehat{y}) \leq v_{\widehat{Q}}(\widehat{y}) \leq E_{\widehat{Q}}[V(\widehat{h})],$$

for $\widehat{Q} \in \mathcal{Q}_e$, $\widehat{y} > 0$ and $\widehat{h} \in D_{\widehat{Q}}(\widehat{y})$. Thus, in view of (3.20), we arrive at (iv). \square

Theorem 3.20. *Suppose that all the assumptions of [Theorem 3.19](#) hold.*

Then, for any $x > 0$, there exist some $\widehat{X} \in \mathfrak{X}(x)$ and $\widehat{Q} \in \mathcal{Q}_e$ such that

$$u^c(x) = \inf_{Q \in \mathcal{Q}} E_Q[U_c(\widehat{X}_T)] = E_{\widehat{Q}}[U_c(\widehat{X}_T)] = u_{\widehat{Q}}^c(x). \quad (3.21)$$

There also exist some $\widehat{y} > 0$ and $\widehat{Y} \in \mathcal{Y}_{\widehat{Q}}(\widehat{y})$ such that

$$v(\widehat{y}) = E_{\widehat{Q}}[V(\widehat{Y}_T)] \text{ and } u^c(x) = v(\widehat{y}) + x\widehat{y}. \quad (3.22)$$

Moreover, $\widehat{X}_T \in -\partial V(\widehat{Y}_T)$. Also, $\widehat{X}\widehat{Y}$ is a \widehat{Q} -martingale.

Proof. The equalities (3.21) follow from [Remark 2.22](#) and [Lemma 2.21](#) for $\widehat{X}_T = \widehat{g}$. In turn, (3.22) follows from [Theorem 3.19](#) with $x_0 = x$ and $\widehat{Y}_T = \widehat{h}$.

To prove the remaining statement, consider, as in [[37](#), proof of [Theorem 2.6](#)], $V(\widehat{h}) + \widehat{g}\widehat{h} - U_c(\widehat{g})$. It follows from the definition of V that

$$V(\widehat{h}) + \widehat{g}\widehat{h} - U_c(\widehat{g}) \geq 0. \quad (3.23)$$

Since $\widehat{h} \in D_{\widehat{Q}}(\widehat{y})$, we obtain for $\widetilde{h} = \widehat{h}\widehat{Z} \in D(\widehat{y})$ that

$$E_{\widehat{Q}}[\widehat{g}\widehat{h}] = \mathbb{E} \left[\widehat{g}\widetilde{h} \mathbf{I}_{\{\widehat{Z} > 0\}} \right].$$

Hence,

$$\begin{aligned} E_{\widehat{Q}}[V(\widehat{h}) + \widehat{g}\widehat{h} - U_c(\widehat{g})] &= v(\widehat{y}) + \mathbb{E} \left[\widehat{g}\widetilde{h} \mathbf{I}_{\{\widehat{Z} > 0\}} \right] - u^c(x) \\ &\stackrel{(3.5)}{\leq} v(\widehat{y}) + x\widehat{y} - u^c(x) = 0. \end{aligned} \quad (3.24)$$

Thus, noting (3.23) obtain $V(\widehat{h}) + \widehat{g}\widehat{h} - U_c(\widehat{g}) = 0$. Using (3.4), we get that

$$\widehat{g} \in -\partial V(\widehat{h}),$$

equivalently,

$$\widehat{X}_T \in -\partial V(\widehat{Y}_T).$$

Thanks to (3.24), we have that $E_{\widehat{Q}}[\widehat{X}_T \widehat{Y}_T] = E_{\widehat{Q}}[\widehat{g}\widehat{h}] = x\widehat{y}$. As a result, the \widehat{Q} -martingale property of $\widehat{X}\widehat{Y}$ is proved as in [Theorem 3.17](#). \square

3.3 Optimal investments for the problem of maximization the robust non-concave utility functional in incomplete market

In this section, we present the conditions under which the optimal strategy for maximizing the concavified utility function is also the solution for maximizing the non-concave utility function. To derive them, we use Reichlin's result [34, Lemma 5.7].

The following theorem applies to both the standard utility maximization problem and the robust utility maximization problem.

Theorem 3.21. *Suppose that Assumption 2.2 and Assumption 3.1 hold.*

Additionally, assume that the optimal solution g^ for the concavified problem (either the standard problem: $\sup_{g \in C(x)} \mathbb{E}[U_c(g)]$, or the robust problem: $\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U_c(g)]$) satisfies $g^* \in -\partial V(\lambda Z)$, where $Z \in \mathcal{Z}$ has a continuous distribution.*

Then, g^ is also an optimal solution for the corresponding non-concave problem.*

Proof. For the standard utility maximization problem, the proof is the same as for the [34, Lemma 5.7].

To prove the statement for the robust utility maximization problem, start by observing that under the assumptions of theorem, [34, Lemma 5.7] yields that $\mathbb{P}[g^* \in \{U < U_c\}] = 0$. Consequently,

$$\inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g^*)] = \inf_{Q \in \mathcal{Q}_e} E_Q[U(g^*)]. \quad (3.25)$$

Since g^* is an optimal solution for the concavified problem, we get

$$\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U_c(g)] = \inf_{Q \in \mathcal{Q}} E_Q[U_c(g^*)].$$

Noting that $\mathcal{Q}_e \subseteq \mathcal{Q}$, we have that

$$\begin{aligned} \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U_c(g)] &= \inf_{Q \in \mathcal{Q}} E_Q[U_c(g^*)] \leq \inf_{Q \in \mathcal{Q}_e} E_Q[U_c(g^*)] \\ &\stackrel{(3.25)}{=} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g^*)] \leq \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)]. \end{aligned} \quad (3.26)$$

[Lemma 3.5](#) together with inequality $U \leq U_c$ imply that

$$\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U_c(g)] \geq \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(g)] = \sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[U(g)].$$

Therefore, all inequalities in (3.26) are equalities. Hence, [Lemma 3.4](#) leads to

$$\sup_{g \in C(x)} \inf_{Q \in \mathcal{Q}} E_Q[U(g)] = \inf_{Q \in \mathcal{Q}} E_Q[U(g^*)],$$

concluding the proof. □

Conclusions to Chapter 3

In this chapter, we have obtained results similar to those of [Chapter 2](#), but with a different approach due to the consideration of an incomplete market model.

We have presented a set of equalities and inequalities that establish the relationship between the robust utility functional of the initial utility function and its concavification. We have also derived the assumption under which the minimax identity holds for the initial utility function.

Using the duality theory approach, we have obtained optimal investment strategies. Specifically, we have presented an optimal strategy for the standard utility maximization problem and derived some of its properties. Additionally, we have proven a similar theorem for the robust maximization functional of the concavified utility function. Under certain conditions, we have demonstrated the existence of an optimal solution for the robust non-concave utility maximization problem and constructed it.

Conclusions

The dissertation work has yielded the following results.

In general:

- Studied the properties of concavified utility functions and investigated a particular case of concavification for a utility related to a call option.
- Investigated the minimax identity for non-concave utility functions and proved that, under certain natural assumptions, the standard utility functional of the initial utility function is equal to that of its concavification.

