UDC 519.2
DOI: https://doi.org/10.17721/1812-5409.2024/1.4

Vadym PONOMAROV, PhD (Phys. & Math.)
ORCID ID: 0000-0002-7813-3298
e-mail: vponomarov@knu.ua
Taras Shevchenko National University of Kyiv, Kyiv, Ukraine

EXACT FORMULAS FOR MARKOV RETRIAL QUEUES CONTROLLED BY HYSTERESIS STRATEGIES

This paper examines the Markov model for multiserver retrial queues with an input flow rate that depends on the number of calls in orbit and is controlled by hysteresis strategies. The system consists of n identical servers. If an incoming call finds a free server, it occupies it and is served for an exponentially distributed time. If all servers are busy upon arrival, the call joins the orbit and returns for service after a random period of time. The system's service process is described by a three-dimensional continuous-time Markov chain. We first establish the conditions for the existence of a stationary regime. Next, we provide exact vector-matrix formulas for steady-state probabilities. Our investigative technique is based on approximating the input system by the system with a truncated state space and contains an effective computational algorithm. For $n = 1$ and $n = 2$, the representations can be simplified to closed scalar formulas for stationary probabilities using the model parameters. These results are consistent with earlier works. To demonstrate practical significance, we present a multi-objective problem of maximizing total income generated by the system. Considering the economic nature of the problem, we utilized the method of linear convolution of criteria. The obtained representations enable us to determine an optimal strategy that maximizes the objective function.

K e y w o r d s : *retrial queue, steady-state probabilities, quasi-birth-and-death process, optimal control.*

AMS 2020 classification: 62H30, 34C60.

Introduction

Classical retrial queues are defined by a specific feature: when a call arrives and all servers are busy, the call leaves the service area and retries after a random period of time. This feature is crucial for modeling computer and communication networks (Das, Lobiyal, & Katti, 2016; Saxena, Chauhan, & Kait, 2016), as well as other scenarios such as stacked aircraft waiting to land, traffic management problems (Rao, 2017), and queues of retrial shoppers. The fundamental works of (Falin, & Templeton,1997; Artalejo, & Gomez-Corral, 2008) have accumulated the results of the retrial queue theory. Most researches on retrial queueing systems focus on analyzing steady-state distributions. Explicit formulas for stationary probabilities enable fundamental analysis of system performance measures, optimal control, and more. However, this problem has only been solved in the simplest cases of one or two servers (Gomez-Corral, 2002; Falin, & Gomez-Corral, 2000) or for special classes of retrial queues (Fiems, 2022; Lebedev, Ponomarov, & Livinska, 2022).

This work assumes that the parameters of the retrial queue may depend on the phase state of the service process (Mandelbaum, & Pats, 1994; Makushenko, Usar, Livinska, & Sharapov, 2023). From a mathematical perspective, the assumed dependence makes the retrial queue model more complex. The method of generating functions, which has been effective for classical models (Falin, & Templeton, 1997), is not applicable. Therefore, we have developed an alternative approach based on approximating the input system characteristics with the corresponding characteristics of the system with a truncated state space. The similar approach was used in (Atencia, Lebedev, Ponomarov, & Livinska, 2019) for queues controlled by threshold policies. The approach developed in this study is applied to state-dependent retrial queues controlled by hysteresis strategies to determine the explicit representation of steady-state probabilities. Subsequently, the obtained results are utilized to identify an optimal set of strategies that maximizes the total income from the system.

1. Mathematical model

Consider a three-dimensional Markov process $Q(t) = \big(Q_1(t), Q_2(t), Q_3(t)\big)$with continuous time in phase set

$$S = S^{(1)} \cup S^{(2)}, S^{(1)} \cap S^{(2)} = \emptyset,$$

where $S^{(1)} = \{(i,j,1): i = 0,\dots,n, j = 0,\dots,H_2 - 1\}$, $S^{(2)} = \{(i,j,2): i = 0,\dots,n, j = H_1,\dots\}$.

Local characteristics $q_{(i',j',r')}^{(i,j,r)}$ of the process $Q(t)$ are defined in the following way:

- For $(i = 0,\dots,n-1, j = 0,\dots,H_2 - 1, r = 1) \vee (i = 0,\dots,n-1, j = H_1 + 1,\dots, r = 2)$

$$q_{(i',j',r')}^{(i,j,r)} = \begin{cases} \lambda_r, when\ (i',j',r') = (i+1,j,r), \\ i\mu, when\ (i',j',r') = (i-1,j,r), \\ j\nu, when\ (i',j',r') = (i+1,j-1,r), \\ -(\lambda_r + i\mu + j\nu), when\ (i',j',r') = (i,j,r), \\ 0, otherwise; \end{cases}$$

- For $(i = n, j = 0,\dots,H_2 - 2, r = 1) \vee (i = n, j = H_1 + 1,\dots, r = 2)$

$$q_{(i',j',r')}^{(i,j,r)} = \begin{cases} \lambda_r, when\ (i',j',r') = (n,j+1,r), \\ n\mu, when\ (i',j',r') = (n-1,j,r), \\ -(\lambda_r + n\mu), when\ (i',j',r') = (i,j,r), \\ 0, otherwise; \end{cases}$$

- For $i = n, j = H_2 - 1, r = 1$

$$q_{(i',j',r')}^{(n,H_2-1,1)} = \begin{cases} \lambda_1, when\ (i',j',r') = (n,H_2,2), \\ n\mu, when\ (i',j',r') = (n-1,H_2-1,1), \\ -(\lambda_1 + n\mu), when\ (i',j',r') = (n,H_2-1,1), \\ 0, otherwise; \end{cases}$$

