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UNIFORM ATTRACTORS AND ASYMPTOTIC GAIN PROPERTY FOR THE PDE-ODE SYSTEM

The paper investigates the qualitative behavior of weak solutions to a dissipative infinite-dimensional non-autonomous perturbed system, which consists of a parabolic reaction-diffusion system and a nonlinear system of ordinary differential equations. Perturbations are modeled by bounded functions included in the right-hand side of both systems. The considered system is known to possess a global attractor in the absence of perturbations, which determines the long-term dynamics of all trajectories. However, the robustness of such attractors under external disturbances remains a challenging issue, especially in the context of nonlinear and infinite-dimensional systems.

In this study, we adopt an approach based on the theory of uniform attractors for non-autonomous dynamical systems (semi-processes). A corresponding family of semi-processes associated with the perturbed PDE–ODE system is constructed. The existence of a uniform attractor is proven, and its convergence to the global attractor of the unperturbed system is established as the amplitude of disturbances tends to zero. This allows us to derive a nonlocal robust estimate of the asymptotic gain (AG) type, which complements the local ISS-based robustness result and provides an upper bound on the deviation of perturbed trajectories from the unperturbed attractor in terms of the perturbation magnitude. The results contribute to the theoretical foundation for robustness analysis in dissipative infinite-dimensional systems with complex long-term dynamics.

K e y w o r d s: ***asymptotic gain, uniform attractor, infinite-dimensional system, PDE-ODE system.***

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Introduction

It is known that the global asymptotically stable equilibrium position of a linear system is robust in the sense that for any bounded perturbations, the solution eventually finds itself in an equilibrium position, and the size of this neighborhood depends only on the norm of the perturbations. For nonlinear systems, this is generally not the case (Khalil, 1992). An estimate of the trajectory divergences for such a system in the case of its perturbation can be obtained within the framework of the Input to State Stability (ISS) theory (Sontag, 1998; Atamas, Dashkovskiy, & Slynko, 2023a; Zheng, Zhu, & Dashkovskiy, 2022; Mironchenko, 2023b, 2023a; Mironchenko, Prieur, & Wirth, 2021). The main tool in both the finite-dimensional and infinite-dimensional cases is the ISS Lyapunov function, in terms of which the desired robust estimates are obtained (Sontag, 1998; Dashkovskiy, & Mironchenko, 2013; Kasyanov, & Zgurovsky, 2018a; Atamas, Dashkovskiy, & Slynko, 2023b; Mironchenko, 2023c). However, for many infinite-dimensional dissipative dynamical systems, a characteristic property is the presence of a non-trivial attracting set in the phase space – a global attractor (Temam, 1998; Robinson, 2001; Kasyanov, Toscano, & Zadoianchuk, 2012; Kapustyan et al., 2014; Chueshov, & Rezounenko, 2015; Chueshov, & Ryzhkova, 2012). The transfer of the basic ISS constructions to parabolic and hyperbolic systems with a non-trivial attractor was carried out in work (Kapustyan, & Dashkovskiy, 2022). At the same time, to prove the property of Asymptotic Gain (AG), an approach related to the analysis of uniform attractors of non-autonomous dynamical systems (semi-processes) and their dependence on perturbations turned out to be productive (Kapustyan et al., 2023; Kasyanov, & Zgurovsky, 2018c, 2018b; Zgurovsky et al., 2017; Dashkovskiy, & Slynko, 2022; Xu, Caraballo, & Valero, 2022b, 2022a).

In this work, this approach is applied to the PDE-ODE type system, the robustness of which in the sense of local ISS was established in the article (Kapustyan, & Yusypiv, 2023). Here, a corresponding family of semi-processes will be constructed, the existence of a uniform attractor will be proven, and (based on the fact of its convergence to the attractor of the unperturbed system) a robust estimate will be established in the sense of AG.

