## АЛГЕБРА, ГЕОМЕТРІЯ ТА ТЕОРІЯ ЙМОВІРНОСТЕЙ

**UDC 512.554.3**
**DOI: https://doi.org/10.17721/1812-5409.2025/2.2**

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# MATRIX REPRESENTATIONS OF ITERATED WREATH PRODUCTS OF ONE-DIMENSIONAL LIE ALGEBRAS

***The wreath product is a classical construction in group theory, known for its ability to generate complex groups from simpler ones and its deep connections to permutation and automorphism groups. Its analogue in Lie algebra theory, however, remains less developed, despite several notable contributions by A. Shmelkin, F. Sullivan, V. Sushchansky, N. Bondarenko, V. Petrogradsky, and others. In particular, the Lie algebras $L_{p,n}$, associated via the Lazard correspondence with Sylow $p$-subgroups of symmetric groups $Sym(p^n)$, admit a decomposition as iterated wreath products of one-dimensional Lie algebras over the field $\mathbb{F}_p$, along with a tableau-based representation of elements. In this paper, we consider the Lie algebras $L_n := L_{2,n}$ and their inverse limit $L_\infty$, which is the infinitely iterated wreath product of one-dimensional Lie algebras over $\mathbb{F}_2$. We construct explicit embeddings of $L_n$ into the Lie algebra $UT_m(\mathbb{F}_2)$ of strictly upper triangular matrices of minimal possible order $m = 2^{n-1}+1$, and extend this to a recursive embedding of the Lie algebra $L_\infty$ into the infinite-dimensional Lie algebra $UT_\infty(\mathbb{F}_2)$. The construction employs matrix schemes inspired by earlier representations of iterated wreath products of cyclic groups. These results provide a concrete realization of $L_\infty$ and further deepen the connection between combinatorial Lie algebra structures and their matrix representations.***

**K e y w o r d s:** ***Lie algebra representation, wreath product, infinitely iterated wreath product, matrix scheme.***

**AMS 2020 classification: 17B10, 17B70, 17B40.**

## Introduction

The wreath product is a fundamental algebraic construction that combines two structures into a more complex one. It is particularly important in group theory, where it provides a method to build intricate groups from simpler components, preserving key structural features and playing a central role in the study of permutation groups, automorphism groups, and the construction of examples and counterexamples. In the context of Lie algebras, the wreath product has been less extensively explored, with notable contributions such as the verbal wreath product introduced in (Shmelkin, 1973), wreath products with abelian Lie algebras studied in (Sullivan, 1985; Sushchansky, & Netreba, 2005b; Bondarenko, 2006b), a general framework for wreath products of Lie algebras presented in (Petrogradsky, Razmyslov, & Shishkin, 2007), and a matrix wreath product of algebras introduced in (Alahmadi et al., 2017, 2019). This have further contributed to the development of self-similarity ideas in the theory of Lie algebras, see (Bartholdi, 2015; Petrogradsky, 2016; Futorny, Kochloukova, & Sidki, 2019).

One of the earliest applications of the wreath product is the description of the Sylow $p$-subgroup $S_{p,n}$ of the symmetric group $Sym(p^n)$ as the $n$-th iterated wreath product of the cyclic group $C_p$, see (Kaloujnine, 1945). This approach allows for representing group elements as special tableaux, which have been used to establish many properties of these groups. In particular, the representation of the group $S_{p,n}$ by unitriangular matrices is constructed in (Leonov, Nekrashevych, & Sushchansky, 2005) and extended to infinitely iterated wreath products of $C_p$ in (Leonov, 2004, 2012).