- For $i = 0, \dots, n-1, j = H_1, r = 2$

$$q_{(i',j',r')}^{(i,H_{1,2})} = \begin{cases} \lambda_2, when\ (i',j',r') = (i+1, H_1, 2), \\ i\mu, when\ (i',j',r') = (i-1, H_1, 2), \\ H_1\nu, when\ (i',j',r') = (i+1, H_1-1, 1), \\ -(\lambda_2 + i\mu + H_1\nu), when\ (i',j',r') = (i, H_1, 2), \\ 0, otherwise. \end{cases}$$

The Markov chain $Q(t)$ describes a service process in the following system. The service facility consists of n identical servers, each with a service rate of $\mu > 0$. The rate of retrial calls flow is $\nu > 0$. Two natural numbers H_1 and H_2 $(H_1 \leq H_2)$ represent the lower and upper thresholds, respectively. If the number of retrial calls $j \leq H_1 - 1$, then the queue operates in the first regime and the input flow rate is equal to λ_1. If $j \geq H_2$, then the queue operates in the second regime with the input flow rate λ_2. If $H_1 \leq j < H_2$ then the queue remains in the regime in which it previously operated. This switching of the operating mode is called a hysteresis control strategy. The first component, $Q_1(t)$, indicates the number of busy servers at time $t \geq 0$, and the second component, $Q_2(t)$, is the number of retrial sources. $Q_3(t)$ indicates the current operating mode.

2. Steady-state analysis

Conditions of stationary regime existence for the process $Q(t)$, $t \geq 0$ are the following.

Lemma 1. Under $\lambda_2/n\mu < 1$ an ergodic distribution of the process $Q(t)$ exists and is the same as single stationary one.

If from the phase set $S = S^{(1)} \cup S^{(2)}$ we eliminate the finite number of points

$$\{(i,j,r): i = 0,1,\dots,n; j = 0,1,\dots,H_2; r = 1,2\}$$

then for the residual part of S and the Lyapunov test functions $\phi(i,j,r) = \alpha i + j$, $(i,j,r) \in S$ the conditions of Tweedy theorem (Falin, & Templeton, 1997, p. 97) are held. This proves a conclusion of the lemma.

To construct calculating schemes and explicit formulas, we consider the truncated $((M\_Q/M)/n)/N$- queue with a finite number of retrial calls, N. As N approaches infinity, this queue approximates the initial model. Therefore, to eliminate details unrelated to the main topic, we assume that N is greater than H_2.

Local characteristics $q_{(i',j',r')}^{(i,j,r)}(N)$, $(i,j,r), (i',j',r') \in S(N)$, $S(N) = S^{(1)}(N) \cup S^{(2)}(N)$, $S^{(1)}(N) = S^{(1)}$, $S^{(2)}(N) = \{(i,j,2): i = 0,1,\dots,n; j = H_1,\dots,N\}$ of the truncated queue are the same as the above except the case $i = n,\ j = N,\ r = 2$ where

$$q_{(i',j',r')}^{(n,N,2)}(N) = \begin{cases} n\mu, when\ (i',j',r') = (n-1, N, 2), \\ -n\mu, when\ (i',j',r') = (n, N, 2), \\ 0, otherwise. \end{cases}$$

Let $Q(t,N) = (Q_1(t,N), Q_2(t,N), Q_3(t,N))^T$ be the homogeneous Markov chain in the phase set $S(N)$ with continuous time $t \geq 0$ and the transition probabilities given by the infinitesimal rates $q_{(i',j',r')}^{(i,j,r)}(N)$, $(i,j,r), (i',j',r') \in S(N)$.

Then for the process $Q(t,N)$ a stationary distribution always exists and we will denote it by

$$\pi_{ij}^{(r)}(N) = \lim_{t\to\infty} P(Q_1(t) = i, Q_2(t) = j, Q_3(t) = r), (i,j,r) \in S(N).$$

Our next goal is to find the stationary probabilities $\pi_{ij}^{(r)}(N)$.

We introduce the notations for vectors and matrices which are given by the model parameters:

$$\pi_j^{(r)}(N) = \left(\pi_{0j}^{(r)}(N), \dots, \pi_{n-1j}^{(r)}(N)\right)^T,$$

$A^{(r)}(j) = \left\| a_{ik}^{(r)}(j) \right\|_{i,k=1}^{n}$, is a matrix with entries, $a_{ii-1}^{(r)}(j) = (j+1)\nu$, $i = 2,\dots,n-1$;

$$a_{nk}^{(r)}(j) = \begin{cases} \dfrac{(j+1)\nu n\mu}{\lambda_r}, k \neq n-1, \\ \dfrac{(j+1)\nu(\lambda_r + n\mu)}{\lambda_r}, k = n-1, \end{cases}$$

and all other entries are equal to 0;

$B^{(r)}(j) = \left\| b_{ik}^{(r)}(j) \right\|_{i,k=1}^{n}$ is a three-diagonal matrix with entries $b_{ii-1}^{(r)}(j) = -\lambda_r$, $i = 2,\dots,n$; $b_{ii}^{(r)}(j) = \lambda_r + j\nu + (i-1)\mu$, $i = 1,2,\dots,n$; $b_{ii+1}^{(r)}(j) = -i\mu$, $i = 1,\dots,n-1$;

$\left[B^{(r)}(j)\right]^{-1} = \bar{B}^{(r)}(j)$, $j = 0,1,\dots,N-1$ (for the sake of reduction of long formulas);

$C(N) = \| c_{ik}(N) \|_{i,k=1}^{n}$, where $\left(c_{11}(N), c_{12}(N), \dots, c_{1n}(N)\right) = e_1^T = (1,0,\dots,0)$; and for $i = 2,\dots,n$ $\left(c_{i1}(N), c_{i2}(N), \dots, c_{in}(N)\right) = \left(b_{i-1i}^{(2)}(N), b_{i-12}^{(2)}(N), \dots, b_{i-1n}^{(2)}(N)\right)$;