1. Problem statement and main result

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Consider the PDE-ODE system

$$\begin{cases} u_t = A\Delta u - f(u) + B(x)v(t) + d_1(t,x), \\ u|_{\partial\Omega} = 0 \end{cases} \tag{1}$$

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$$\frac{dv}{dt} = -F(v) + \int_{\Omega} G(x)u(x,t)dx + d_2(t) \tag{2}$$

where A is given $N \times N$–matrix for which $\frac{1}{2}(A + A^*) \geqslant \nu_1 I$, $u = u(x,t) = (u^1, ..., u^N)$, $v = v(t) = (v^1, ..., v^M)$ are unknown functions, $B, G \in L^2(\Omega)$ are given matrices of corresponding dimensions, $d_1 \in L^\infty(0, +\infty; (L^2(\Omega))^N)$, $d_2 \in L^\infty(0, +\infty; \mathbb{R}^M)$ are incoming "disturbing" signals, and for all $y, w \in \mathbb{R}^N, u \in \mathbb{R}^M$ the following conditions hold:

$$\begin{aligned} f \in \mathbb{C}^1(\mathbb{R}^N; \mathbb{R}^N), F \in \mathbb{C}^1(\mathbb{R}^M; \mathbb{R}^M), \\ \sum_{i=1}^N f^i(y) y^i \geqslant \nu_2 \cdot \sum_{i=1}^N |y^i|^{p_i} - c_1, \\ \sum_{i=1}^N |f^i(y)|^{\frac{p_i}{p_i - 1}} \leqslant c_2 (\sum_{i=1}^N |y^i|^{p_i} + 1), \\ (Df(u)w, w)_{\mathbb{R}^N} \geqslant -c_3 \cdot \|w\|^2_{\mathbb{R}^N}, \\ \sum_{i=1}^M F^i(u) u^i \geqslant \nu_3 \|u\|^2_{\mathbb{R}^M} - c_4, \end{aligned} \tag{3}$$

where $\nu_1, \nu_2, \nu_3, c_1, c_2, c_3, c_4$ are given constants, $p_i \geqslant 2, i = \overline{1, N}$.

Hereafter, we will use the notation:

$$p = (p_1, ..., p_N), q = (q_1, ..., q_N), q_i = \frac{p_i}{p_i - 1},$$

$$L^p(\Omega) = L^{p_1}(\Omega) \times ... \times L^{p_N}(\Omega),$$
$$L^p(0, T; L^p(\Omega)) = L^{p_1}(0, T; L^{p_1}(\Omega)) \times ... \times L^{p_N}(0, T; L^{p_N}(\Omega)),$$
$$H = (L^2(\Omega))^N, \ V = (H^1_0(\Omega))^N$$

$AC([0, T]; \mathbb{R}^M)$ is the space of absolutely continuous functions from $[0, T]$ to $\mathbb{R}^M$.

Using Galerkin approximations (Robinson, 2001; Kapustyan et al., 2014), it is standardly established that under conditions (3), the problem (1),(2) for arbitrary perturbations $d = \{d_1, d_2\} \in L^\infty(0, +\infty; (L^2(\Omega))^N) \times L^\infty(0, +\infty; \mathbb{R}^M)$ and for arbitrary initial data $z_0 = \{u_0, v_0\}$ from the phase space $X = (L^2(\Omega))^N \times \mathbb{R}^M$ has a unique weak solution

$$z = \{u, v\} \in (L^2(0, T; V) \cap L^p(0, T; L^p(\Omega))) \times AC([0, T]; \mathbb{R}^M)$$

on $(0, T)$ with $z(0) = z_0$. Extending each such solution to $[0, +\infty)$, we define

$$S_d(t, z_0) := \{z(t), \ z(\cdot) \text{ is a solution to (1),(2) with perturbation } d, \ z(0) = z_0\}.$$

In the paper (Kapustyan, & Yusypiv, 2023), it was proven that in the absence of perturbations (i.e., $d \equiv 0$), the semigroup S_0 has a global attractor $\Theta \subset X$, meaning there exists a compact set $\Theta \subset X$ that is invariant:

$$S_0(t, \Theta) = \Theta \ \forall t > 0,$$

and uniformly attracting:

$$dist(S_0(t, B), \Theta) \to 0, t \to \infty \text{ for any bounded } B \subset X,$$

where, here and hereafter,

$$dist(A, B) = \inf_{a \in A} \sup_{b \in B} \|a - b\|_X.$$

In the presence of perturbations, S_d is no longer a semigroup, and the question arises as to how the deviation of the trajectories of the perturbed system S_d from the set Θ depends on the magnitude

$$\|d\|_\infty := \max\{esssup_{t \geqslant 0} \|d_1(t)\|_H, esssup_{t \geqslant 0} \|d_2(t)\|_{\mathbb{R}^M}\}.$$