For the complete market model:

- Extended existing results to obtain the minimax identity for non-concave utility functions. Derived equalities and inequalities that establish the relationship between the robust utility functional of the initial utility function and its concavification. Presented cases where all the inequalities become equalities, leading to the minimax identity for the initial non-concave utility function.
- Demonstrated the existence of the “worst-case scenario” measure in two cases both for expected value of the initial utility function and its concavification.
- Proved the existence and uniqueness of the optimal solution for the standard non-concave utility maximization problem. Constructed the explicit form of the optimal solution under standard budget constraints. Showed that this solution is also a unique optimal solution for the maximization problem of the concavified utility function. Examined the case of implementing an additional upper bound given by a random variable, considering both discrete and continuous cases.
- Established the existence and uniqueness of the optimal solution for the robust non-concave utility maximization problem and presented a method for constructing the optimal solution.

For the incomplete market model:

- Obtained similar results to the complete market case, including equalities and inequalities relating the robust utility functional of the initial utility function and its concavification. Derived assumptions under which the minimax identity holds for the initial utility function.
- Employed the duality theory approach to obtain optimal investment strategies. Presented an optimal strategy for the standard utility maximization problem and derived its properties. Established a similar theorem for the robust maximization functional of the concavified utility function. Showed the existence of an optimal solution for the robust non-concave utility maximization problem and provided a construction method.

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Appendix A

Supplementary proofs and script

A.1 Proofs concerning the Model Setup

Proof of Proposition 2.3. Consider sequence $\{c_n\}_{n \in \mathbb{N}} \geq 0$, such that $\sum_{n \in \mathbb{N}} c_n = 1$ and $\{Q_n\}_{n \in \mathbb{N}} \in \mathcal{Q}$.

We need to show that

$$\tilde{Q} := \sum_{n \in \mathbb{N}} c_n Q_n \in \mathcal{Q}. \quad (\text{A.1})$$

Consider $\tilde{Z} = \frac{d\tilde{Q}}{dP}$ and $Z_n = \frac{dQ_n}{dP} \in \mathcal{Z}$. For all $A \in \mathcal{F}$, we have

$$\int_A \tilde{Z} dP = \int_A \frac{d\tilde{Q}}{dP} dP = \int_A \frac{d(\sum_{n \in \mathbb{N}} c_n Q_n)}{dP} dP = \sum_{n \in \mathbb{N}} c_n Q_n(A).$$

Moreover,

$$\begin{aligned} \int_A \sum_{n \in \mathbb{N}} c_n Z_n dP &= \int_A \sum_{n \in \mathbb{N}} c_n \frac{dQ_n}{dP} dP \\ &\stackrel{\text{monotone convergence theorem}}{=} \sum_{n \in \mathbb{N}} c_n \int_A \frac{dQ_n}{dP} dP = \sum_{n \in \mathbb{N}} c_n Q_n(A). \end{aligned}$$

Hence, for all $A \in \mathcal{F}$

$$\int_A \tilde{Z} dP = \int_A \sum_{n \in \mathbb{N}} c_n Z_n dP.$$

Thus, it follows that $\tilde{Z} = \sum_{n \in \mathbb{N}} c_n Z_n$ almost surely. Hence, (A.1) is equivalent to

$$\tilde{Z} = \sum_{n \in \mathbb{N}} c_n Z_n \in \mathcal{Z}.$$

To prove it consider another sequence $\tilde{Z}_n := \sum_{i \leq n} c_i^n Z_i$ such that $c_i^n = c_i$ for all $i < n$ and $c_n^n = 1 - \sum_{i < n} c_i^n$.

Since \mathcal{Q} is convex, it follows that \mathcal{Z} is convex. Hence, $\tilde{Z}_n \in \mathcal{Z}$ as a finite convex combination and $\lim_{n \rightarrow \infty} \tilde{Z}_n = \tilde{Z}$.

From [Assumption 2.2](#) (iii) follows the proof. □

Lemma A.1. *Under [Assumption 2.2](#) (i) and (iii), condition (ii) is equivalent to the assumption:*

$$Q \ll \mathbb{P} \text{ for all } Q \in \mathcal{Q} \text{ and } \mathcal{Q}_e \neq \emptyset. \quad (\text{A.2})$$

Proof. Assume [\(A.2\)](#) holds. The first part of the [\(A.2\)](#) is the only if part of [\(ii\)](#), i.e.

$$\mathbb{P}(A) = 0 \implies Q(A) = 0 \text{ for all } Q \in \mathcal{Q}.$$

If $Q(A) = 0$ for all $Q \in \mathcal{Q}$, then for $Q \in \mathcal{Q}_e$ holds $Q(A) = 0$. Hence, $\mathbb{P}(A) = 0$.

Assume [Assumption 2.2](#) (ii) holds. Note that [\(i\)](#) and [\(iii\)](#) implies by [Proposition 2.3](#) that the set \mathcal{Q} is closed under countable convex combinations.

It follows from [[19](#), Lemma 7] that \mathcal{Q} has an equivalent countable subset, we denote it by \mathcal{Q}_0 , i.e.

$$Q_0(A) = 0 \text{ for all } Q_0 \in \mathcal{Q}_0 \iff Q(A) = 0 \text{ for all } Q \in \mathcal{Q}.$$

Since set \mathcal{Q}_0 is countable, it means that $\mathcal{Q}_0 = \{Q_i\}_{i \in \mathbb{N}}$ let us consider measure

$$Q_c := \sum_{i \in \mathbb{N}} c_i \cdot Q_i, \text{ where } c_i \geq 0 \text{ and } \sum_{i \in \mathbb{N}} c_i = 1.$$

For all $A \in \mathcal{F}$ holds

$$\begin{aligned} Q_c(A) = 0 &\iff Q_i(A) = 0, \text{ for all } i \in \mathbb{N} \iff \\ &Q(A) = 0 \text{ for all } Q \in \mathcal{Q} \stackrel{\text{(ii)}}{\iff} \mathbb{P}(A) = 0. \end{aligned}$$

Noting that $Q_c \in \mathcal{Q}$, we have that $\mathcal{Q}_e \neq \emptyset$. □

A.2 Concavification for utility function U of the call option

Proof of Lemma 2.8. The (i) follows from the definition of \widehat{U} .

Noting that for $x > K$ function $\widehat{U}(x)$ is continuously differentiable, consider the derivative of $f(x)$:

$$f'(x) = \frac{\widehat{U}'(x) \cdot x - \widehat{U}(x) + U(0)}{x^2}, \quad x > K.$$

Due to Lemma 2.7 $f'(\widehat{x}) = 0$.

To prove (ii) we need to show that $f'(x) \geq 0$, for $x \in (K, \widehat{x})$, which is equivalent to

$$\widehat{U}'(x) \cdot x - \widehat{U}(x) + U(0) \geq 0, \quad \text{for } x \in (K, \widehat{x}). \quad (\text{A.3})$$

Let us fix $x_0 \in (K, \widehat{x})$ and consider a linear function $y(x)$, such that $y(0) = U(0)$ and $y(x_0) = \widehat{U}(x_0)$.