$D^{(r)} = \left\| d_{ik}^{(r)} \right\|_{i,k=1}^{n}$, where

$$d_{ik}^{(r)} = \begin{cases} \dfrac{H_1 \nu n\mu}{\lambda_r}, i = n, k = 1,2,\dots,n; \\ 0, \quad otherwise; \end{cases}$$

$$F^{(1)}(j) = \begin{cases} \left(\prod_{i=j}^{H_1-1} \bar{B}^{(1)}(i) A^{(1)}(i)\right)\left[E + \sum_{k=H_1}^{H_{2-1}} \left(\prod_{i=j}^{k-1} \bar{B}^{(1)}(i) A^{(1)}(i)\right) \bar{B}^{(1)}(k) D^{(1)}\right], j = 0,1,\dots,H_1, \\ \sum_{k=j}^{H_2-1} \left(\prod_{i=j}^{k-1} \bar{B}^{(1)}(i) A^{(1)}(i)\right) \bar{B}^{(1)}(k) D^{(1)}, j = H_1,\dots,H_2-1; \end{cases}$$

$$F^{(2)}(j,N-1)=\left\{E-\left[\sum_{k=j}^{H_2-1}\left(\prod_{i=j}^{k-1}\bar{B}^{(2)}(i)A^{(2)}(i)\right)\bar{B}^{(2)}(k)D^{(2)}\right]\times\right.$$
$$\left.\times\left[E+\sum_{k=H_1}^{H_2-1}\left(\prod_{i=H_1}^{k-1}\bar{B}^{(2)}(i)A^{(2)}(i)\right)\bar{B}^{(2)}(k)D^{(2)}\right]^{-1}\left(\prod_{i=H_1}^{j-1}\bar{B}^{(2)}(i)A^{(2)}(i)\right)\right\}\times\prod_{i=j}^{N-1}\bar{B}^{(2)}(i)A^{(2)}(i)\,, j=H_1,\dots,N-1.$$

By convention $\sum_{i=k}^{m}\dots$ is equal to the null-matrix under $k>m$.

$$F^{(2)}(j,N)=F^{(2)}(j,N-1)C^{-1}(N), j=H_1,\dots,N-1.$$
$$\Delta_j^{(1)}(N)=\frac{F^{(1)}(j)F^{(2)}(H_1,N)e_1}{e_1^TF^{(1)}(0)F^{(2)}(H_1,N)e_1},\qquad j=0,1,\dots,H_2-1.$$
$$\Delta_j^{(2)}(N)=\frac{F^{(2)}(j,N)e_1}{e_1^TF^{(1)}(0)F^{(2)}(H_1,N)e_1},\qquad j=H_1,\dots,N.$$

We also set

$$\chi_m(j)=\begin{cases}1, j\le m,\\0,\text{otherwise},\end{cases}$$
$$\bar{\chi}_m(j)=1-\chi_m(j).$$

Explicit representations of vector-matrix form for $\pi_{ij}^{(r)}(N),(i,j,r)\in S(N)$can be obtained using the above notations.

Theorem 1. If $\lambda_1,\lambda_2>0$ then for any $N>H_2$ the following formulas for stationary probabilities of the process $Q(t,N)$ hold true:

$$\pi_j^{(1)}(N)=\pi_{00}^{(1)}(N)\Delta_j^{(1)}(N), j=0,1,\dots,H_2-1,$$
$$\pi_j^{(2)}(N)=\pi_{00}^{(1)}(N)\Delta_j^{(2)}(N), j=H_1,\dots,N,$$
$$\pi_{nj}^{(1)}(N)=\pi_{00}^{(1)}(N)\left[\tfrac{(j+1)\nu}{\lambda_1}\bar{1}^T(n)\Delta_{j+1}^{(1)}(N)\chi_{H_2-2}(j)+\tfrac{H_1\nu}{\lambda_1}\bar{1}^T(n)\Delta_{H_1}^{(2)}(N)\bar{\chi}_{H_1-2}(j)\right],\quad(1)$$
$$j=0,1,\dots,H_2-1;$$
$$\pi_{nj}^{(2)}(N)=\pi_{00}^{(1)}(N)\left[\tfrac{(j+1)\nu}{\lambda_2}\bar{1}^T(n)\Delta_{j+1}^{(2)}(N)-\tfrac{H_1\nu}{\lambda_2}\bar{1}^T(n)\Delta_{H_1}^{(2)}(N)\chi_{H_2-1}(j)\right],\quad(2)$$
$$j=H_1,\dots,N-1;$$
$$\pi_{nN}^{(2)}(N)=\pi_{00}^{(1)}(N)\frac{([\lambda_2+(n-1)\mu+N\nu]e_{n-1}^T-\lambda_2e_{n-2}^T)\Delta_N^{(2)}(N)}{n\mu},\quad(3)$$

and

$$\pi_{00}^{(1)}(N)=\left\{\sum_{j=0}^{H_2-1}\bar{1}^T(n)\left(\Delta_j^{(1)}(N)+\frac{(j+1)\nu}{\lambda_1}\Delta_{j+1}^{(1)}(N)\chi_{H_2-2}(j)+\frac{H_1\nu}{\lambda_1}\Delta_{H_1}^{(2)}(N)\bar{\chi}_{H_1-2}(j)\right)+\right.$$
$$+\sum_{j=H_1}^{N-1}\bar{1}^T(n)\left(\Delta_j^{(2)}(N)+\tfrac{(j+1)\nu}{\lambda_2}\Delta_{j+1}^{(2)}(N)-\tfrac{H_1\nu}{\lambda_2}\Delta_{H_1}^{(2)}(N)\chi_{H_2-1}(j)\right)+\quad(4)$$
$$\left.+\bar{1}^T(n)\Delta_N^{(2)}(N)+\frac{1}{n\mu}([\lambda_2+(n-1)\mu+N\mu]e_{n-1}^T-\lambda_2e_{n-2}^T)\Delta_N^{(2)}(N)\right\}^{-1}.$$