The deviation of a point $z \in X$ from the set $\Theta \subset X$ will be evaluated using the quantity:

$$\|z\|_\Theta := \inf_{\theta \in \Theta} \|z - \theta\|_X.$$

In the paper (Kapustyan, & Yusypiv, 2023), by constructing an ISS-Lyapunov function in a neighborhood of the attractor, the local stability of Θ with respect to perturbations d in the local ISS sense was established, that is,

$$\exists \, r > 0, \ \exists \, \beta \in \mathcal{KL}, \ \exists \, \gamma \in \mathcal{K} \text{ such that}$$

$$\forall \, \|z_0\|_\Theta \leqslant r \ \forall \, \|d\|_\infty \leqslant r, \forall \, t \geqslant 0,$$

$$\|S_d(t, z_0)\|_\Theta \leqslant \beta(\|z_0\|_\Theta, t) + \gamma(\|d\|_\infty),$$

where $\mathcal{K}$ is the class of continuous, strictly increasing functions $\gamma : [0, +\infty) \mapsto [0, +\infty)$ with $\gamma(0) = 0$, and $\mathcal{KL}$ is the class of continuous functions $\beta : [0, +\infty) \times [0, +\infty) \mapsto [0, +\infty)$, such that $\forall s \geqslant 0 \; \beta(\cdot, s) \in \mathcal{K}, \forall t \geqslant 0 \; \beta(t, s) \to 0, s \to +\infty$.

In this paper, we will establish the nonlocal property of AG only under the conditions (3):

$$\begin{gathered} \exists \gamma \in \mathcal{K}, \forall z_0 \in X, \\ \varlimsup_{t\to\infty} \|S_d(t, z_0)\|_\Theta \leqslant \gamma(\|d\|_\infty). \end{gathered} \tag{4}$$

To prove (4), we use the semiprocess technique: we immerse problem (1),(2) into the family of semiprocesses $\{(1)_\sigma\}_{\sigma\in\Sigma}$, where $(1)_{\{0\}} = (1)$, and Σ is a specially constructed parameter space that depends on the perturbation d. Next, we will prove that the family of semiprocesses $\{(1)_\sigma\}_{\sigma\in\Sigma}$ has a uniform attractor Θ_Σ and

$$dist(\Theta_\Sigma, \Theta) \to 0, \; \|d\|_\infty \to 0.$$

The last convergence just allows us to prove estimate (4).

2. Semiprocesses and their attractors

Let Σ be a translation-invariant set of functional parameters, i.e.

$$\forall h \geqslant 0, \; \forall \sigma(\cdot) \in \Sigma, \; \sigma(\cdot + h) \in \Sigma.$$

Let a given family $S_\sigma(t, \tau, x)$, where $\sigma \in \Sigma, t \geqslant \tau \geqslant 0, x \in X$.

Definition 1. A family $\{S_\sigma\}_{\sigma\in\Sigma}$ is called a family of semiprocesses if for all $\sigma \in \Sigma, x \in X$ the following conditions are satisfied:

$$S_\sigma(\tau, \tau, x) = x, \forall \tau \geqslant 0,$$
$$S_\sigma(t, s, S_\sigma(s, \tau, x)) = S_\sigma(t, \tau, x), \forall t \geqslant s \geqslant \tau \geqslant 0,$$
$$S_\sigma(t + h, \tau + h, x) = S_{\sigma(\cdot+h)}(t, \tau, x), \forall t \geqslant \tau \geqslant 0, \forall h \geqslant 0.$$

Remark 1. When $\sigma = 0$ (undisturbed case) we have

$$S_0(t_1 + t_2, 0, x) = S_0(t_1, 0, S_0(t_2, 0, x)).$$

In other words, S_0 satisfies the semigroup property.