Hence,

$$y(x) = \frac{\widehat{U}(x_0) - U(0)}{x_0} \cdot x + U(0).$$

Due to the same argument as in Lemma 2.7 there exists $x_1 > \widehat{x}$ such that $y(x_1) = \widehat{U}(x_1)$. Hence, $\widehat{U}(x) > y(x)$, $x_0 < x < x_1$, because of the strict concavity of the function $\widehat{U}(x)$, for $x > K$.

Therefore,

$$\begin{aligned} \widehat{U}'(x_0) &= \lim_{x \rightarrow x_0^+} \frac{\widehat{U}(x) - \widehat{U}(x_0)}{x - x_0} = \lim_{x \rightarrow x_0^+} \frac{\widehat{U}(x) - y(x_0)}{x - x_0} \\ &\geq \lim_{x \rightarrow x_0^+} \frac{y(x) - y(x_0)}{x - x_0} = y'(x) = \frac{\widehat{U}(x_0) - U(0)}{x_0}. \end{aligned}$$

Thus, for all $x_0 \in (K, \widehat{x})$ the inequality (A.3) holds. Which gives us (ii).

The (iv) proves similarly. □

A.3 Useful statements

Proposition A.2. Consider a sequence $(h_n)_{n \in \mathbb{N}} \in \mathbb{R}$ such that exists finite $\lim_{n \rightarrow \infty} h_n$. Then, for a sequence $(\widehat{h}_n)_{n \in \mathbb{N}} \in \text{conv}(h_n, h_{n+1}, \dots)$ holds that $\lim_{n \rightarrow \infty} \widehat{h}_n = \lim_{n \rightarrow \infty} h_n$.

Proof. For each $n \in \mathbb{N}$ exist such a sequence $\alpha_i^n \in \mathbb{R}^+ : \sum_{i \in \mathbb{N}} \alpha_i^n = 1$ and $\widehat{h}_n = \sum_{i \in \mathbb{N}} \alpha_i^n h_{n+i-1}$.

It holds that

$$\sum_{i \in \mathbb{N}} \alpha_i^n h_{n+i-1} \geq \inf_{i \in \mathbb{N}} h_{n+i-1}. \quad (\text{A.4})$$

For all $\epsilon > 0$ exists such a number $N_0 \in \mathbb{N}$ that for all $n \geq N_0$, one has

$$h_n \geq \lim_{n \rightarrow \infty} h_n - \epsilon.$$

Hence, for all $n \geq N_0$ it holds that

$$\inf_{i \in \mathbb{N}} h_{n+i-1} \geq \lim_{n \rightarrow \infty} h_n - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we obtain

$$\liminf_{n \rightarrow \infty} \inf_{i \in \mathbb{N}} h_{n+i-1} \geq \lim_{n \rightarrow \infty} h_n.$$

Thus, noting (A.4)

$$\liminf_{n \rightarrow \infty} \widehat{h}_n \geq \lim_{n \rightarrow \infty} h_n. \quad (\text{A.5})$$

Now, we are going to use similar arguments for the lim sup . It holds that

$$\sum_{i \in \mathbb{N}} \alpha_i^n h_{n+i-1} \leq \sup_{i \in \mathbb{N}} h_{n+i-1}. \quad (\text{A.6})$$

For all $\epsilon > 0$ exists such a number $N \in \mathbb{N}$ that for all $n \geq N$

$$h_n \leq \lim_{n \rightarrow \infty} h_n + \epsilon.$$

Hence, for all $n \geq N$

$$\sup_{i \in \mathbb{N}} h_{n+i-1} \leq \lim_{n \rightarrow \infty} h_n + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, one has

$$\limsup_{n \rightarrow \infty} \sup_{i \in \mathbb{N}} h_{n+i-1} \leq \lim_{n \rightarrow \infty} h_n.$$

Thus, due to (A.6)

$$\limsup_{n \rightarrow \infty} \widehat{h}_n \leq \lim_{n \rightarrow \infty} h_n. \quad (\text{A.7})$$

From (A.5) and (A.7) follows the proof. \square

A.4 Properties for robust utility functionals

Proof of Lemma 2.24. Let us prove (i). The proof for (ii) is the same.

First, note that for all $g \in C(x)$

$$\inf_{Q \in \mathcal{Q}_e} E_Q[U(g)] = \inf_{Z \in \mathcal{Z}_e} \mathbb{E}[ZU(g)].$$

From the definition of infimum follows that exists such a sequence $(Z_n)_{n \in \mathbb{N}} \in \mathcal{Z}_e$ that

$$\lim_{n \rightarrow \infty} \mathbb{E}[Z_n U(g)] = \inf_{Z \in \mathcal{Z}_e} \mathbb{E}[ZU(g)].$$

Using the Kolmos-type argument [13, Lemma A1.1] there exists a sequence $(\widehat{Z}_n)_{n \in \mathbb{N}} \in \text{conv}(Z_n, Z_{n+1}, \dots)$ such that \widehat{Z}_n converges a.s to some \widehat{Z} .

Moreover, $(\widehat{Z}_n)_{n \in \mathbb{N}} \subset \mathcal{Z}_e$, because $(Z_n)_{n \in \mathbb{N}} \subset \mathcal{Z}_e \subset \mathcal{Z}$. Proposition 2.3 implies that $(\widehat{Z}_n)_{n \in \mathbb{N}} \subset \mathcal{Z}$ and also for all integer $n : \widehat{Z}_n \approx \mathbb{P}$.

From Assumption 2.2 (iv) follows that $\widehat{Z} \in \mathcal{Z}_e$.

For each $n \in \mathbb{N}$ exist such a sequence $\alpha_i^n \in \mathbb{R}^+$ that $\sum_{i \in \mathbb{N}} \alpha_i^n = 1$ and $\widehat{Z}_n = \sum_{i \in \mathbb{N}} \alpha_i^n Z_{n+i-1}$.

Thus,

$$\limsup_{n \rightarrow \infty} \mathbb{E}[\widehat{Z}_n U(g)] = \limsup_{n \rightarrow \infty} \sum_{i \in \mathbb{N}} \alpha_i^n \mathbb{E}[Z_{n+i-1} U(g)] = \inf_{Z \in \mathcal{Z}_e} \mathbb{E}[ZU(g)].$$

Where the last equality holds due to the Proposition A.2. Note, that from Assumption 2.11 follows that for all $g \in C(x)$ holds $\inf_{Z \in \mathcal{Z}_e} \mathbb{E}[ZU(g)] < \infty$.

