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О. Д. Борисенко[1], *к.ф.-м.н., доц.*
О. В. Борисенко[2], *к.ф.-м.н., доц.*

Вимирання та виживання у залежній від щільності популяції хижака стохастичній моделі хижак-жертва із стрибками

[1] Київський національний університет імені Тараса Шевченка, 01033, Київ, вул. Володимирська,64.
e-mail: borysenko13101953@gmail.com
[2] Національний технічний університет України "КПІ" 03056, г.Київ-56, проспект Перемоги, 37
e-mail: oborisenko1373@gmail.com

O. D. Borysenko[1], *Ph.D.,Associate Professor*
O. V. Borysenko[2], *Ph.D.,Associate Professor*

Extinction and persistence in stochastic predator population density-dependent predator-prey model with jumps

[1]Taras Shevchenko National University of Kyiv, 01033, Kyiv, 64 Volodymyrska st.
e-mail: borysenko13101953@gmail.com
[2]National Technical University of Ukraine "KPI", 37 Prospect Peremogy, Kyiv 03056
e-mail: oborisenko1373@gmail.com

Одержано достатні умови вимирання, не виживання у середньому, слабкого та сильного виживання у середньому популяцій у стохастичній, залежній від щільності хижаків моделі хижак-жертва з функціональною відповіддю типу Холлінг II.

Ключові слова: стохастична модель хижак-жертва, залежність від щільності хижаків, вимирання, не виживання у середньому, слабке та сильне виживання у середньому.

The non-autonomous stochastic density dependent predator-prey model with Holling-type II functional response disturbed by white noise, centered and non-centered Poisson noises is investigated. Corresponding system of stochastic differential equations has a unique, positive, global (no explosions in a finite time) solution. Sufficient conditions are obtained for extinction, non-persistence in the mean, weak and strong persistence in the mean of a predator and prey population densities in the considered stochastic predator-prey model.

Key Words: stochastic predator-prey model, predator density dependence, extinction, non-persistence in the mean, weak and strong persistence in the mean.

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1 Introduction

The deterministic autonomous Rosenzweig-Mac'Arthur predator-prey model ([1]) has a form

$$\begin{aligned} dx_1(t) &= x_1(t)\left(a_1 - b_1x_1(t) - \tfrac{cx_2(t)}{1+mx_1(t)}\right)dt, \\ dx_2(t) &= x_2(t)\left(-a_2 + \tfrac{\kappa c x_1(t)}{1+mx_1(t)}\right)dt, \end{aligned} \quad (1)$$

where $x_1(t)$ and $x_2(t)$ are the prey and predator population densities at time t, respectively; $a_1 > 0$ is the growth rate of prey x_1; $b_1 > 0$ measures the strength of competition among individuals of species x_1; c is the maximum ingestion rate; $m > 0$ is the half-saturation; $a_2 > 0$ is the death rate of predator x_2, and $\kappa > 0$ is the conversion factor.

In the paper [2] the stochastic version of model (1) is considered. The authors studied the non-autonomous density dependent predator-prey model with Holling-type II functional response, disturbed by white noise and jumps generated by centered and non-centered Poisson measures. So the authors take into account the influence on the predator-prey model of the predator population density dependence and of such random factors as fires, earthquakes, hurricanes etc. This model is driven by the system of stochastic differential equations

$$\begin{aligned} dx_i(t) = {} & x_i(t)\left[(-1)^{i-1}\left(a_i(t) - \tfrac{c_i(t)x_{3-i}(t)}{1+m(t)x_1(t)}\right)\right. \\ & \left. - b_i(t)x_i(t)\right]dt + \sigma_i(t)x_i(t)dw_i(t) \\ & + \int_{\mathbb{R}} \gamma_i(t,z)x_i(t^-)\tilde{\nu}_1(dt,dz) \\ & + \int_{\mathbb{R}} \delta_i(t,z)x_i(t^-)\nu_2(dt,dz), \\ & x_i(0) = x_{i0} > 0, \ i = 1,2. \end{aligned} \quad (2)$$