Let us put

$$S_\Sigma(t, 0, x) := \bigcup_{\sigma\in\Sigma} S_\sigma(t, 0, x).$$

Definition 2. A compact set $\Theta_\Sigma \subset X$ is called a uniform attractor of the family $\{S_\sigma\}_{\sigma\in\Sigma}$, if for an arbitrary bounded $B \subset X$

$$dist(S_\Sigma(t, 0, B), \Theta_\Sigma) \to 0, \; t \to \infty,$$

and Θ_Σ is the minimal closed set satisfying this property.

Lemma 1. (Kapustyan et al., 2014) *Let Σ be a compact set, and the following properties hold:*

1) exists bounded $B_0 \subset X$ such that $\forall$ bounded $B \subset X$

$$\exists T = T(B), \forall t \geqslant T, S_\Sigma(t, 0, B) \subset B_0;$$

2) $\forall\{\sigma_n\} \subset \Sigma$, bounded $\{x_n\} \subset X, \{t_n \nearrow \infty\}$

the sequence $\{S_{\sigma_n}(t, 0, x_n)\}$ is precompact in X.

Then $\{S_\sigma\}_{\sigma\in\Sigma}$ has a uniform attractor $\Theta_\Sigma \subset B_0$.
If an additional condition is met:
3) $\forall t > 0$ mapping $(x, \sigma) \to S_\sigma(t, 0, x)$ is continuous,
then Θ_Σ is invariant in the following sense:

$$\Theta_\Sigma \subset S_\Sigma(t, 0, \Theta_\Sigma), \forall t \geqslant 0.$$

Remark 2. For $\Sigma = \{0\}$ conditions 1)-3) guarantee the existence of a global attractor Θ for the semigroup S_0.

Lemma 2. (Kapustyan, & Dashkovskiy, 2022) *Let $\Sigma = \Sigma(d), \|d\|_\infty \leqslant R$ satisfies the conditions of the Lemma 1 with the same set B_0, $\Sigma(0) = \{0\}, d \in \Sigma(d)$ and the following conditions are fulfilled:*

1) $\forall\|d_n\|_\infty \to 0, \forall\{x_n\} \subset B_0, \forall t_n \nearrow \infty, \forall \sigma_n \subset \Sigma(d_n)$

sequence $\{S_{\sigma_n}(t_n, 0, x_n)\}$ is precompact in X;

2) $\forall t > 0, \forall\|d_n\|_\infty \to 0, \forall \sigma_n \in \Sigma(d_n), \forall x_n \to x$ up to a subsequence

$$S_{\sigma_n}(t, 0, x_n) \to S_0(t, 0, x).$$

Then $\exists\gamma \in K, \forall x \in X, \forall \|d\|_\infty \leqslant R$

$$\varlimsup_{t\to\infty} dist(S_{\Sigma(d)}(t,0,x),\Theta) \leqslant \gamma(\|d\|_\infty). \tag{5}$$

Remark 3. Since $d \in \Sigma(d)$, then (5) implies

$$\varlimsup_{t\to\infty} \|S_d(t,0,x)\|_\Theta \leqslant \gamma(\|d\|_\infty).$$

3. Uniform attractors of the semiprocesses generated by (1),(2)

Let $d \in L^\infty(\mathbb{R}_+;X)$ with $\|d\|_\infty \leqslant R_0$. We consider the set

$$\Sigma(d) = cl_{L^{2,w}_{loc}(\mathbb{R}_+;X)}\{d(\cdot+\tau)\,|\,\tau \geqslant 0\},$$

where cl means the closure, and $L^{2,w}_{loc}(\mathbb{R}_+;X)$ is the space of locally integrable X-valued functions supplied with the local weak convergence topology. It is known that $\Sigma(d)$ is translation-invariant and compact in $L^{2,w}_{loc}(\mathbb{R}_+;X)$. Moreover,

$$\forall\sigma \in \Sigma(d)\ \|\sigma\|^2_+ := \sup_{t\geqslant 0}\int_t^{t+1} \|\sigma(s)\|^2_X ds \leqslant \|d\|^2_\infty. \tag{6}$$

As in the case of bounded perturbations, using Galerkin approximations (Robinson, 2001; Kapustyan et al., 2014), it is standardly established that under conditions (3), the problem (1),(2) for arbitrary perturbations of the form $\sigma = \{\sigma_1,\sigma_2\} \in L^2_{loc}(0,+\infty;X)$ and for arbitrary $\tau \geqslant 0$ and $z_\tau \in X$ has a unique weak solution on $[\tau,+\infty)$, satisfying the condition $z(\tau) = z_\tau$. Moreover, the family of mappings

$$S_\sigma(t,\tau,z_\tau) := \{z(t),\ z(\cdot) \text{ is solution to (1),(2) on } (\tau,+\infty) \text{ with perturbation } \sigma \in \Sigma(d),\ z(\tau) = z_\tau\}$$

satisfies the conditions of Definition 1, thus forming a family of semi-processes.