An important connection between discrete groups and Lie algebras is provided by the construction of a graded Lie algebra associated with the lower central series of a group. This construction is especially effective for nilpotent groups and $p$-groups — a relation known as the Lazard correspondence. The group $S_{p,n}$ is a $p$-group, and we denote the associated graded Lie algebra by $L_{p,n}$. It is shown in (Sushchansky, & Netreba, 2005a, 2005b) that the algebra $L_{p,n}$ decomposes as the $n$-th iterated wreath product of one-dimensional Lie algebras over the field $\mathbb{F}_p$. In particular, elements of $L_{p,n}$ admit a special tableau representation, analogous to that of the group $S_{p,n}$. This representation was used to describe characteristic subalgebras, as well as the lower and upper central series of the Lie algebra $L_{p,n}$. Subsequently, iterated wreath products of $n$-dimensional abelian Lie algebras over the field $\mathbb{F}_p$ were studied using the same tableau-based approach in (Bondarenko, 2006b), and the Lie algebra associated with the group of finitary automorphisms of the $p$-adic tree was described in (Bondarenko, Gupta, & Sushchansky, 2010). Recently, the algebra $L_{p,n}$ has found applications in algebraic combinatorics related to integer partitions with a special role of the algebra $L_{2,n}$, see (Aragona, Civino, & Gavioli, 2024a, 2024b).

In this paper, we consider matrix representations of the Lie algebras $L_n := L_{2,n}$ and the infinite-dimensional Lie algebra $L_\infty$, which is the inverse limit of $L_n$, the infinitely iterated wreath product of one-dimensional Lie algebras over the field $\mathbb{F}_2$. We construct an explicit embedding of $L_n$ into the Lie algebra $UT_m(\mathbb{F}_2)$ of strictly upper triangular matrices for the minimal possible order $m$. As a consequence, we obtain an explicit embedding of the Lie algebra $L_\infty$ into the Lie algebra $UT_\infty(\mathbb{F}_2)$ of infinite strictly upper triangular matrices. Further, we show that this embedding can be constructed recursively using matrix schemes, introduced in (Leonov, 2004). It is worth noting that an embedding of the Lie algebra $UT_m(\mathbb{F}_p)$ into the Lie algebra $L_{p,n}$ was previously constructed in (Bondarenko, 2006a).

**1. The matrix representation of the $n$-th iterated wreath product of one-dimensional Lie algebras**

Let $L_n = \wr_{i=1}^n A_1$ be the $n$-th iterated wreath product of the one-dimensional Lie algebras $A_1$ over the field $\mathbb{F}_2$. The elements of $L_n$ are tableaux of the form:

$$u = [u_1, u_2(x_1), u_3(x_1, x_2), \ldots, u_n(x_1, \ldots, x_{n-1})],$$

where $u_1 \in \mathbb{F}_2$ and $u_i(x_1, \ldots, x_{i-1}) \in \mathbb{F}_2[x_1, \ldots, x_{i-1}]/\langle x_1^2, x_2^2, \ldots, x_{i-1}^2\rangle$. The elements of the quotient ring are uniquely represented by polynomials, called reduced polynomials, in which each variable $x_i$ appears with exponent either zero or one. The $k$-th coordinate of the tableau $u$ is denoted as $u_k$. The operations in $L_n$ are defined as follows:

1) The sum of two elements $u$ and $v$ is given component-wise by $(u + v)_k = u_k + v_k$.
2) The multiplication of an element $u$ by a scalar $\alpha \in \mathbb{F}_2$ is defined as $(\alpha u)_k = \alpha u_k$.
3) The Lie bracket operation of two elements $u$ and $v$ is defined by the rule

$$(u, v)_k = \sum_{i=1}^{k-1} \left( \frac{\partial v_k}{\partial x_i} u_i + v_i \frac{\partial u_k}{\partial x_i} \right).$$

An important concept in studying the properties of the Lie algebra $L_n$ is the notion of height. The height of a reduced monomial $x_1^{k_1} x_2^{k_2} \ldots x_{n-1}^{k_{n-1}}$, $k_i \in \{0, 1\}$, is the number

$$h(x_1^{k_1} x_2^{k_2} \ldots x_{n-1}^{k_{n-1}}) = 1 + k_1 + k_2 \cdot 2 + k_3 \cdot 4 + \ldots + k_{n-1} \cdot 2^{n-2}.$$

Note that for every height $i \in \{1, 2, \ldots, 2^{n-1}\}$ there exists a unique monomial of height $i$, which we denote by $t(i)$. A few first monomials:

$$\begin{array}{llll} t(1) = 1, & t(2) = x_1, & t(3) = x_2, & t(4) = x_1 x_2, \\ t(5) = x_3, & t(6) = x_1 x_3, & t(7) = x_2 x_3, & t(8) = x_1 x_2 x_3. \end{array}$$

The height of a reduced polynomial is defined as the largest height among the heights of its monomials. Every reduced polynomial $u(x_1, x_2, \ldots, x_{n-1})$ can be uniquely represented as a sum of monomials $t(i)$ for different values of $i \in \{1, 2, \ldots, 2^{n-1}\}$.