where $x_1(t)$ and $x_2(t)$ are the prey and predator population densities at time t, respectively, $x_i(t^-)$, $i = 1,2$ is a left limit of $x_i(t), i = 1,2$, $b_2(t) > 0$ is the predator density dependence rate at time t, $c_2(t) = \kappa c_1(t)$, $w_i(t), i = 1,2$ are independent standard one-dimensional Wiener processes, $\nu_i(t,A), i = 1,2$ are independent Poisson measures, which are independent on $w_i(t), i = 1,2$, $\tilde{\nu}_1(t,A) = \nu_1(t,A) - t\Pi_1(A)$, $E[\nu_i(t,A)] = t\Pi_i(A), i = 1,2$, $\Pi_i(A), i = 1,2$ are finite measures on the Borel sets A in $\mathbb{R}$. In [2] the authors proved that there is a unique positive global (no explosion in a finite time) solution to the system (2) and that this solution is stochastically ultimately bounded. It is deduced the sufficient conditions for stochastic permanence of the system (2).

The impact of centered and non-centered Poisson noises to the stochastic non-autonomous predator-prey models is studied in the papers [3], [4].

In this paper, we derive the sufficient conditions for the extinction, weak and strong persistence in the mean of predator and prey population densities, driven by system (2).

In the following we will use the notations $X(t) = (x_1(t), x_2(t))$, $X_0 = (x_{10}, x_{20})$, $|X(t)| = \sqrt{x_1^2(t) + x_2^2(t)}$, $\mathbb{R}^2_+ = \{X \in \mathbb{R}^2 :\ x_1 > 0, x_2 > 0\}$,

$$\beta_i(t) = \frac{\sigma_i^2(t)}{2} + \int_{\mathbb{R}} [\gamma_i(t,z) - \ln(1+\gamma_i(t,z))]\Pi_1(dz) - \int_{\mathbb{R}} \ln(1+\delta_i(t,z))]\Pi_2(dz), i = 1,2.$$

$$\bar{f}(t) = \frac{1}{t}\int_0^t f(s)\,ds,$$

$$f_* = \liminf_{t\to\infty} f(t),\ \ f^* = \limsup_{t\to\infty} f(t).$$

For the bounded, continuous functions $f_i(t), t \in [0, +\infty), i = 1,2$, let us denote

$$f_{i\sup} = \sup_{t\geqslant 0} f_i(t), f_{i\inf} = \inf_{t\geqslant 0} f_i(t), i = 1,2.$$

2 Auxiliary results

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, $w_i(t), i = 1,2, t \geqslant 0$ are independent standard one-dimensional Wiener processes on $(\Omega, \mathcal{F}, \mathbf{P})$, and $\nu_i(t,A), i = 1,2$ are independent Poisson measures defined on $(\Omega, \mathcal{F}, \mathbf{P})$ independent on $w_i(t), i = 1,2$. Here $\mathsf{E}[\nu_i(t,A)] = t\Pi_i(A), i = 1,2$, $\tilde{\nu}_i(t,A) = \nu_i(t,A) - t\Pi_i(A), i = 1,2$, $\Pi_i(\cdot), i = 1,2$ are finite measures on the Borel sets in $\mathbb{R}$. On the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ we consider an increasing, right continuous family of complete sub-σ-algebras $\{\mathcal{F}_t\}_{t\geqslant 0}$, where $\mathcal{F}_t = \sigma\{w_i(s), \nu_i(s,A), s \leqslant t, i = 1,2\}$.

We need the following assumption.

Assumption 1. It is assumed, that $a_i(t), b_i(t), c_i(t), \sigma_i(t), \gamma_i(t,z), \delta_i(t,z), i = 1,2$, $m(t)$ are bounded, continuous on t functions, $a_i(t) > 0, i = 1,2$, $b_{i\inf} > 0$, $c_{i\inf} > 0, i = 1,2$, $m_{\inf} > 0$, and $\ln(1+\gamma_i(t,z)), \ln(1+\delta_i(t,z)), i = 1,2$ are bounded, $\Pi_i(\mathbb{R}) < \infty, i = 1,2$.

In what follows we will assume that Assumption 1 holds. In [2] the following result is proved.