Theorem 1. *For every perturbation* d, $\|d\|_\infty \leqslant R_0$, *the family of mappings* $\{S_\sigma\}_{\sigma\in\Sigma(d)}$ *possesses the uniform attractor* $\Theta_{\Sigma(d)}$, *and*

$$\Theta_{\Sigma(d)} \subset S_{\Sigma(d)}(t,0,\Theta_{\Sigma(d)})\ \forall t \geqslant 0.$$

Proof. Let us verify the fulfillment of the conditions of Lemma 1. To this end, we establish some a priori estimates for $z(t) = \{u(t),v(t)\} = S_\sigma(t,0,z_0)$, where $\sigma = \{\sigma_1,\sigma_2\} \in \Sigma(d)$ satisfies (6). Since for $u \in L^2(0,T;V) \cap L^p(0,T;L^p(\Omega))$, $v \in C([0,T];\mathbb{R}^M)$, by virtue of (3),

$$A\Delta u - f(u) + B\cdot v + D\cdot d \in L^2(0,T;V^*) + L^q(0,T;L^q(\Omega)) + L^\infty(0,T;L^2(\Omega)) \subset L^2(0,T;V^*) + L^q(0,T;L^q(\Omega)),$$

then, according to (Robinson, 2001), $z : [0,T] \mapsto X$ is an absolutely continuous function, and after multiplying (1) by u scalarly in H, and (2) by v scalarly in $\mathbb{R}^m$, we obtain: for almost all (a.e.) $t > 0$

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_H + \nu_1\|u(t)\|^2_V + \nu_2\|u(t)\|^p_{L^p} \leqslant c_1|\Omega| + \|B\|_{L^2}\cdot\|v(t)\|_{\mathbb{R}^M} + \|\sigma_1(t)\|_H\|u(t)\|_H, \tag{7}$$

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{\mathbb{R}^M} + \nu_3\|v(t)\|^2_{\mathbb{R}^M} \leqslant c_4 + \|G\|_{L^2}\cdot\|u(t)\|_H + \|\sigma_2(t)\|_{\mathbb{R}^M}\|v(t)\|_{\mathbb{R}^M}. \tag{8}$$

From (7), (8), Young's inequality, and the Poincaré inequality, we deduce that there exist $\nu > 0$, $c > 0$, depending only on the constants of the problem, such that for a.a. $t > 0$

$$\frac{d}{dt}(\|u(t)\|^2_H + \|v(t)\|^2_{\mathbb{R}^M}) + \nu(\|u(t)\|^2_V + \|u(t)\|^p_{L^p} + \|u(t)\|^2_H + \|v(t)\|^2_{\mathbb{R}^M}) \leqslant$$

$$\leqslant c(1 + \|\sigma_1(t)\|^2_H + \|\sigma_2(t)\|^2_{\mathbb{R}^M}).$$

Next, using the inequality

$$\int_0^t \|h(s)\|^2 e^{-a(t-s)}ds \leqslant (1-e^{-a})^{-1}\sup_{t\geqslant 0}\int_t^{t+1}\|h(s)\|^2 ds,$$

and due to the continuity of $z. = \{u(\cdot),v(\cdot)\} : [0,+\infty) \mapsto X$, we deduce:

$$\forall t \geqslant 0\ \|z(t)\|^2_X \leqslant \|z_0\|^2_X e^{-\nu t} + \frac{c}{\nu} + c\|d\|^2_\infty. \tag{9}$$

From (9), we obtain condition 1) of Lemma 1 with the set

$$B_0 = \{z \in X \mid \|z\|^2_X \leqslant 1 + \frac{c}{\nu} + cR_0^2\}.$$

Let us verify condition 2): the precompactness of the sequence $\{\xi_n = S_{\sigma_n}(t_n,0,z_0^n)\}$, where $\{\sigma_n\} \subset \Sigma(d)$, $t_n \to +\infty$, and $\{z_0^n\}$ is bounded in X.