Proof. To determine the probabilities $\pi_{ij}^{(r)}(N)$, we utilize the equality of probability flows through a separation boundary of the phase set into two subsets in the steady state (Walrand, 1988, Section II). For each $j=0,1,\dots,H_2-1$, we divide the phase set into two subsets: $S(N)=S_j^{(1)}(N)\cup\bar{S}_j^{(1)}(N)$, where $S_j^{(1)}(N)=\{(i_1,i_2,1):i_1=0,1,\dots,n;i_2\le j\}$. By equating probability flows through the separation boundary, we obtain:

$$\lambda_1\pi_{nj}^{(1)}(N)=(j+1)\nu\textstyle\sum_{i=0}^{n-1}\pi_{ij+1}^{(1)}(N)\,, j=0,\dots,H_1-2,\quad(5)$$
$$\lambda_1\pi_{nj}^{(1)}(N)=(j+1)\nu\sum_{i=0}^{n-1}\pi_{ij+1}^{(1)}(N)+H_1\nu\sum_{i=0}^{n-1}\pi_{iH_1}^{(2)}(N),$$
$$j=H_1-1,\dots,H_2-2,\quad(6)$$
$$\lambda_1\pi_{nH_2-1}^{(1)}(N)=H_1\nu\textstyle\sum_{i=0}^{n-1}\pi_{iH_1}^{(2)}(N)\,, j=H_2-1.\quad(7)$$

For each $j=H_1,\dots,N-1$ we consider the decomposition of the phase set $S(N)=S_j^{(2)}(N)\cup\bar{S}_j^{(2)}(N)$, where $S_j^{(2)}(N)=\{(i_1,i_2,2):i_1=0,1,\dots,n;H_1\le i_2\le j\}$. Equating probability flows through the separation boundary we arrive at the following set of equations:

$$\lambda_2\pi_{nj}^{(2)}(N)+H_1\nu\textstyle\sum_{i=0}^{n-1}\pi_{iH_1}^{(2)}(N)=(j+1)\nu\sum_{i=0}^{n-1}\pi_{ij+1}^{(2)}(N),\quad(8)$$
$$j=H_1,\dots,H_2-1,$$
$$\lambda_2\pi_{nj}^{(2)}(N)+H_1\nu\textstyle\sum_{i=0}^{n-1}\pi_{iH_1}^{(2)}(N)=(j+1)\nu\sum_{i=0}^{n-1}\pi_{ij+1}^{(2)}(N)+\lambda_1\pi_{nH_2-1}^{(1)}(N),\quad(9)$$
$$j=H_2,\dots,N-1.$$

Now for the phase points $(i,j,1)$, $i=0,1,\dots,n-1$, $j=0,1,\dots,H_2-1$, we write the Kolmogorov equations:

$$[\lambda_1+i\mu+j\nu]\pi_{ij}^{(1)}(N)=\lambda_1\pi_{i-1j}^{(1)}(N)+(i+1)\mu\pi_{i+1j}^{(1)}(N)+(j+1)\nu\pi_{i-1j+1}^{(1)}(N),\quad(10)$$
$$i=0,1,\dots,n-1, j=0,\dots,H_2-2, j\ne H_1-1,$$
$$[\lambda_1+i\mu+(H_2-1)\nu]\pi_{iH_2-1}^{(1)}(N)=\lambda_1\pi_{i-1H_2-1}^{(1)}(N)+(i+1)\mu\pi_{i+1H_2-1}^{(1)}(N),\quad(11)$$
$$i=0,1,\dots,n-1, j=H_2-1;$$

$$[h_1 + i\mu + (H_1 - 1)\nu]\pi^{(1)}_{iH_1-1}(N) = h_1\pi^{(1)}_{i-1H_1-1}(N) + (i+1)\mu\pi^{(1)}_{i+1H_1-1}(N) + \\ + H_1\nu\pi^{(1)}_{i-1H_1}(N) + H_1\nu\pi^{(2)}_{i-1H_1}(N), \\ i = 0,1,\dots,n-1, j = H_1 - 1. \tag{12}$$

In a similar manner for the phase points $(i, j, 2)$, $i = 0,1,\dots,n-1$, $j = H_1,\dots,N$, we find:

$$[\lambda_2 + i\mu + j\nu]\pi^{(2)}_{ij}(N) = \lambda_2\pi^{(2)}_{i-1j}(N) + (i+1)\mu\pi^{(2)}_{i+1j}(N) + (j+1)\nu\pi^{(2)}_{i-1j+1}(N), \\ i = 0,1,\dots,n-1, j = H_1,\dots,N-1, \tag{13}$$

$$[\lambda_2 + i\mu + N\nu]\pi^{(2)}_{iN}(N) = \lambda_2\pi^{(2)}_{i-1N}(N) + (i+1)\mu\pi^{(2)}_{i+1N}(N), \\ i = 0,1,\dots,n-1, j = N. \tag{14}$$

Reducing the equation (13) with use (7), we obtain

$$\pi^{(2)}_{nj}(N) = \frac{(j+1)\nu}{\lambda_2}\bar{1}^T(n)\pi^{(2)}_{j+1}(N), j = H_2,\dots,N-1.$$

Let us substitute the last expression in (13) for $j = H_2,\dots,N-1$ and write the result in a vector-matrix form

$$B^{(2)}(j)\pi^{(2)}_j(N) = A^{(2)}(j)\pi^{(2)}_{j+1}(N), j = H_2,\dots,N-1.$$

It follows that

$$\pi^{(2)}_j(N) = \left(\prod_{i=j}^{N-1}\bar{B}^{(2)}(i)A^{(2)}(i)\right)\pi^{(2)}_N(N), j = H_2,\dots,N-1. \tag{15}$$

Let us rewrite (8) in the following form:

$$\pi^{(2)}_{nj}(N) = \frac{(j+1)\nu}{\lambda_2}\bar{1}^T(n)\pi^{(2)}_{j+1}(N) - \frac{H_1\nu}{\lambda_2}\bar{1}^T(n)\pi^{(2)}_{H_1}(N), \\ j = H_1,\dots,H_2-1.$$