Let us construct the matrix representation of $L_n$ within the Lie algebra $UT_m$ of strictly upper triangular matrices over the field $\mathbb{F}_2$. The Lie algebra $L_n$ is nilpotent of class $2^{n-1}$, while $UT_m$ is nilpotent of class $m - 1$. Therefore, it is required that $m \geqslant 2^{n-1} + 1$, and we will demonstrate that this lower bound is achievable.

For a reduced polynomial $g(x_1, x_2, \ldots, x_{n-1})$ and a tableau $u \in L_n$, define the polynomial

$$g^u = \sum_{k=1}^{n-1} \frac{\partial g}{\partial x_k} u_k.$$

It was shown in (Sushchansky, & Netreba, 2005b) that $h(g^u) \leqslant h(g) - 1$.

We define the mapping $\phi : L_n \to UT_{2^{n-1}+1}$ as follows. Given a tableau $u \in L_n$, consider the monomials $t(j)$ for $j \in \{1, 2, \ldots, 2^{n-1} + 1\}$, apply $u$, and express the result as a linear combination of monomials:

$$t(j)^u = \sum_{k=1}^{n-1} \frac{\partial t(j)}{\partial x_k} u_k = \sum_{i=1}^{j-1} a_{ij} \cdot t(i), \ a_{ij} \in \mathbb{F}_2.$$

The coefficients $a_{ij}$ form an upper triangular matrix $A^u = (a_{i,j})$ of order $2^{n-1} + 1$. We define $\phi(u) = A^u$.

**Theorem 1.** *The mapping $\phi : L_n \to UT_{2^{n-1}+1}$, $\phi(u) = A^u$, is a monomorphism.*

*Proof.* The mapping $\phi$ is linear by construction. We need to verify that $\phi$ preserves the Lie bracket:

$$\phi((u, v)) = [\phi(u), \phi(v)] \quad \text{for all } u, v \in L_n.$$

It is sufficient to verify the equality on the basis elements of $L_n$, which are tableaux where all coordinates are zero except for one, which contains the monomial corresponding to all possible heights at the respective coordinate.

Consider the element

$$u = [0, \ldots, 0, t(l), 0, \ldots, 0] \in L_n,$$

where the monomial $t(l)$ of the variables $x_1, x_2, \ldots, x_{p-1}$ is located at position $p$, here $l \leqslant 2^{p-1}$. Let us describe the matrix $A = \phi(u)$. We compute:

$$t(j)^u = \frac{\partial t(j)}{\partial x_p} t(l) = t(i), \ \text{ where } i = h\left(\frac{\partial t(j)}{\partial x_p} t(l)\right) < j.$$

It follows that every row and every column of $A$ contains at most one non-zero coefficient. The $ij$-coefficient of $A$ is non-zero if and only if the variable $x_p$ appears in the monomial $t(j)$, and the monomials $t(j)$ and $t(l)$ do not have common variables. Let $I_{p,l}$ be the set of heights of the monomials $t(j)$ satisfying these conditions. Then

$$A_{ij} = 1 \text{ if and only if } i = j + l - 2^{p-1} - 1 \text{ for every } j \in I_{p,l}.$$

Consider another element $v = [0, \dots, 0, t(m), 0, \dots, 0] \in L_n$, where the monomial $t(m)$ of the variables $x_1, x_2, \dots, x_{q-1}$ is located at position $q$. Let $B = \phi(v)$ and consider $(u, v)$ and $[A, B] = AB - BA$.