Since, by the properties of the semiprocesses and condition 1), for sufficiently large $n \geqslant 1$, we have

$$\xi_n = S_{\sigma_n}(t_n - 1 + 1, t_n - 1, S_{\sigma_n}(t_n - 1, 0, z_0^n)) = S_{\sigma_n(\cdot + t_n - 1)}(1, 0, S_{\sigma_n}(t_n - 1, 0, z_0^n)) \in S_\Sigma(1, 0, B_0),$$

it is sufficient to show the precompactness of the $\{S_{\sigma_n}(1, 0, z_0^n)\}$, where $\{\sigma_n\} \subset \Sigma(d)$, $\|z_0^n\|_X \leqslant r$.

Let $z_n(t) = \{u_n(t), v_n(t)\} = S_{\sigma_n}(t, 0, z_0^n)$, $\|z_0^n\|_X \leqslant r$.

The boundedness, and hence the precompactness, of the $\{v_n(1)\}$ in $\mathbb{R}^M$ follows from (9).

Let us prove that

$$\|u_n(1)\|_V \leqslant K(r) \tag{10}$$

for some constant $K(r)$ independent of n. Then, by the compactness of the embedding $V \subset H$, we obtain the precompactness of the $\{u_n(1)\}$ in H.

Let us derive (10) following the arguments from (Temam, 1998), which are justified by passing to Galerkin approximations of problem (1),(2). We multiply equation (1) by Δu_n in H. Taking into account conditions (3) and the smoothness of f, we obtain the inequality:

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|_V^2 + \nu_1\|\Delta u_n(t)\|_H^2 \leqslant c_3\|u_n(t)\|_V^2 + \|B\|_{L^2}\|v_n(t)\|_{\mathbb{R}^M}\|\Delta u_n(t)\|_H + \|\sigma_1^n(t)\|_H\|\Delta u_n(t)\|_H. \tag{11}$$

Since, by (9)

$$\sup_{t\geqslant 0}\|v_n(t)\|_{\mathbb{R}^M} \leqslant \sqrt{r^2 + \frac{c}{\nu} + cR_0^2},$$

we deduce from (11):

$$\frac{d}{dt}\|u_n(t)\|_V^2 + \lambda\|u_n(t)\|_V^2 \leqslant c(r)(\|u_n(t)\|_V^2 + 1 + \|\sigma_1^n(t)\|_H^2) \tag{12}$$

for some positive constants λ і $c(r)$.

Multiplying (12) by $te^{\lambda t}$ yields:

$$\frac{d}{dt}(t\|u_n(t)\|_V^2 e^{\lambda t}) \leqslant c(r)(\|u_n(t)\|_V^2 + 1 + \|\sigma_1^n(t)\|_H^2)te^{\lambda t} + \|u_n(t)\|_V^2 e^{\lambda t}.$$

Integrating the last inequality over $[0, 1]$, we obtain:

$$\|u_n(1)\|_V^2 \leqslant (c(r) + 1)\int_0^1 \|u_n(t)\|_V^2 dt + c(r)R_0^2. \tag{13}$$

From estimates (7) and (9) we have:

$$\nu_1\int_0^1 \|u_n(t)\|_V^2 dt \leqslant \frac{1}{2}r^2 + c_1|\Omega| + (\|B\|_{L^2} + R_0^2)\sqrt{r^2 + \frac{c}{\nu} + R_0^2}. \tag{14}$$

Combining (13) and (14), we derive estimate (10).

Thus, the family of semiprocesses $\{S_\sigma\}_{\sigma\in\Sigma(d)}$ possesses the uniform attractor $\Theta_{\Sigma(d)}$.

The theorem is proved. □

4. Robust stability of the global attractor of the system (1),(2)

Now we turn to the formulation of the main result regarding the robust stability of the global attractor.