Using the representation of $\pi^{(2)}_{nj}(N)$ we write the system (13) as

$$B^{(2)}(j)\pi^{(2)}_j(N) = A^{(2)}(j)\pi^{(2)}_{j+1}(N) - D^{(2)}\pi^{(2)}_{H_1}(N), j = H_1,\dots,H_2-1.$$

From here we obtain

$$\pi^{(2)}_j(N) = \left(\prod_{i=j}^{N-1}\bar{B}^{(2)}(i)A^{(2)}(i)\right)\pi^{(2)}_N(N) - \left[\sum_{k=j}^{H_2-1}\left(\prod_{i=j}^{k-1}\bar{B}^{(2)}(i)A^{(2)}(i)\right)\bar{B}^{(2)}(k)D^{(2)}\right]\pi^{(2)}_{H_1}(N), \\ j = H_1,\dots,H_2-1.$$

From the last expression under $j = H_1$ we find $\pi^{(2)}_{H_1}(N)$:

$$\pi^{(2)}_{H_1}(N) = \left[E + \sum_{k=H_1}^{H_2-1}\left(\prod_{i=H_1}^{k-1}\bar{B}^{(2)}(i)A^{(2)}(i)\right)\bar{B}^{(2)}(k)D^{(2)}\right]^{-1} \cdot \left(\prod_{i=H_1}^{N-1}\bar{B}^{(2)}(i)A^{(2)}(i)\right)\pi^{(2)}_N(N).$$

Thus, we end up with the formula:

$$\pi^{(2)}_j(N) = \left\{E - \left[\sum_{k=j}^{H_2-1}\left(\prod_{i=j}^{k-1}\bar{B}^{(2)}(i)A^{(2)}(i)\right)\bar{B}^{(2)}(k)D^{(2)}\right]\times \right. \\ \left. \times\left[E + \sum_{k=H_1}^{H_2-1}\left(\prod_{i=H_1}^{k-1}\bar{B}^{(2)}(i)A^{(2)}(i)\right)\bar{B}^{(2)}(k)D^{(2)}\right]^{-1}\left(\prod_{i=H_1}^{j-1}\bar{B}^{(2)}(i)A^{(2)}(i)\right)\right\}\times \\ \times\left(\prod_{i=j}^{N-1}\bar{B}^{(2)}(i)A^{(2)}(i)\right)\pi^{(2)}_N(N), \\ j = H_1,\dots,H_2-1. \tag{16}$$

Combining the equations (15) and (16) we obtain

$$\pi^{(2)}_j(N) = F^{(2)}(j, N-1)\pi^{(2)}_N(N), j = H_1,\dots,N-1.$$

The equations (10) for $j = H_1,\dots,H_2-2$ and (13) are converted into

$$B^{(1)}(j)\pi^{(1)}_j(N) = A^{(1)}(j)\pi^{(1)}_{j+1}(N) + D^{(1)}\pi^{(2)}_{H_1}(N), j = H_1,\dots,H_2-2, \\ B^{(1)}(H_2-1)\pi^{(1)}_{H_2-1}(N) = D^{(1)}\pi^{(2)}_{H_1}(N).$$

A solution of the last equations is

$$\pi^{(1)}_j(N) = \left[\sum_{k=j}^{H_2-1}\left(\prod_{i=j}^{k-1}\bar{B}^{(1)}(i)A^{(1)}(i)\right)\bar{B}^{(1)}(k)D^{(1)}\right]\pi^{(2)}_{H_1}(N), \\ j = H_1,\dots,H_2-1.$$

From (12) we find

$$\pi^{(1)}_{H_1-1}(N) = \bar{B}^{(1)}(H_1-1)A^{(1)}(H_1-1)\pi^{(1)}_{H_1}(N) + \bar{B}^{(1)}(H_1-1)A^{(1)}(H_1-1)\pi^{(2)}_{H_1}(N) = \\ = \bar{B}^{(1)}(H_1-1)A^{(1)}(H_1-1)\left[\sum_{k=H_1}^{H_2-1}\left(\prod_{i=j}^{k-1}\bar{B}^{(1)}(i)A^{(1)}(i)\right)\bar{B}^{(1)}(k)D^{(1)}\right]\pi^{(2)}_{H_2}(N) + \\ + \bar{B}^{(1)}(H_1-1)A^{(1)}(H_1-1)\pi^{(2)}_{H_1}(N) = \\ = \bar{B}^{(1)}(H_1-1)A^{(1)}(H_1-1)\left[E + \sum_{k=H_1}^{H_2-1}\left(\prod_{i=j}^{k-1}\bar{B}^{(1)}(i)A^{(1)}(i)\right)\bar{B}^{(1)}(k)D^{(1)}\right]\pi^{(2)}_{H_1}(N).$$

Let us consider (10) for $j = 0,1, \dots, H_1 - 2$. It may be written in the form

$$B^{(1)}(j)\pi_j^{(1)}(N) = A^{(1)}(j)\pi_{j+1}^{(1)}(N), j = 0,1, \dots, H_1 - 2.$$

The solution of this recurrent relation is

$$\pi_j^{(1)}(N) = \left(\prod_{i=j}^{H_1-2} \bar{B}^{(1)}(i)A^{(1)}(i)\right)\pi_{H_1-1}^{(1)}(N) =$$

$$= \left(\prod_{i=j}^{H_1-1} \bar{B}^{(1)}(i)A^{(1)}(i)\right)\left[E + \sum_{k=H_1}^{H_2-1}\left(\prod_{i=j}^{k-1} \bar{B}^{(1)}(i)A^{(1)}(i)\right)\bar{B}^{(1)}(k)D^{(1)}\right]\pi_{H_1}^{(2)}(N),$$
$$j = 0,1, \dots, H_1 - 2.$$