1. If $p = q$, then $(u, v) = 0$. Consider the matrix $AB$. The $ij$-coordinate of $AB$ is non-zero if and only if the following monomials are non-zero:
$$t(j)^u = \frac{\partial t(j)}{\partial x_p} t(l) = t(s) \text{ and } t(s)^v = \frac{\partial t(s)}{\partial x_p} t(m) = t(i).$$
Note, however, that the first equality implies that $t(s)$ does not contain $x_p$, and therefore the second equation is always equal to zero. It follows that $AB = 0$. Similarly, $BA = 0$ and $[A, B] = 0$.
2. Let $p < q$. The tableau $(u, v)$ has the following coordinates:
$$(u, v)_k = 0 \text{ for } k \neq q \text{ and } (u, v)_q = \frac{\partial v_q}{\partial x_p} u_p = \frac{\partial t(m)}{\partial x_p} t(l).$$
Then the $ij$-coefficient of the matrix $C = \phi(w)$ for $w = (u, v)$ is non-zero if and only if the following equality holds:
$$t(i) = t(j)^w = \sum_{k=1}^{n-1} \frac{\partial t(j)}{\partial x_k} w_k = \frac{\partial t(j)}{\partial x_q} w_q = \frac{\partial t(j)}{\partial x_q} \cdot \frac{\partial t(m)}{\partial x_p} \cdot t(l).$$
Let us compare this with the matrix $[A, B] = AB - BA$. The $ij$-coordinate of $AB$ is non-zero if and only if the following equalities hold:
$$t(j)^u = \frac{\partial t(j)}{\partial x_p} t(l) = t(s),$$
$$t(i) = t(s)^v = \frac{\partial t(s)}{\partial x_q} t(m) = \frac{\partial^2 t(j)}{\partial x_q \partial x_p} \cdot t(l) \cdot t(m) + \frac{\partial t(j)}{\partial x_p} \cdot \frac{\partial t(l)}{\partial x_q} \cdot t(m).$$
Since $p < q$, the monomial $t(l)$ does not contain the variable $x_p$; therefore, the last term in the above sum is equal to zero. Similarly, for the $ij$-coordinate of $BA$, we get:
$$t(j)^v = \frac{\partial t(j)}{\partial x_q} t(m) = t(s),$$
$$t(i) = t(s)^u = \frac{\partial t(s)}{\partial x_p} t(l) = \frac{\partial^2 t(j)}{\partial x_p \partial x_q} \cdot t(m) \cdot t(l) + \frac{\partial t(j)}{\partial x_q} \cdot \frac{\partial t(m)}{\partial x_p} \cdot t(l).$$
Since we are working over the field $\mathbb{F}_2$, the $ij$-coordinate of $[A, B] = AB - BA$ is non-zero (equal to one) if and only if we have
$$t(i) = \frac{\partial t(j)}{\partial x_q} \cdot \frac{\partial t(m)}{\partial x_p} \cdot t(l),$$
which coincides with the condition for $C$. Therefore, $C = [A, B]$.

It remains to prove that $\phi$ is injective. Let us take a nonzero element $u \in L_n$, and let $k$ be the smallest position such that $u_k \neq 0$. For the monomial $t(2^{k-1}+1) = x_k$ we have $t(2^{k-1}+1)^u = u_k$. Hence, for every monomial $t(i)$ appearing in $u_k$, the $(i, 2^{k-1}+1)$-coefficient of $\phi(u)$ is equal to one. This implies that the kernel of $\phi$ is trivial, and thus $\phi$ is a monomorphism. □

**2. The matrix representation of the infinitely iterated wreath product of one-dimensional Lie algebras**

The sequence of Lie algebras $L_n$ naturally forms an inverse system,

$$\dots \to L_3 \to L_2 \to L_1,$$

where the transition map $\pi_n : L_n \to L_{n-1}$ is defined by restricting tableaux with $n$ components to their first $n-1$ components. The inverse limit of this system is called the *infinitely iterated wreath product* $L_\infty = \varprojlim L_n = \wr_{i=1}^{n} A_1$ of the one-dimensional Lie algebras $A_1$ over the field $\mathbb{F}_2$. The elements of $L_\infty$ are infinite tableaux of the form:

$$u = [u_1, u_2(x_1), u_3(x_1, x_2), \dots, u_n(x_1, \dots, x_{n-1}), \dots],$$

where $u_1 \in \mathbb{F}_2$ and $u_i(x_1, \dots, x_{i-1}) \in \mathbb{F}_2[x_1, \dots, x_{i-1}]/\langle x_1^2, x_2^2, \dots, x_{i-1}^2 \rangle$, $i \geqslant 2$.