Moreover,

$$\liminf_{n \rightarrow \infty} \mathbb{E}[\widehat{Z}_n U(g)] \geq \inf_{Z \in \mathcal{Z}_e} \mathbb{E}[ZU(g)].$$

Thus,

$$\inf_{Z \in \mathcal{Z}_e} \mathbb{E}[ZU(g)] = \lim_{n \rightarrow \infty} \mathbb{E}[\widehat{Z}_n U(g)]. \quad (\text{A.8})$$

As before, $U(g)$ is bounded from below, hence, it follows that exists $c > 0$ such that $U(g) + c > 0$. From Fatou's lemma follows that

$$\lim_{n \rightarrow \infty} \mathbb{E}[\widehat{Z}_n (U(g) + c)] \geq \mathbb{E} \left[\liminf_{n \rightarrow \infty} \widehat{Z}_n (U(g) + c) \right].$$

Furthermore,

$$\begin{aligned}\liminf_{n \rightarrow \infty} \mathbb{E} \left[\widehat{Z}_n \cdot c \right] &= c, \\ \mathbb{E} \left[\liminf_{n \rightarrow \infty} \widehat{Z}_n \cdot c \right] &= c.\end{aligned}$$

Hence, one has

$$\inf_{Z \in \mathcal{Z}_e} \mathbb{E} [ZU(g)] \stackrel{(A.8)}{=} \lim_{n \rightarrow \infty} \mathbb{E} [\widehat{Z}_n U(g)] \geq \mathbb{E} \left[\liminf_{n \rightarrow \infty} \widehat{Z}_n U(g) \right] = \mathbb{E} [\widehat{Z}U(g)].$$

Also, holds

$$\inf_{Z \in \mathcal{Z}_e} \mathbb{E} [ZU(g)] \leq \mathbb{E} [\widehat{Z}U(g)] = E_{\widehat{Q}}[U(g)].$$

Which concludes the proof. □

A.5 Auxiliary statements for the constrained case of Section 2.3

In what follows $U: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a non-decreasing upper-semicontinuous function satisfying a mild growth condition, $U^v(y) = U(y \wedge v)$, $v > 0$, and U_c^v is the concavification of U^v . For $v > y > 0$, let

$$a(v, y) = \begin{cases} \inf\{z \leq y : U_c^v(x) > U^v(x) \text{ on } [z, y]\}, & U^v(y) < U_c^v(y), \\ y, & U^v(y) = U_c^v(y) \end{cases}$$

and

$$b(v, y) = \begin{cases} \sup\{z \leq y : U_c^v(x) > U^v(x) \text{ on } [y, z]\}, & U^v(y) < U_c^v(y), \\ y, & U^v(y) = U_c^v(y) \end{cases}$$

be the left and right endpoints of the interval around y , in which where $U^v < U_c^v$ (or just y in the case where $U^v(y) = U_c^v(y)$). Observe that $U^v(a(v, y)) = U_c^v(a(v, y))$ and $U^v(b(v, y)) = U_c^v(b(v, y))$: in the case of inequality we would have it in some open interval, contradicting the definition of infimum or supremum.

Lemma A.3. *The functions a and b defined above are measurable.*

Proof. We will show only measurability of a , that of b can be shown similarly.

Note that a is obviously non-decreasing in y . It is also right-continuous in y . Indeed, let $y_n \geq y_0$, $y_n \rightarrow y_0$, $n \rightarrow \infty$. If $U^v(y_0) < U_c^v(y_0)$, then, thanks to continuity of U_c^v and upper-semicontinuity of U^v , this inequality holds in an open interval around y_0 , which means that $a(v, y_n) = a(v, y_0)$ for all n large enough. Otherwise, if $U^v(y_0) = U_c^v(y_0)$, then $a(v, y_n) \in (y_0, y_n]$ for all $n \geq 1$, whence $a(v, y_n) \rightarrow y_0 = a(v, y_0)$, $n \rightarrow \infty$.

Further, since for $v_1 < v_2$, $U_c^{v_2}$ dominates U^{v_1} on $[0, v_1]$, we have that $U_c^{v_2} \geq U_c^{v_1}$. Consequently, a is non-increasing in v . Now the proof follows from the following lemma.

Lemma A.4. *Let a function $f: (0, \infty)^2 \rightarrow \mathbb{R}$ be such that for each $x > 0$, $f(x, \cdot)$ is non-decreasing and right-continuous, and for each $y > 0$, $f(\cdot, y)$ is non-decreasing. Then, f is measurable.*

Proof. For arbitrary $t \in \mathbb{R}$ consider the set $A_t = f^{-1}((-\infty, t))$. Thanks to monotonicity, $(x, y) \in A_t \Rightarrow (x', y') \in A_t$ for all $x' \leq x, y' \leq y$. Moreover, thanks to right-continuity in y , the x -sections $A_{t,x} = \{y > 0 : (x, y) \in A_t\}$ are open intervals.

Define $A_{t,x+} = \bigcup_{z>x} A_{t,z}$. We claim that the set $A_t^o := \{(x, y) \in (0, \infty)^2 : y \in A_{t,x+}\}$ is open (it is actually the interior of A_t). Indeed, if $(x, y) \in A_t^o$, then $y \in A_{t,z}$ for some $z > x$. Since $A_{t,z}$ is open, for some $\varepsilon > 0$, $(y - \varepsilon, y + \varepsilon) \subset A_{t,z}$. Then, thanks to monotonicity, $(0, z) \times (y - \varepsilon, y + \varepsilon) \subset A_t^o$.

By the definition of A_t^o ,

$$A_t \setminus A_t^o = \bigcup_{x>0} \{x\} \times (A_{t,x} \setminus A_{t,x+}).$$

For any $x > 0$, $A_{t,x} \setminus A_{t,x+}$ is a difference of two open intervals, so it's either a half-open interval or empty. Since the half-open intervals for different x are disjoint, there are at most countable number of them. Therefore, $A_t \setminus A_t^o$ is Borel as a countable union of Borel sets, which finishes the proof. \square

Lemma A.5. *Let $\{P_v, v \in (0, \infty)\}$ be a family of atomless probability measures on a standard Borel space (Ω, \mathcal{F}) , such that for any $A \in \mathcal{F}$, $P_v(A)$ is a measurable, and $\xi: (0, \infty) \times \Omega \rightarrow \mathbb{R}$ be measurable. Then, there exist measurable functions*

$\zeta: (0, \infty) \times \Omega \rightarrow \mathbb{R}$ and $q: (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ such that for all $v \in (0, \infty)$, $\zeta(v, \cdot)$ has a uniform distribution on $(0, 1)$ with respect to P_v , $q(v, \cdot)$ is non-decreasing, and $q(v, \zeta(v, \omega)) = \xi(v, \omega)$ P_v -a.s.

Proof. Since (Ω, \mathcal{F}) carries an atomless measure, it is uncountable. Then it is well known that it is isomorphic to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, i.e. there exists a measurable bijection $\tau: \Omega \rightarrow \mathbb{R}$ such that τ^{-1} is measurable as well. Therefore we can assume without loss of generality that $(\Omega, \mathcal{F}) = ((0, 1), \mathcal{B}((0, 1)))$.