Thus

$$\pi_j^{(1)}(N) = F^{(1)}(j)\pi_{H_1}^{(2)}(N) = F^{(1)}(j)F^{(2)}(H_1, N-1)\pi_N^{(2)}(N), j = 0,1, \dots, H_2 - 1.$$

We take into consideration the equations (14) for $i = 0,1, \dots, n-2$ and append them with an identity $\pi_{0N}^{(2)}(N) = \pi_{0N}^{(2)}(N)$. These equations may be represented in a vector-matrix form:

$$C(N)\pi_N^{(2)}(N) = \pi_{0N}^{(2)}(N)e_1$$

or

$$\pi_N^{(2)}(N) = \pi_{0N}^{(2)}(N)C^{-1}(N)e_1.$$

From here we find

$$\pi_j^{(2)}(N) = \pi_{0N}^{(2)}(N)F^{(2)}(j, N-1)C^{-1}(N)e_1 = \pi_{0N}^{(2)}(N)F^{(2)}(j, N)e_1,$$
$$j = H_1, \dots, N$$

and

$$\pi_j^{(1)}(N) = \pi_{0N}^{(2)}(N)F^{(1)}(j)F^{(2)}(H_1, N-1)C^{-1}(N)e_1 = \pi_{0N}^{(2)}(N)F^{(1)}(j)F^{(2)}(H_1, N)e_1, \quad (17)$$
$$j = 0,1, \dots, H_2 - 1.$$

Let us write out $\pi_{0N}^{(2)}(N)$ via $\pi_{00}^{(2)}(N)$. From (17) under $j = 0$

$$\pi_0^{(1)}(N) = \pi_{0N}^{(2)}(N)F^{(1)}(0)F^{(2)}(H_1, N)e_1$$

and

$$\pi_{0N}^{(2)}(N) = \pi_{00}^{(1)}(N)\left[e_1^T F^{(1)}(0)F^{(2)}(H_1, N)e_1\right]^{-1}.$$

Thus

$$\pi_j^{(2)}(N) = \pi_{00}^{(1)}(N)\Delta_j^{(2)}(N), j = H_1, \dots, N,$$

and

$$\pi_j^{(1)}(N) = \pi_{00}^{(1)}(N)\Delta_j^{(1)}(N), j = 0,1, \dots, H_2 - 1.$$

Now the formulas (1) and (2) may be deduced from (5)–(9) and the formula (3) from (14) for $i = n - 1$. Obviously, (4) is an implication of a normalization condition.

The theorem is proved.

The conditions of Theorem 1 from Falin (Falin, 1988) are met for the Markov chains $Q(t, N), N > H_2$ and $Q(t)$. This implies a stochastic ordering of the probability distributions related to $Q(t, N)$ and $Q(t)$. Additionally, the ergodicity condition ensures the existence of the stationary distribution $\pi_{ij}^{(r)}, (i, j, r) \in S$ for $Q(t)$ and

$$\lim_{N\to\infty} \pi_{ij}^{(r)}(N) = \pi_{ij}^{(r)}. \quad (18)$$

As per the given equation,

$$\lim_{N\to\infty} \pi_j^{(r)}(N)/\pi_{00}^{(r)}(N) = \pi_j^{(r)}/\pi_{00}^{(r)} = \Delta_j^{(r)} = (\Delta_{0j}^{(r)}, \dots, \Delta_{n-1j}^{(r)})^T,$$

where $\pi_j^{(r)} = (\pi_{0j}^{(r)}, \dots, \pi_{n-1j}^{(r)})^T, j = 0,1, \dots; r = 1,2$.

Now the result of Theorem 1 may be converted in the similar representation for $\pi_{ij}^{(r)}, (i, j, r) \in S$.

Theorem 2. If $\lambda_1, \lambda_2 > 0$ and the ergodicity condition $\frac{\lambda_2}{n\mu} < 1$ is true, then a stationary regime of the process $Q(t)$ exists and the stationary probabilities are:

$$\pi_j^{(1)} = \pi_{00}^{(1)}\Delta_j^{(1)}, j = 0,1, \dots, H_2 - 1, \quad (19)$$

$$\pi_j^{(2)} = \pi_{00}^{(1)}\Delta_j^{(2)}, j = H_1, \dots, N, \quad (20)$$

$$\pi_{nj}^{(1)} = \pi_{00}^{(1)}\left[\frac{(j+1)\nu}{\lambda_1}\bar{1}^T(n)\Delta_{j+1}^{(1)}\chi_{H_2-2}(j) + \frac{H_1\nu}{\lambda_1}\bar{1}^T(n)\Delta_{H_1}^{(2)}\chi_{H_1-2}(j)\right], \quad (21)$$
$$j = 0,1, \dots, H_2 - 1;$$

$$\pi_{nj}^{(2)} = \pi_{00}^{(1)}\left[\frac{(j+1)\nu}{\lambda_2}\bar{1}^T(n)\Delta_{j+1}^{(2)} - \frac{H_1\nu}{\lambda_2}\bar{1}^T(n)\Delta_{H_1}^{(2)}\chi_{H_2-1}(j)\right], j = H_1, \dots, N - 1, \quad (22)$$

and

$$\pi_{00}^{(1)} = \left\{\sum_{j=0}^{H_2-1}\bar{1}^T(n)\left(\Delta_j^{(1)} + \frac{(j+1)\nu}{\lambda_1}\Delta_{j+1}^{(1)}\chi_{H_2-2}(j) + \frac{H_1\nu}{\lambda_1}\Delta_{H_1}^{(2)}\chi_{H_1-2}(j)\right) + \right.$$
$$\left. + \sum_{j=H_1}^{\infty}\bar{1}^T(n)\left(\Delta_j^{(2)} + \frac{(j+1)\nu}{h_2}\Delta_{j+1}^{(2)} - \frac{H_1\nu}{h_2}\Delta_{H_1}^{(2)}\chi_{H_2-1}(j)\right)\right\}^{-1}. \quad (23)$$

The results of Theorem 2 provide an explicit representation of a vector-matrix type for the steady-state distribution of the $((M\_Q/M)/n)/\infty$-queue under the hysteresis control strategy. For the specific cases of $n = 1$ and $n = 2$, they can be simplified to closed scalar formulas of an explicit form for stationary probabilities via the model parameters. These results are consistent with the known results obtained in earlier works. It is important to note that the widely used threshold control strategy is a specific case of hysteresis when $H_1 = H_2$. The theorem's proof provides insight into constructing an efficient calculation scheme for the stationary probabilities $\pi_{ij}^{(r)}, (i,j,r) \in S$.