The representations $\phi_n : L_n \to UT_{2^{n-1}+1}$ constructed in the previous section are compatible with the inverse system of $L_n$ in the following sense. The Lie algebras $UT_m$ form an inverse system, where the transition map $\tau_m : UT_m \to UT_{m-1}$ is given by restricting a matrix to its $(m-1) \times (m-1)$ submatrix. The inverse limit is the Lie algebra $UT_\infty$ of infinite strictly upper triangular matrices over the field $\mathbb{F}_2$. The definition of $\phi_n$ implies that the following diagram is commutative:

$$\begin{array}{ccccccccc} \dots & \xrightarrow{\pi_5} & L_4 & \xrightarrow{\pi_4} & L_3 & \xrightarrow{\pi_3} & L_2 & \xrightarrow{\pi_2} & L_1 \\ & & \downarrow \phi_4 & & \downarrow \phi_3 & & \downarrow \phi_2 & & \downarrow \phi_1 \\ \dots & \xrightarrow{\tau_{17}} & UT_9 & \xrightarrow{\tau_9} & UT_5 & \xrightarrow{\tau_5} & UT_3 & \xrightarrow{\tau_3} & UT_1 \end{array}.$$

Passing to the limit, we get a monomorphism $\phi : L_\infty \to UT_\infty$. Explicitly, given a tableau $u \in L_\infty$ and a column index $j \geqslant 1$, compute $t(j)^u$ and express it as a linear combination of monomials:

$$t(j)^u = \sum_{k=1}^{\infty} \frac{\partial t(j)}{\partial x_k} u_k = \sum_{i=1}^{j-1} a_{ij} \cdot t(i), \ a_{ij} \in \mathbb{F}_2.$$

Then, we have

$$\phi(u) = A, \ \text{ where } A = (a_{i,j})_{i,j \geqslant 1} \in UT_\infty.$$

We will show that the representation $\phi$ could be described recursively (or self-similarly) using the notion of matrix schemes introduced in (Leonov, 2004; Leonov, Nekrashevych, & Sushchansky, 2005). Let $M_\infty$ be the set of infinite matrices over the field $\mathbb{F}_2$. For a matrix $A = (a_{ij})_{i,j \geqslant 1} \in M_\infty$, we define four submatrices $A_{11}, A_{12}, A_{21}, A_{22}$ as follows:

$$A_{i_0 j_0} = (b_{i,j}), \ \text{ where } b_{i,j} = a_{i_0+2(i-1), j_0+2(j-1)}, \ i_0, j_0 \in \{1, 2\},$$

$$A_{i_0 j_0} = \begin{pmatrix} a_{i_0,j_0} & a_{i_0,j_0+2} & a_{i_0,j_0+4} & \dots \\ a_{i_0+2,j_0} & a_{i_0+2,j_0+2} & a_{i_0+2,j_0+4} & \dots \\ a_{i_0+4,j_0} & a_{i_0+4,j_0+2} & a_{i_0+4,j_0+4} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

*Definition* 1. The (parity) *scheme* of a matrix $A \in M_\infty$ is the $2 \times 2$ matrix, whose coefficients are matrices:

$$sh(A) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

*Example.* Let us construct the scheme of the matrix $A = \phi(u)$ for the table $u = [1, 0, 0, \dots] \in L_\infty$. The $ij$-coefficient of $A$ is determined by the equation:

$$t(j)^u = \sum_{k=1}^{\infty} \frac{\partial t(j)}{\partial x_k} u_k = \frac{\partial t(j)}{\partial x_1} = t(i).$$

We have $a_{ij} = 1$ if and only if $t(j)$ contains the variable $x_1$, which occurs when $j$ is even. In this case, $i = j - 1$ and $a_{j-1,j} = 1$. Consequently, we get the matrix and its scheme:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad sh(A) = \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}.$$