Assume first that the distribution of $\xi(v, \omega)$ is continuous for all $v \in (0, \infty)$. The cumulative distribution function $F_\xi(v, x) = P_v(\{\xi(v, \omega) \leq x\})$ is jointly measurable (see e.g. [39, Lemma 4.1]), so the quantile function $q_\xi(v, r) = \inf\{x \in \mathbb{R} : F_\xi(v, x) \geq r\}$ is jointly measurable as well. So in this case we can set $\zeta(v, \omega) = F_\xi(v, \xi(v, \omega))$ and $q(v, r) = q_\xi(v, r)$; by the quantile transformation theorem, ζ and q are as required.

For general ξ , define

$$\kappa(v, x) = P_v(\{\omega : \xi(v, \omega) < x\}) + P_v(\{\omega \leq x : \xi(v, \omega) = \xi(v, x)\}), x \in (0, 1),$$

which is jointly measurable thanks to [39, Lemma 4.1]. It is easy to see that for any $v \in (0, \infty)$, κ has continuous distribution under P_v , and

$$q_\xi(v, \kappa(v, \omega)) = \xi(v, \omega),$$

where, as above, q_ξ is the quantile function of ξ . Then we can set $\zeta(v, \omega) = F_\kappa(v, \kappa(v, \omega))$ and $q(v, r) = q_\xi(v, q_\kappa(v, r))$, arriving at the desired statement. \square

Lemma A.6. *Let $\{P_v, v \in (0, \infty)\}$ be a family of atomless probability measures on a standard Borel space (Ω, \mathcal{F}) , such that for any $A \in \mathcal{F}$, $P_v(A)$ is measurable. Also let $Y, \phi: (0, \infty) \times \Omega \rightarrow [0, \infty)$ be jointly measurable functions such that $Y(v, \omega) \leq v$ for all $v > 0, \omega \in \Omega$. Then, there exists a jointly measurable function $Y^*(v, \omega)$ such that for all $v > 0$, $E_{P_v}[Y^*(v, \omega)\phi(v, \omega)] \leq E_{P_v}[Y(v, \omega)\phi(v, \omega)]$ and*

$$E_{P_v}[U^v(Y^*(v, \omega))] = E_{P_v}[U_c^v(Y^*(v, \omega))] = E_{P_v}[U_c^v(Y(v, \omega))].$$

Proof. We will adapt the construction used in the proof of [34, Proposition 5.3] so that it has the desired measurability property.

Define

$$S = \{(v, \omega) \in (0, \infty) \times \Omega : U^v(Y(v, \omega)) < U_c^v(Y(v, \omega))\}$$

and for $(v, \omega) \in S$, let

$$\begin{aligned}\alpha(v, \omega) &:= \inf\{z : U^v(x) < U_c^v(x) \text{ on } (z, X(\omega)]\}, \\ \beta(v, \omega) &:= \sup\{z : U^v(x) < U_c^v(x) \text{ on } [X(\omega), z)\}.\end{aligned}$$

be the left and the right ends of the interval where $U^v < U_c^v$. These functions are measurable thanks to [Lemma A.3](#).

For $(v, \omega) \in S$, define

$$\lambda(v, \omega) = \frac{\beta(v, \omega) - X(\omega)}{\beta(v, \omega) - \alpha(v, \omega)}$$

so that $X(v, \omega) = \lambda(v, \omega)\alpha(v, \omega) + (1 - \lambda(v, \omega))\beta(v, \omega)$. Due to [Lemma A.5](#), there exist measurable functions $\zeta, q: (0, \infty) \times \Omega \rightarrow \mathbb{R}$ such that for all $v > 0$, $\phi(v, \omega) = q(v, \zeta(v, \omega))$ P_v -a.s. and $\zeta(v, \omega)$ is uniformly distributed on $(0, 1)$ under P_v . For $s \in [0, 1]$, $v > 0$ and $\omega, \omega' \in \Omega$, define

$$h(s, v, \omega, \omega') = \mathbb{I}_{(v, X(\omega)) \in S, (v, X(\omega')) \in S, a(v, X(\omega)) = a(v, X(\omega'))} (\mathbb{I}_{\zeta(v, \omega') < s} - \lambda(v, \omega'))$$

and

$$f(s, v, \omega) = \int_{\Omega} h(v, \omega, \omega', s) P_v(d\omega').$$

Since $\lambda(v, \omega) \in (0, 1)$ and ζ has continuous distribution under P_v , we have that for all $(v, \omega) \in S$, f is continuous in s and $f(0, v, \omega) < 0 < f(1, v, \omega)$. Denoting $\sigma(v, \omega) = \inf\{s \in (0, 1) : f(s, v, \omega) \geq 0\}$, we have $f(\sigma(v, \omega), v, \omega) = 0$. Also for any $s \in (0, 1)$, $\{(v, \omega) : \sigma(v, \omega) \leq s\} = \{(v, \omega) : f(s, v, \omega) \geq 0\}$, so $\sigma(v, \omega)$ is measurable.

Now set

$$Y^*(v, \omega) = \begin{cases} Y(v, \omega), & (v, \omega) \notin S; \\ \alpha(v, \omega), & (v, \omega) \in S \cap \{\zeta(v, \omega) < \sigma(v, \omega)\}; \\ \beta(v, \omega), & (v, \omega) \in S \cap \{\zeta(v, \omega) \geq \sigma(v, \omega)\}. \end{cases}$$

Since for any fixed $v > 0$, the construction coincides with that given in [[34](#), Proposition 5.3], the rest of the proof follows. \square

A.6 Python script for the [Example 2.28](#)

```

import numpy as np
from scipy import optimize as opt
Q_0 = 0.1          #lower bound
Q_1 = 0.9          #higher bound
options = {'maxiter':1000, 'disp':True}

#determine the utility function
def utility(x):
return -np.exp(max(x-0.6,0))

#minus expectation of the utility function
def expectation(x, params):
q, x_0, y_b, y_g, func, r = params

#q-probability of good state of economics
#x_0 - initial capital,
#y_b -growth rate at a bad state of economy,
#y_g -growth rate at a good state of economy,
# func - utility function,
#r - fixed interest rate.

x1, x20, x21 = x

#x_1 - capital that we put in the stock at the time t=0,
#x20 - capital that we put in the stock at the time t=1,
#if at time t=1 economy was in bad state,
#x21 - capital that we put in the stock at the time t=1,
# if at time t=1 economy was in a good state.

s=\
func(((x_0-x1)*(1+r)+(1+y_b)*x1-x20)*(1+r)+(1+y_b)*x20)
*((1-q)**2)+\
func(((x_0-x1)*(1+r)+(1+y_g)*x1-x21)*(1+r)+(1+y_b)*x21)