3. Optimization problem

To demonstrate the importance and usefulness of the obtained results, let us consider the problem of maximizing the total profit from the system. A multi-objective problem is the most natural and straightforward approach.

Let $f_1(t, H_1, H_2)$ represent the number of calls served in the system, $f_2(t, H_1, H_2)$ represent the number of customers rejected for service and became sources of retrials, and $f_3(t, H_1, H_2)$ represent the number of switches in the rate of the input flow.

If the conditions of Lemma 1 are met, there exist limits $\lim_{t\to\infty} t^{-1} f_k(t, H_1, H_2), k = 1,2,3$.. Let us denote these limits as $f_k(H_1, H_2), \mathrm{k} = 1,2,3.$

The objective of maximizing the profit from the system's functioning is to find thresholds H_1 and H_2 $(H_1 \le H_2)$ that solve the multi-objective problem.

$$f_1(H_1, H_2) \to \max,$$
$$f_2(H_1, H_2) \to \min,$$
$$f_3(H_1, H_2) \to \min,$$
$$H_1, H_2 \in \{0,1,\dots\},$$
$$H_1 \le H_2.$$

Solving a multi-objective optimization problem is not as straightforward as solving a conventional single-objective problem. Various techniques are widely used, such as the method of global criterion and the utility function method. When considering the economic nature of the problem, the most convenient method to solve it in practice is the method of linear convolution of criteria. The solution to the original multi-objective problem is obtained by solving the one-criterion problem.

$$W(H_1, H_2) = C_1 f_1(H_1, H_2) - C_2 f_2(H_1, H_2) - C_3 f_3(H_1, H_2) \to \max,$$
$$H_1, H_2 \in \{0,1,\dots\},$$
$$H_1 \le H_2,$$

where C_1 is the profit from the servicing of a customer; C_2 is a fine for the service rejection; C_3 is a fine for switching the service rate.

The limit functionals $f_k(H_1, H_2), \mathrm{k} = 1,2,3$ can be expressed using the stationary distribution of the system $\pi_{ij}^{(r)}$, $(i,j,r) \in S$:

$$f_1(H_1, H_2) = \sum_{j=0}^{H_2-1} \sum_{i=1}^{n} i\,\mu\pi_{ij}^{(1)} + \sum_{j=H_1}^{\infty} \sum_{i=1}^{n} i\,\mu\pi_{ij}^{(2)},$$

$$f_2(H_1, H_2) = \sum_{j=0}^{H_2-1} \lambda_1\,\pi_{nj}^{(1)} + \sum_{j=H_1}^{\infty} \lambda_2\,\pi_{nj}^{(2)},$$

$$f_3(H_1, H_2) = \lambda_1\,\pi_{nH_2-1}^{(1)} + H_1 \nu \sum_{i=0}^{n-1} \pi_{iH_1}^{(2)},$$

which together with the result of Theorem 2 gives the algorithm of solving of the described optimization problem.

To demonstrate the obtained results, let us consider an example of solving the problem of selecting the optimal hysteresis control strategy for the $((M/M)/2)/15$ queue with the following parameters: $\nu = 1.1$; $\mu = 0.1$; $\lambda_1(j) = 1 + \frac{j}{1.3}$, $\lambda_2(j) = \frac{j+2}{2j+1}$. We will solve the above-described multicriteria optimization problem using the method of linear convolution of criteria with the cost coefficients $C_1 = 8$; $C_2 = 2$; $C_3 = 1.5$.

Figure 1 displays the graph of the objective function's dependence on the threshold values.

The objective function attains its maximum value of $W(2,9) = 4.513$ when the thresholds are set to $H_1 = 2$ and $H_2 = 9$.

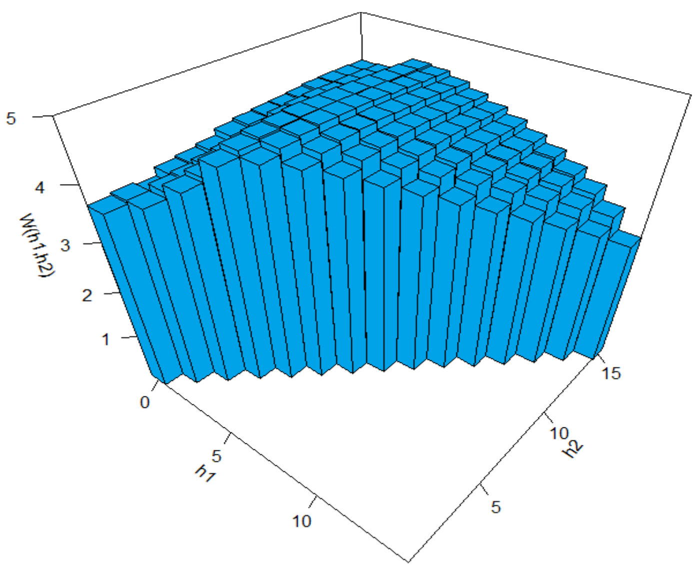

Fig. 1. Objective function for $((M/M)/2)/15$ queue

Discussion and conclusions

The paper examines the steady state of the Markov model for retrial queue systems with input flow of customers controlled by hysteresis strategies. The study derives explicit vector-matrix formulas for stationary probabilities based on system parameters. In certain scenarios, such as uncontrolled queues or threshold control strategies, these representations align with previously established findings by other authors. The investigative technique is based on approximating the initial queue with the truncated state space queue. This approach provides an efficient computational algorithm for computing steady state distributions. Explicit formulae of stationary probabilities are useful for analyzing the system, calculating performance measures, and solving optimal control problems. As an illustration of the results' application, we present and solve a multi-objective optimization problem aimed at maximizing the total income from the system.