*Definition* 2. For a tableau $u = [u_1, u_2(x_1), u_3(x_1, x_2), \dots] \in L_\infty$, define two tableaux $u|_0$ and $u|_1$ as follows:

$$u|_c = [u'_1, u'_2(x_1), u'_3(x_1, x_2), \dots] \in L_\infty, \ c = 0, 1,$$

where $u'_k(x_1, \dots, x_{k-1})$ is obtained from $u_{k+1}(x_1, \dots, x_k)$ by applying the substitution:

$$x_1 \mapsto c, \ x_2 \mapsto x_1, \ x_3 \mapsto x_2, \ \dots, \ x_k \mapsto x_{k-1}. \tag{1}$$

**Theorem 2.** *The mapping $\phi : L_\infty \to UT_\infty$ satisfies the recursion:*

$$sh(\phi(u)) = \begin{pmatrix} \phi(u|_0) & u_1 E \\ \phi(u|_0) + \phi(u|_1) & \phi(u|_0) \end{pmatrix}, \ u \in L_\infty.$$

*Proof.* Let $A = \phi(u) = (a_{i,j})_{i,j \geqslant 1}$ and $B = \phi(u|_0) = (b_{i,j})_{i,j \geqslant 1}$. We need to show that $A_{11} = B$, which is equivalent to the condition $a_{1+2i,1+2j} = b_{1+i,1+j}$ for all $i, j \geqslant 0$. The coefficients of $A$ and $B$ comes from the equations:

$$t(1+2j)^u = \sum_{k=1}^{\infty} \frac{\partial t(1+2j)}{\partial x_k} u_k = \sum_{i=1}^{2j} a_{i,1+2j} \cdot t(i),$$

$$t(1+j)^{u|_0} = \sum_{k=1}^{\infty} \frac{\partial t(1+j)}{\partial x_k} (u|_0)_k = \sum_{i=1}^{j} b_{i,1+j} \cdot t(i).$$

Note that a monomial has odd height if and only if it does not contain the variable $x_1$. Consequently, monomials of even height appearing in $u_k$ do not contribute to the coefficients $a_{i,1+2j}$ for odd $i$. The key observation we need is that, the construction of $(u|_0)_k$ from $u_{k+1}$ via the substitution (1) implies a bijective correspondence between the monomials $t(1+2l)$ appearing in $u_{k+1}$ and the monomials $t(1+l)$ appearing in $(u|_0)_k$. The monomial $t(1+2l)$ of $u_{k+1}$ contributes to the coefficient $a_{1+2i,1+2j}$, where

$$1+2i = h\left(\frac{\partial t(1+2j)}{\partial x_{k+1}} t(1+2l)\right) = 1+2j+2l-2^k \quad \Rightarrow \quad i = j+l-2^{k-1}.$$

The monomial $t(1+l)$ of $(u|_0)_k$ contributes to the coefficient $b_{1+i,1+j}$, where

$$1+i = h\left(\frac{\partial t(1+j)}{\partial x_k} t(1+l)\right) = 1+j+l-2^{k-1}.$$

This produces the same value of $i$ as above, implying $a_{1+2i,1+2j} = b_{1+i,1+j}$, and hence $A_{11} = B$. The proof that $A_{22} = B$ follows in a similar manner.

Let us prove that $A_{12} = u_1 E$. For an even $j$, the monomial $t(j)$ contains the variable $x_1$. It follows that every monomial appearing in the last sum of the equation

$$t(2+2j)^u = \sum_{k=1}^{\infty} \frac{\partial t(2+2j)}{\partial x_k} u_k = \frac{\partial t(2+2j)}{\partial x_1} u_1 + \sum_{k>1}^{\infty} \frac{\partial t(2+2j)}{\partial x_k} u_k,$$

has even height and does not contribute to $A_{12}$. Consequently, if $u_1 = 0$, $A_{12} = 0$, and if $u_1 = 1$, then $A_{12} = E$.