```

```

*q*(1-q)+\
func(((x_0-x1)*(1+r)+(1+y_b)*x1-x20)*(1+r)+(1+y_g)*x20)
*q*(1-q)+\
func(((x_0-x1)*(1+r)+(1+y_g)*x1-x21)*(1+r)+(1+y_g)*x21)
*(q**2)
return -s # minus expectation of the utility function

#sup of expectation
def sup1(q, params):
paramss = [q] + params
constraints = [{'type': 'ineq', 'fun': const_1,
'args': paramss},
{'type': 'ineq', 'fun': const_2, 'args': paramss},
{'type': 'ineq', 'fun': const_3, 'args': paramss},
{'type': 'ineq', 'fun': const_4, 'args': paramss},
{'type': 'ineq', 'fun': const_5, 'args': paramss},
{'type': 'ineq', 'fun': const_6, 'args': paramss}
]
res = opt.minimize(expectation, [0,0,0], args=paramss,
method='COBYLA', constraints=constraints, options=options)
ut_q = -res.fun
return ut_q

def const_1(x, q, x_0, y_b, y_g, func, r): #constraints
x1, x20, x21 = x
s = (x_0 - x1) * (1 + r) + (1 + y_b) * x1
return s

def const_2(x, q, x_0, y_b, y_g, func, r): #constraints
x1, x20, x21 = x
s = (x_0 - x1) * (1 + r) + (1 + y_g) * x1
return s

```

```

def const_3(x, q, x_0, y_b, y_g, func, r): #constraints
x1, x20, x21 = x
s = ((x_0 - x1)*(1 + r)+(1+y_b)*x1 - x20)*(1 + r)
+(1+y_b)*x20
return s

def const_4(x, q, x_0, y_b, y_g, func, r): #constraints
x1, x20, x21 = x
s = ((x_0-x1)*(1 + r)+(1+y_g)*x1 - x21)*(1 + r)
+(1+y_b)*x21
return s

def const_5(x, q, x_0, y_b, y_g, func, r): #constraints
x1, x20, x21 = x
s = ((x_0-x1)*(1 + r)+(1+y_b)*x1 - x20)*(1 + r)
+(1+y_g)*x20
return s

def const_6(x, q, x_0, y_b, y_g, func, r): #constraints
x1, x20, x21 = x
s = ((x_0-x1)*(1 + r)+(1+y_g)*x1-x21)*(1 + r)
+(1+y_g)*x21
return s

params = [0.9, 1, -0.5, 0.2, utility, 0]
params2 = [1, -0.5, 0.2, utility, 0]

constraints = [{'type': 'ineq', 'fun': const_1, 'args': params},
{'type': 'ineq', 'fun': const_2, 'args': params},
{'type': 'ineq', 'fun': const_3, 'args': params},
{'type': 'ineq', 'fun': const_4, 'args': params},
{'type': 'ineq', 'fun': const_5, 'args': params},
{'type': 'ineq', 'fun': const_6, 'args': params}]

```

```

]
constraints2 = [{'type': 'ineq', 'fun': lambda x:(x-Q_0)},
{'type': 'ineq', 'fun': lambda x:(Q_1-x)}
]

#infsup of expectation
res1 = opt.minimize(sup1, 0.5, args=params2, method='COBYLA',
constraints=constraints2, options=options)

#similarly to the previous expectation
def expectation2(q, params):
x, x_0, y_b, y_g, func, r = params
x1, x20, x21 = x
s=\
func(((x_0-x1)*(1+r)+(1+y_b)*x1-x20)*(1+r)+(1+y_b)*x20)
*((1-q)**2)+\
func(((x_0-x1)*(1+r)+(1+y_g)*x1-x21)*(1+r)+(1+y_b)*x21)
*q*(1-q)+\
func(((x_0-x1)*(1+r)+(1+y_b)*x1-x20)*(1+r)+(1+y_g)*x20)
*q*(1-q)+\
func(((x_0-x1)*(1+r)+(1+y_g)*x1-x21)*(1+r)+(1+y_g)*x21)
*(q**2)
return s

# minus infimum of expectation
def inf1(x, params):
paramss = [x] + params
constraints2 = [{'type': 'ineq', 'fun': lambda x:(x-Q_0)},
{'type': 'ineq', 'fun': lambda x:(Q_1-x)}
]
res = opt.minimize(expectation2, 0.5, args=paramss,
method='COBYLA',
constraints=constraints2, options=options)

```

```

return -res.fun

params3 = [1, -0.5, 0.2, utility, 0]

# -supinf of expectation
res2 = opt.minimize(inf1, [0,0,0], args=params3,
method='COBYLA',
constraints=constraints, options=options)

# difference between RHS and LHS of (4.18)
res1.fun+res2.fun

```

A.7 Properties of concavification U_c in the setup of Section 2.4

The next lemma is certainly well-known, but for the convenience, we present it here.

Lemma A.7. *For the concave function U_c , such that it is differentiable in $y \in \mathbb{R}$ and for all $x \in \mathbb{R}$ holds*

$$U_c(x) - U_c(y) \leq U'_c(y)(x - y). \quad (\text{A.9})$$

Proof. Consider three cases:

- If $x = y$, then it is obvious.
- If $x > y$, then (A.9) is equivalent to

$$\frac{U_c(x) - U_c(y)}{x - y} \leq U'_c(y) = \lim_{z \downarrow y} \frac{U_c(z) - U_c(y)}{z - y}.$$

To prove this we will show that function $\frac{U_c(z) - U_c(y)}{z - y}$ is decreasing with respect to z , for $z > y$. Hence, we need to show that for all $y < z_1 \leq z_2$ the following holds

$$\frac{U_c(z_1) - U_c(y)}{z_1 - y} \geq \frac{U_c(z_2) - U_c(y)}{z_2 - y}.$$

This is equivalent to

$$\begin{aligned} & z_2 U_c(z_1) - y U_c(z_1) - z_2 U_c(y) + y U_c(y) \\ & \geq z_1 U_c(z_2) - y U_c(z_2) - z_1 U_c(y) + y U_c(y). \end{aligned}$$

Hence, we need to prove

$$(z_2 - y)U_c(z_1) \geq (z_1 - y)U_c(z_2) + (z_2 - z_1)U_c(y). \quad (\text{A.10})$$

Since $y < z_1 \leq z_2$, there exist such $0 \leq \lambda := \frac{z_2 - z_1}{z_2 - y} < 1$ that $z_1 = \lambda y + (1 - \lambda)z_2$.

Then

$$\frac{z_2 - z_1}{z_2 - y} = \lambda; \quad \frac{z_1 - y}{z_2 - y} = 1 - \lambda.$$

Thus, (A.10) is equivalent to

$$U_c(\lambda y + (1 - \lambda)z_2) \geq (1 - \lambda)U_c(z_2) + \lambda U_c(y), \quad (\text{A.11})$$

which is true due to the concavity of U_c .