References

Artalejo, J. R., & Gomez-Corral, A. (2008). *Retrial queueing systems.* Springer-Verlag.

Atencia, I., Lebedev, E., Ponomarov, V., & Livinska, H. (2019). Special retrial queues with state-dependent input rate. *Communications in Computer and Information Science*, 1109, 73–85. https://doi.org/10.1007/978-3-030-33388-1_7

Das, I., Lobiyal, D. K., & Katti, C. P. (2016). Queuing effect on multipath routing in mobile ad hoc networks. *International Journal of Information Engineering and Electronic Business* (IJIEEB), 8, 62–68. https://doi.org/10.5815/ijieeb.2016.01.07

Falin, G. I. (1988). Comparability of migration process. *Theory Probability and Its Application,* 33, 392–396.

Falin, G. I., & Gómez-Corral, A. (2000). On a bivariate Markov process arising in the theory of single-server retrial queues. *Statistica Neerlandica*, 54, 67–78.

Falin, G. I., & Templeton, J. G. C. (1997). *Retrial queues*. Chapman & Hall.

Fiems, D. (2022). Retrial queues with constant retrial times. *Queueing Systems*, 103, 1–19.

Gómez-Corral, A. (2002). Analysis of a single-server retrial queue with quasi-random input and nonpreemptive priority. *Computers & Mathematics with Applications*, 43, 767–782.

Lebedev, E., Ponomarov, V., & Livinska, H. (2022). On explicit formulas of steady-state probabilities for the $[M/M/c/c+m]$-type retrial queue. *Springer Proceedings in Mathematics & Statistics*, 408, 211–222.

Makushenko, I., Usar, I., Livinska, H., & Sharapov, M. (2023). Optimal threshold strategies for retrial systems with queue. *Journal of Computational and Applied Mathematics*, 427, 115–136. https://doi.org/10.1016/j.cam.2023.115136

Mandelbaum, A., & Pats, G. (1994). *State-dependent queues: approximations and applications.* Haifa : Technion Institute.

Rao, A. S. (2017) Improving the serviceability of a prepaid autorickshaw counter using queuing model: an optimization approach. *International Journal of Information Technology and Computer Science (IJITCS)*, 9, 19–27. https://doi.org/10.5815/ijitcs.2017.12.03

Saxena, D., Chauhan, R. K., & Kait, R. (2016). Dynamic fair priority optimization task scheduling algorithm in cloud computing: concepts and implementations. *International Journal of Computer Network and Information Security (IJCNIS)*, 8, 41–48. https://doi.org/10.5815/ijcnis.2016.02.05

Walrand, J. (1988). *An introduction to queueing networks*. Prentice Hall, Englewood Cliffs.

Отримано редакцією журналу / Received: 10.03.24
Прорецензовано / Revised: 26.04.24
Схвалено до друку / Accepted: 24.05.24

Вадим ПОНОМАРЬОВ, канд. фіз.-мат. наук
ORCID ID: 0000-0002-7813-3298
e-mail: vponomarov@knu.ua
Київський національний університет імені Тараса Шевченка, Київ, Україна

ТОЧНІ ФОРМУЛИ ДЛЯ МАРКОВСЬКИХ МОДЕЛЕЙ СИСТЕМ З ПОВТОРНИМИ ВИКЛИКАМИ У ГІСТЕРЕЗИСНИХ СТРАТЕГІЯХ КЕРУВАННЯ ПАРАМЕТРАМИ

Досліджено модель Маркова для багатосерверних систем з повторними викликами, вхідний потік у яких залежить від кількості викликів на орбіті і контролюється гістерезисними стратегіями. Система складається з n-однакових серверів. Якщо вимога надходить у момент, коли в системі є вільний сервер, то вона займає його й обслуговується протягом експоненціально розподіленого часу. Якщо всі сервери зайняті, то вимога потрапляє на орбіту і повертається на обслуговування через випадковий проміжок часу. Процес обслуговування системи описується тривимірним ланцюгом Маркова з неперервним часом. Спочатку ми встановлюємо умови існування стаціонарного режиму. Далі наводимо явні формули векторно-матричного типу для стаціонарних імовірностей процесу. Метод дослідження базується на апроксимації вихідної системи системою з урізаним простором станів і містить ефективний обчислювальний алгоритм. Для n = 1 та n = 2 отримані подання можна спростити до явних скалярних формул для стаціонарних імовірностей через параметри моделі. Ці результати узгоджуються з відомими роботами інших авторів. Для демонстрації практичної значущості отриманих результатів ми формулюємо багатоцільову задачу максимізації сумарного доходу, що генерується системою. Ураховуючи економічну природу задачі, ми використали метод лінійної згортки критеріїв. Отримані представлення дають змогу визначити оптимальну стратегію, що максимізує цільову функцію.

Ключові слова: ***системи з повторними викликами, стаціонарні ймовірності, процеси квазінародження та загибелі, оптимальне керування.***

Автор заявляє про відсутність конфлікту інтересів. Спонсори не брали участі в розробленні дослідження; у зборі, аналізі чи інтерпретації даних; у написанні рукопису; в рішенні про публікацію результатів.

The author declares no conflicts of interest. The funders had no role in the design of the study; in the collection, analyses or interpretation of data; in the writing of the manuscript; in the decision to publish the results.