Let $C = \phi(u|_1) = (c_{i,j})_{i,j\geqslant 1}$. We are left to show that $A_{21} = B + C$, i.e.,

$$a_{2+2i,1+2j} = b_{1+i,1+j} + c_{1+i,1+j} \text{ for all } i,j \geqslant 0.$$

Only monomials of even heights in $u_k$ contribute to the coefficient $a_{i,1+2j}$ for an even $i$. The $k$-th component of $u|_1$ is constructed from $u_{k+1}$ via the substitution (1). By construction, every monomial $t(2+2l)$ appearing in $u_{k+1}$ produces the monomial $t(1+l)$ in $(u|_1)_k$, but also every monomial $t(1+2l)$ in $u_{k+1}$ produces the monomial $t(1+l)$ in $(u|_1)_k$. The monomial $t(2+2l)$ of $u_{k+1}$ contributes to the coefficient $a_{2+2i,1+2j}$, where

$$2+2i = h\left(\frac{\partial t(1+2j)}{\partial x_{k+1}} t(2+2l)\right) = 1+2j+1+2l-2^k \quad \Rightarrow \quad i = j+l-2^{k-1}.$$

The monomial $t(1+l)$ of $(u|_1)_k$ contributes to the coefficient $c_{1+i,1+j}$, where

$$1+i = h\left(\frac{\partial t(1+j)}{\partial x_k} t(1+l)\right) = 1+j+l-2^{k-1}.$$

Note that this produces the same value of $i$ as above.

A possible monomial $t(1+l)$ in $(u|_1)_k$ coming from $t(1+2l)$ in $u_{k+1}$ also contributes to the coefficient $c_{1+i,1+j}$, while $t(1+2l)$ does not contribute to $a_{2+2i,1+2j}$. Note that $t(1+2l)$ in $u_{k+1}$ produces $t(1+l)$ in $(u|_0)_k$. Therefore, this possible additional term in $t(2+2j)^{u|_1}$ has a corresponding term coming from $t(1+l)$ appearing in $(u|_0)_k$ that contributes to the coefficient $b_{1+i,1+j}$. Consequently, $a_{2+2i,1+2j} = b_{1+i,1+j} + c_{1+i,1+j}$, and hence $A_{21} = B + C$. □

*Example.* Consider the tableau $u = [1,1,1,\ldots] \in L_\infty$. Then

$$u|_0 = u|_1 = [1,1,1,\ldots] = u.$$

Therefore, we obtain the following scheme recursion for $A = \phi(u)$:

$$sh(A) = \begin{pmatrix} A & E \\ A+A & A \end{pmatrix} = \begin{pmatrix} A & E \\ 0 & A \end{pmatrix}.$$

By expanding this recursion, we obtain the matrix:

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

*Example.* Consider the tableau $u = [1, x_1, x_2, \ldots] \in L_\infty$. We compute:

$$v := u|_0 = [0, x_1, x_2, x_3, \ldots], \qquad u|_1 = [1, x_1, x_2, x_3, \ldots] = u,$$
$$v|_0 = [0, x_1, x_2, x_3, \ldots] = v, \qquad v|_1 = [1, x_1, x_2, x_3, \ldots] = u.$$

Therefore, we obtain the following scheme recursion for $A = \phi(u)$ and $B = \phi(v)$:

$$sh(A) = \begin{pmatrix} B & E \\ A+B & B \end{pmatrix}, \qquad sh(B) = \begin{pmatrix} B & 0 \\ A+B & B \end{pmatrix}.$$

We can simplify the recursion using:

$$sh(A+B) = sh(A) + sh(B) = \begin{pmatrix} B+B & E \\ A+B+A+B & B+B \end{pmatrix} = \begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}.$$

By expanding this recursion, we obtain the matrices:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

*Example.* Consider the tableau $u = [1, x_1, x_1x_2, x_1x_2x_3, \ldots] \in L_\infty$. We compute:

$$u|_0 = [0, 0, 0, \ldots], \qquad u|_1 = [1, x_1, x_1x_2, x_1x_2x_3, \ldots] = u.$$

Therefore, we obtain the following scheme recursion for $A = \phi(u)$:

$$sh(A) = \begin{pmatrix} 0 & E \\ A & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