- If $x < y$, then (A.9) is equivalent to

$$\frac{U_c(x) - U_c(y)}{x - y} \geq U'_c(y) = \lim_{z \uparrow y} \frac{U_c(z) - U_c(y)}{z - y}.$$

To prove this we will show that function $\frac{U_c(z) - U_c(y)}{z - y}$ is decreasing with respect to z , for $z < y$. Hence, we need to show that for all $z_2 \leq z_1 < y$ the following holds

$$\frac{U_c(z_1) - U_c(y)}{z_1 - y} \leq \frac{U_c(z_2) - U_c(y)}{z_2 - y}.$$

This is equivalent to

$$\begin{aligned} & z_2 U_c(z_1) - y U_c(z_1) - z_2 U_c(y) + y U_c(y) \\ & \leq z_1 U_c(z_2) - y U_c(z_2) - z_1 U_c(y) + y U_c(y). \end{aligned}$$

Hence, we need to prove

$$(z_2 - y)U_c(z_1) \leq (z_1 - y)U_c(z_2) + (z_2 - z_1)U_c(y). \quad (\text{A.12})$$

Since $z_2 \leq z_1 < y$, there exist such $0 \leq \lambda := \frac{z_1 - z_2}{y - z_2} < 1$ that $z_1 = \lambda y + (1 - \lambda)z_2$. Therefore,

$$\frac{z_2 - z_1}{z_2 - y} = \lambda; \quad \frac{z_1 - y}{z_2 - y} = 1 - \lambda.$$

Thus, (A.12) is equivalent to

$$U_c(\lambda y + (1 - \lambda)z_2) \geq (1 - \lambda)U_c(z_2) + \lambda U_c(y), \quad (\text{A.13})$$

which is true due to the concavity of U_c . □

Lemma A.8. *Consider the concave function U_c , such that it is differentiable in $y \in \mathbb{R}$ and strictly concave for some open interval containing y , then for all $x \in \mathbb{R}$ such that $x \neq y$ holds*

$$U_c(x) - U_c(y) < U'_c(y)(x - y). \quad (\text{A.14})$$

Proof. Consider two cases:

- If $x > y$, then (A.14) is equivalent to

$$\frac{U_c(x) - U_c(y)}{x - y} < U'_c(y) = \lim_{z \downarrow y} \frac{U_c(z) - U_c(y)}{z - y}.$$

From assumptions of the lemma follows that there is such an $\varepsilon > 0$ that $x > y + \varepsilon > y$ and function U_c is strictly concave on the interval $(y, y + \varepsilon)$.

From the proof of [Lemma A.7](#) follows that

$$\frac{U_c(x) - U_c(y)}{x - y} \leq \frac{U_c(y + \varepsilon) - U_c(y)}{\varepsilon}.$$

Now, it is enough to prove that

$$\frac{U_c(y + \varepsilon) - U_c(y)}{\varepsilon} < U'_c(y) = \lim_{z \downarrow y} \frac{U_c(z) - U_c(y)}{z - y}.$$

This is true because for all $y < z_1 < z_2 \leq y + \varepsilon$ we have

$$\frac{U_c(z_1) - U_c(y)}{z_1 - y} > \frac{U_c(z_2) - U_c(y)}{z_2 - y},$$

since U_c is strictly concave in the interval $(y, y + \varepsilon)$ and using the same arguments as in [Lemma A.7](#).

- If $x < y$, then the proof is similar to the previous case with interval $(y - \varepsilon, y)$. \square

Proof of Lemma 2.20. Since U is strictly increasing, it follows that U_c is also strictly increasing. Since U_c is locally affine on the set $\{U < U_c\}$ it is continuously differentiable there. It is also continuously differentiable in the interior of the set $\{U = U_c\}$, since U is. So we have to show that $U_c(\cdot)$ is continuously differentiable at the points a_i and b_i (excluding a_1 if it equals 0). Fix i . First, we show that derivative exists in the point a_i . The left derivative of the U_c in the point a_i coincides with the left derivative of the U in the point a_i . Thus, $U'_c(a_i^-) = U'(a_i)$.

On the other hand, we know that the function U_c is linear from a_i to b_i .

Hence,

$$U_c(x) = \frac{U(b_i) - U(a_i)}{b_i - a_i}(x - a_i) + U(a_i), \text{ for } x \in [a_i, b_i].$$

So, we need to show that

$$U'_c(a_i^+) = \frac{U(b_i) - U(a_i)}{b_i - a_i} = U'(a_i).$$

It means that the linear function $y(x) := \frac{U(b_i) - U(a_i)}{b_i - a_i}(x - a_i) + U(a_i)$ is tangent to the U in the point a_i . Lets prove that $y(x) \geq U(x)$, $x \in (b_{i-1}, b_i]$.

Remark. By b_0 we denote point 0.

It is known that $y(x) = U_c(x) \geq U(x)$, $x \in [a_i, b_i]$. Assume that exists such a point $x_i \in (b_{i-1}, a_i)$, that

$$y(x_i) = \frac{U(b_i) - U(a_i)}{b_i - a_i}(x_i - a_i) + U(a_i) < U(x_i). \quad (\text{A.15})$$

Consider, $0 < \lambda = \frac{a_i - x_i}{b_i - x_i} < 1$.

Then,

$$\begin{aligned} U_c(a_i) &= U_c(\lambda b_i + (1 - \lambda)x_i) \geq \lambda U_c(b_i) + (1 - \lambda)U_c(x_i) \\ &\geq \lambda U(b_i) + (1 - \lambda)U(x_i) \stackrel{(\text{A.15})}{>} \lambda U(b_i) + (1 - \lambda) \cdot \left(\frac{U(b_i) - U(a_i)}{b_i - a_i}(x_i - a_i) + U(a_i) \right). \end{aligned}$$

Noting that $U_c(a_i) = U(a_i)$ and plugging in the $\lambda = \frac{a_i - x_i}{b_i - x_i}$, one has

$$U(a_i) > \frac{a_i - x_i}{b_i - x_i} \cdot U(b_i) + \frac{b_i - a_i}{b_i - x_i} \cdot \left(\frac{U(b_i) - U(a_i)}{b_i - a_i}(x_i - a_i) + U(a_i) \right).$$

Which is equivalent to

$$U(a_i) > \frac{a_i - x_i}{b_i - x_i} \cdot U(a_i) + \frac{b_i - a_i}{b_i - x_i} \cdot U(a_i) = U(a_i).$$

Contradiction. Hence, $y(x) \geq U(x)$, $x \in (b_{i-1}, b_i]$.

Thus,

$$\begin{aligned} U'(a_i^+) &= \lim_{a \downarrow a_i} \frac{U(a) - U(a_i)}{a - a_i} \leq \lim_{a \downarrow a_i} \frac{y(a) - U(a_i)}{a - a_i} \\ &= \lim_{a \downarrow a_i} \frac{y(a) - y(a_i)}{a - a_i} = y'(a_i) = \frac{U(b_i) - U(a_i)}{b_i - a_i}; \\ U'(a_i^-) &= \lim_{a \uparrow a_i} \frac{U(a_i) - U(a)}{a_i - a} \geq \lim_{a \uparrow a_i} \frac{U(a_i) - y(a)}{a_i - a} \\ &= \lim_{a \uparrow a_i} \frac{y(a_i) - y(a)}{a_i - a} = y'(a_i) = \frac{U(b_i) - U(a_i)}{b_i - a_i}. \end{aligned}$$

Which gives us

$$U'_c(a_i) = \frac{U(b_i) - U(a_i)}{b_i - a_i} = U'(a_i).$$

The proof for point b_i is similar.

The right derivative of the U_c in the point b_i coincides with the right derivative of the U in the point b_i . Thus, $U'_c(b_i^+) = U'(b_i)$.