Theorem 2. *The attractor Θ of the unperturbed system (1),(2) (when $d \equiv 0$) is stable in the sense of AG, i.e., $\exists \gamma \in \mathcal{K}$ such that $\forall\, z_0 \in X\ \forall\, \|d\|_\infty \leqslant R_0$*

$$\varlimsup_{t\to\infty}\|S_d(t, z_0)\|_\Theta \leqslant \gamma(\|d\|_\infty),$$

where we denote $S_d(t, z_0) = S_d(t, 0, z_0)$.

Proof. Suppose that the limit equality holds

$$dist(\Theta_{\Sigma(d)}, \Theta) \to 0, \|d\|_\infty \to 0. \tag{15}$$

Let us prove that (15) implies (4). Indeed, by construction $\Sigma(0) = \{0\}$ and $d \in \Sigma(d)$. Therefore, for $z_0 \in X$, $z \in \Theta_{\Sigma(d)}$, $t > 0$, $z(t) \in S_d(t, z_0)$ we have the following implications

$$\|z(t) - \theta\|_X \leqslant \|z(t) - z\|_X + \|z - \theta\|_X \ \Rightarrow$$

$$\Rightarrow \inf_{\theta\in\Theta}\|z(t) - \theta\|_X \leqslant \|z(t) - z\|_X + \inf_{\theta\in\Theta}\|z - \theta\|_X \Rightarrow$$

$$\Rightarrow \inf_{\theta\in\Theta}\|z(t) - \theta\|_X \leqslant \inf_{z\in\Theta_{\Sigma(d)}}\|z(t) - z\|_X + \sup_{z\in\Theta_{\Sigma(d)}}\inf_{\theta\in\Theta}\|z - \theta\|_X \Rightarrow$$

$$\Rightarrow \|z(t)\|_\Theta \leqslant dist(z(t), \Theta_{\Sigma(d)}) + dist(\Theta_{\Sigma(d)}, \Theta) \Rightarrow$$

$$\Rightarrow \|S_d(t, z_0)\|_\Theta \leqslant dist(S_{\Sigma(d)}(t, z_0), \Theta_{\Sigma(d)}) \ + dist(\Theta_{\Sigma(d)}, \Theta).$$

Since $\Theta_{\Sigma(d)}$ is the uniform attractor, then from Definition 2 for each d we have:

$$dist(S_{\Sigma(d)}(t, z_0), \Theta_{\Sigma(d)}) \to 0, t \to +\infty.$$

Let us put $\gamma(s) := \sup\limits_{\|d\|_\infty \leqslant s} dist(\Theta_{\Sigma(d)}, \Theta) + s$. Then $s \mapsto \gamma(s)$ is increasing and $\gamma(s) \to 0$ as $s \to 0+$. Therefore (Clarke, Ledyaev, & Stern, 1998), Lemma 5 implies the existence of $\gamma_1 \in K$ such that $dist(\Theta_{\Sigma(d)}, \Theta) \leqslant \gamma_1(\|d\|_\infty)$, which means that (4) is satisfied.

So, according to the previous arguments, we need to verify assumptions 1), 2) of Lemma 2.

Since $\forall \|d_n\|_\infty \to 0$, $\forall \sigma_n \in \Sigma(d_n)$ satisfies $\|\sigma_n\|_+ \leqslant \|d_n\| \leqslant R_0$, condition 1) follows from the arguments of Theorem 1 and the estimate (10).

Now consider $z_1 - z_2$, where $z_1 = \{u_1, v_1\}, z_2 = \{u_2, v_2\}$ are solutions to (1),(2) with perturbation $d = \{d_1, d_2\}$, $\|d\|_\infty \to 0$, and without perturbations, respectively. Using the estimate for each solution

$$\sup_{t \geqslant 0} \|z(t)\|_X^2 \leqslant \|z(0)\|_X^2 + \frac{c}{\nu} + cR_0^2,$$

we derive for a.a. $t > 0$:

$$\frac{1}{2}\frac{d}{dt}\|u_1(t) - u_2(t)\|_H^2 + \nu_1\|u_1(t) - u_2(t)\|_V^2 \leqslant c_3\|u_1(t) - u_2(t)\|_H^2 +$$