## 3. Discussion and conclusions

We have constructed explicit matrix representations of the Lie algebras $L_n$, defined as the $n$-th iterated wreath products of one-dimensional Lie algebras over the field $\mathbb{F}_2$, as embeddings into the Lie algebra $UT_m(\mathbb{F}_2)$ of strictly upper triangular matrices with minimal possible order $m = 2^{n-1} + 1$. Furthermore, we have extended this construction to a recursive embedding of their inverse limit $L_\infty$, the infinitely iterated wreath product, into the infinite-dimensional Lie algebra $UT_\infty(\mathbb{F}_2)$, using matrix schemes inspired by similar constructions for iterated wreath products of groups. The recursive nature of the embeddings may provide insights into automorphism groups, growth functions, and homological invariants of these algebras, while also suggesting potential applications in representation theory, combinatorics, and the theory of self-similar Lie algebras. In future work, it would be natural to explore the representations of iterated wreath products of higher-dimensional abelian Lie algebras over $\mathbb{F}_p$, their cohomological dimensions, and potential applications to $p$-adic and profinite Lie theory.

**Authors' contribution:** Nataliia Bondarenko — problem formulation, conducting research, literature review, and writing the original draft; Ievgen Bondarenko — conducting research, writing, and editing of the manuscript.

**Sources of funding.** Funding is partially provided by Taras Shevchenko National University of Kyiv.

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**Отримано редакцією журналу / Received: 09.04.25**
**Прорецензовано / Revised: 24.06.25**
**Схвалено до друку / Accepted: 10.10.25**

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## МАТРИЧНІ ЗОБРАЖЕННЯ ІТЕРОВАНИХ ВІНЦЕВИХ ДОБУТКІВ ОДНОВИМІРНИХ АЛГЕБР ЛІ

***Вінцевий добуток є класичною конструкцією в теорії груп, відомою своєю здатністю утворювати складні групи з простіших, а також глибокими зв'язками з групами підстановок та автоморфізмів. Його аналог у теорії алгебр Лі, однак, залишається менш дослідженим, незважаючи на низку вагомих внесків, зроблених А. Шмелькіним, Ф. Салліваном, В. Сущанським, Н. Бондаренко, В. Петроградським та ін. Зокрема, алгебри Лі $L_{p,n}$, що пов'язані із силовськими $p$-підгрупами симетричних груп $Sym(p^n)$ через відповідність Лазара, допускають декомпозицію як ітеровані вінцеві добутки одновимірних алгебр Лі над полем $\mathbb{F}_p$, а також мають табличне зображення елементів. У цій роботі ми розглядаємо алгебри Лі $L_n := L_{2,n}$ та їхню зворотну границю $L_\infty$, що є нескінченно ітерованим вінцевим добутком одновимірних алгебр Лі над полем $\mathbb{F}_2$. Ми будуємо явні занурення алгебри $L_n$ в алгебру Лі строго верхньотрикутних матриць $UT_m(\mathbb{F}_2)$ мінімально можливого порядку $m = 2^{n-1}+1$ та поширюємо це на рекурсивне занурення алгебри Лі $L_\infty$ в нескінченновимірну алгебру Лі $UT_\infty(\mathbb{F}_2)$. Конструкція використовує матричні шаблони, побудовані на основі попередніх зображень ітерованих вінцевих добутків циклічних груп. Ці результати забезпечують конкретну реалізацію $L_\infty$ і ще більше поглиблюють зв'язок між комбінаторною структурою алгебр Лі та їх матричними зображеннями.***

**Ключові слова:** ***зображення алгебри Лі, вінцевий добуток, нескінченно ітерований вінцевий добуток, шаблон матриці.***

Автори заявляють про відсутність конфлікту інтересів. Спонсори не брали участі в розробленні дослідження (у зборі, аналізі чи інтерпретації даних, якщо це мало місце), у написанні рукопису та в рішенні про публікацію результатів.

The authors declare no conflicts of interest. The funders had no role in the design of the study (in the collection, analyses or interpretation of data if applicable), in the writing of the manuscript as well as in the decision to publish the results.