On the other hand, we know that the function U_c is linear from a_i to b_i .

Hence,

$$U_c(x) = \frac{U(b_i) - U(a_i)}{b_i - a_i} (x - b_i) + U(b_i), \text{ for } x \in [a_i, b_i].$$

So, we need to show that

$$U'_c(b_i^-) = \frac{U(b_i) - U(a_i)}{b_i - a_i} = U'(b_i).$$

It means that the linear function $y(x) := \frac{U(b_i) - U(a_i)}{b_i - a_i} (x - b_i) + U(b_i)$ is tangent to the U in the point b_i . Lets prove that $y(x) \geq U(x)$, $x \in [a_i, a_{i+1})$. We already know that $y(x) = U_c(x) \geq U(x)$, $x \in [a_i, b_i]$. Assume that exists such a point $x_i \in (b_i, a_{i+1})$, that

$$y(x_i) = \frac{U(b_i) - U(a_i)}{b_i - a_i} (x_i - b_i) + U(b_i) < U(x_i). \quad (\text{A.16})$$

Consider, $0 < \lambda = \frac{x_i - b_i}{x_i - a_i} < 1$.

Then,

$$\begin{aligned} U_c(b_i) &= U_c(\lambda a_i + (1 - \lambda)x_i) \geq \lambda U_c(a_i) + (1 - \lambda)U_c(x_i) \\ &\geq \lambda U(a_i) + (1 - \lambda)U(x_i) \stackrel{(A.16)}{>} \lambda U(a_i) + (1 - \lambda) \cdot \left(\frac{U(b_i) - U(a_i)}{b_i - a_i} (x_i - b_i) + U(b_i) \right) \end{aligned}$$

Noting that $U_c(b_i) = U(b_i)$ and plugging in the $\lambda = \frac{x_i - b_i}{x_i - a_i}$, we obtain

$$U(b_i) > \frac{x_i - b_i}{x_i - a_i} \cdot U(a_i) + \frac{b_i - a_i}{x_i - a_i} \cdot \left(\frac{U(b_i) - U(a_i)}{b_i - a_i} (x_i - b_i) + U(b_i) \right).$$

Which is equivalent to

$$U(b_i) > \frac{x_i - b_i}{x_i - a_i} \cdot U(b_i) + \frac{b_i - a_i}{x_i - a_i} \cdot U(b_i) = U(b_i).$$

Contradiction. Hence, $y(x) \geq U(x)$, $x \in [a_i, a_{i+1})$.

Thus,

$$\begin{aligned} U'(b_i^+) &= \lim_{b \downarrow b_i} \frac{U(b) - U(b_i)}{b - b_i} \leq \lim_{b \downarrow b_i} \frac{y(b) - U(b_i)}{b - b_i} \\ &= \lim_{b \downarrow b_i} \frac{y(b) - y(b_i)}{b - b_i} = y'(b_i) = \frac{U(b_i) - U(a_i)}{b_i - a_i}; \\ U'(b_i^-) &= \lim_{b \uparrow b_i} \frac{U(b_i) - U(b)}{b_i - b} \geq \lim_{b \uparrow b_i} \frac{U(b_i) - y(b)}{b_i - b} \\ &= \lim_{b \uparrow b_i} \frac{y(b_i) - y(b)}{b_i - b} = y'(b_i) = \frac{U(b_i) - U(a_i)}{b_i - a_i}. \end{aligned}$$

Which gives us

$$U'_c(b_i) = \frac{U(b_i) - U(a_i)}{b_i - a_i} = U'(b_i).$$

□

Appendix B

List of publications of the recipient by the topic of the dissertation

Scientific works in which the main scientific results of the dissertation are published

1. Bahchedjioglou O., Shevchenko G. Minimax identity with robust utility functional for a nonconcave utility // *Modern Stochastics: Theory and Applications*. 2022. Vol. 10, no. 1. P. 19–35. Access mode: <https://doi.org/10.15559/22-VMSTA215>.
2. Bahchedjioglou O., Shevchenko G. Optimal investments for the standard maximization problem with non-concave utility function in complete market model // *Math Meth Oper Res*. 2022. Vol. 95, no. 5. P. 163–181. Access mode: <https://doi.org/10.1007/s00186-022-00774-0>.
3. Bahchedjioglou O. O. Duality theory for concavification of utility functions in incomplete market model // *Bulletin of Taras Shevchenko National University of Kyiv. Series: Physics and Mathematics*. 2021. no. 2. P. 10–17. Access mode: <https://doi.org/10.17721/1812-5409.2021/2>.
4. Kharytonova O. Duality theory under model uncertainty for non-concave utility functions // *Bulletin of Taras Shevchenko National University of Kyiv. Series: Physics and Mathematics*. 2019. no. 4. P. 50–56. Access mode: <https://doi.org/10.17721/1812-5409.2019/4>.

Publications certifying the approval of the dissertation materials

5. Bahchedjioglou O. Characterization of minimax identity with robust non-concave utility functions for constrained case of random endowments // Proceeding of XX International Scientific – Practical Conference “Shevchenkivska Vesna – 2022”. Kyiv, Ukraine. April 14, 2022. С. 5.
6. Bahchedjioglou O. Properties of utility maximization functionals for non-concave utility function in complete market model // 15ème Colloque Bachelier en Mathématiques Financières et Calcul Stochastique. Metabief, France. January 16-21, 2023.
7. Bahchedjioglou O. Optimal investments for the standard problem of maximization the non-concave utility functional in complete market model // Conference Materials of International Conference “Modern Stochastics: Theory and Applications. V”. Kyiv, Ukraine. June 1-4, 2021. P. 92.
8. Kharytonova O. Characterization of minimax identity for robust non-concave utility functions // Proceeding of XIX International Scientific – Practical Conference “Shevchenkivska Vesna – 2021”. Kyiv, Ukraine. April 2021. С. 30.
9. Kharytonova O. Characterization of minimax identity for robust non-concave utility functions // Proceedings of Scientific conference “Actual Problems of Stochastic Analysis”. Tashkent, Uzbekistan. February 20-21, 2021. P. 59–63.

Information about the approval of the results of the dissertation

Conferences

1. Республиканская научная конференция «Актуальные проблемы стохастического анализа», 20-21 февраля, 2021, г. Ташкент, Узбекистан.
2. XIX міжнародна науково-практична конференція «Шевченківська весна – 2021», 15-16 квітня, 2021, м. Київ, Україна.
3. Міжнародна конференція «Modern Stochastics: Theory and Applications. V», 1 – 4 червня, 2021, м. Київ, Україна.

4. XX міжнародна науково-практична конференція «Шевченківська весна – 2022», 14 квітня, 2022, м. Київ, Україна.
5. 15ème Colloque Bachelier en Mathématiques Financières et Calcul Stochastique, January 16-21, 2023, Metabief, France.

Scientific seminars

1. Seminar held by the department of probability theory, statistics and actuarial mathematics of faculty of Mechanics and Mathematics Taras Shevchenko National University of Kyiv (Kyiv, 2020).