$$+\|B\|_{L^2}\|v_1(t) - v_2(t)\|_{\mathbb{R}^M}\|u_1(t) - u_2(t)\|_H + \|d_1(t)\|_H\|u_1(t) - u_2(t)\|_H,$$

$$\frac{1}{2}\frac{d}{dt}\|v_1(t) - v_2(t)\|_{\mathbb{R}^M}^2 \leqslant \sup_{\|v\|_{\mathbb{R}^M} \leqslant r_0} \|DF(v)\| \cdot \|v_1(t) - v_2(t)\|_{\mathbb{R}^M}^2 +$$

$$+\|G\|_{L^2}\|u_1(t) - u_2(t)\|_H\|v_1(t) - v_2(t)\|_{\mathbb{R}^M} + \|d_2(t)\|_{\mathbb{R}^M}\|v_1(t) - v_2(t)\|_{\mathbb{R}^M},$$

where $r_0 = \sqrt{max\{\|z_1(0)\|_X^2, \|z_2(0)\|_X^2\} + \frac{c}{\nu} + cR_0^2}$.

Thus, with some constant $c(r_0) > 0$, we obtain

$$\forall t \geqslant 0 \ \|z_1(t) - z_2(t)\|_X^2 \leqslant e^{c(r_0)t}(\|z_1(0) - z_2(0)\|_X^2 + \|d\|_\infty^2).$$

From the last inequality, the desired property 2) of Lemma 2 follows.

The theorem is proved. □

Discussions and conclusions

This study provides important insights into the behavior of complex, infinite-dimensional PDE-ODE systems under external disturbances. We constructed a family of semiprocesses for the perturbed system and proved the existence of a uniform attractor, showing that it approaches the global attractor of the unperturbed system as the disturbance size decreases. This establishes a nonlocal asymptotic gain property, which measures how far perturbed trajectories deviate from the unperturbed attractor. Building on earlier local stability results, our findings use mathematical techniques like Galerkin approximations to confirm the system's robustness, enhancing our understanding of its long-term dynamics under perturbations.

These results are valuable for applications like reaction-diffusion processes, where such systems are common. The AG property offers a way to predict and control system behavior despite external influences, aiding in the design of effective control strategies. Future work could explore systems with time-varying or random disturbances and test these findings in practical settings. The methods used here, combining semiprocesses and stability analysis, can also be applied to other complex systems, opening new avenues for research into their robustness and stability.

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РІВНОМІРНІ АТРАКТОРИ ТА ВЛАСТИВІСТЬ АСИМПТОТИЧНОГО ПІДСИЛЕННЯ ДЛЯ СИСТЕМИ ТИПУ PDE-ODE

У статті досліджено якісну поведінку слабких розв'язків нескінченновимірної дисипативної неавтономної системи із зовнішніми збуреннями, що складається з параболічної системи типу реакції-дифузії та нелінійної системи звичайних диференціальних рівнянь. Збурення моделюються обмеженими функціями, що входять у праві частини обох підсистем. Відомо, що в незбуреному випадку така система має глобальний атрактор, який визначає довготривалу динаміку всіх траєкторій. Водночас питання робастності таких атракторів за наявності зовнішніх збурень залишається складним, особливо у випадку нелінійних та нескінченновимірних систем.

У пропонованому дослідженні застосовано підхід, що базується на теорії рівномірних атракторів для неавтономних динамічних систем (теорія напівпроцесів). Побудовано відповідну сім'ю напівпроцесів, що відповідає збуреній системі типу PDE–ODE. Доведено

існування рівномірного атрактора, а також встановлено його збіжність до глобального атрактора незбуреної системи за наближення амплітуди збурень до нуля. Це дає змогу вивести нелокальну робастну оцінку типу асимптотичного підсилення (AG), що доповнює робастну оцінку в сенсі локальної ISS і забезпечує верхню межу відхилення збурених траєкторій від незбуреного атрактора в термінах норми збурень. Отримані результати сприяють розвитку теоретичних засад аналізу робастності дисипативних нескінченновимірних систем зі складною довготривалою поведінкою.

Ключові слова: *асимптотичне підсилення, рівномірний атрактор, нескінченновимірна система, система типу PDE-ODE.*

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