Lemma 2.1. *(Theorem 2.1, [2]) There exists a unique global solution $X(t)$ to the system* (2) *for any initial value $X(0) = X_0 \in \mathbb{R}^2_+$, and* $\mathbf{P}\{X(t) \in \mathbb{R}^2_+\} = 1$.

Lemma 2.2. *For the density of the population $x_i(t), i = 1,2$ we have*

$$\limsup_{t\to\infty} \frac{\ln x_i(t)}{t} \leqslant 0,\ i = 1,2 \qquad a.s.$$

Proof. By the Itô's formula we have for $i = 1,2$

$$e^t \ln x_i(t) - \ln x_{i0} = \int_0^t e^s \Big\{ \ln x_i(s) + (-1)^{i-1}\Big[a_i(s) - \frac{c_i(s)x_{3-i}(s)}{1+m(s)x_1(s)}\Big] - b_i(s)x_i(s) - \frac{\sigma_i^2(s)}{2} + \int_{\mathbb{R}} [\ln(1+\gamma_i(s,z)) - \gamma_i(s,z)]\Pi_1(dz)\Big\} ds + \psi_i(t), \quad (3)$$

where

$$\psi_i(t) = \int_0^t e^s \sigma_i(s) dw_i(s) + \int_0^t\int_{\mathbb{R}} e^s \ln(1+\gamma_i(s,z))\tilde{\nu}_1(ds,dz) + \int_0^t\int_{\mathbb{R}} e^s \ln(1+\delta_i(s,z))\nu_2(ds,dz),\ i = 1,2.$$

From the exponential inequality ([5], Lemma 2.2) we have

$$\mathbf{P}\Big\{\sup_{0\leqslant t\leqslant T} \zeta_i(\mu,t) > \beta\Big\} \leqslant e^{-\mu\beta}, \quad \forall 0 < \mu \leqslant 1,\ \beta > 0,\ i = 1,2$$

where

$$\zeta_i(\mu,t)=\psi_i(t)-\frac{\mu}{2}\int_0^t e^{2s}\sigma_i^2(s)ds -\frac{1}{\mu}\iint_{0\mathbb{R}}^{t}\Big[(1+\gamma_i(s,z))^{\mu e^s}-1 -\mu e^s\ln(1+\gamma_i(s,z))\Big]\Pi_1(dz)ds -\frac{1}{\mu}\iint_{0\mathbb{R}}^{t}\left[(1+\delta_i(s,z))^{\mu e^s}-1\right]\Pi_2(dz)ds,$$

$i=1,2$.

Choosing $T=k\tau, k\in\mathbb{N}, \tau>0, \mu=e^{-k\tau}$, $\beta=\theta e^{k\tau}\ln k$, $\theta>1$ we get

$$\mathbf{P}\left\{\sup_{0\leqslant t\leqslant k\tau}\zeta_i(\mu,t)>\theta e^{k\tau}\ln k\right\}\leqslant\frac{1}{k^\theta},$$

$i=1,2$. By the Borel-Cantelli lemma for almost all $\omega\in\Omega$, there is a random integer $k_0(\omega)$, such that for $\forall k\geqslant k_0(\omega)$ and $0\leqslant t\leqslant k\tau$

$$\begin{aligned}\psi_i(t)&\leqslant\frac{1}{2e^{k\tau}}\int_0^t e^{2s}\sigma_i^2(s)ds\\&+e^{k\tau}\iint_{0\mathbb{R}}^{t}\Big[(1+\gamma_i(s,z))^{e^{s-k\tau}}-1-e^{s-k\tau}\\&\times\ln(1+\gamma_i(s,z))\Big]\Pi_1(dz)ds\\&+e^{k\tau}\iint_{0\mathbb{R}}^{t}\Big[(1+\delta_i(s,z))^{e^{s-k\tau}}-1\Big]\Pi_2(dz)ds\\&+\theta e^{k\tau}\ln k,\ i=1,2.\end{aligned}\tag{4}$$

Appling the inequality $x^r\leqslant 1+r(x-1)$, $\forall x\geqslant 0$, $0\leqslant r\leqslant 1$ with $x=1+\gamma_i(s,z)$, $r=e^{s-k\tau}$, then with $x=1+\delta_i(s,z)$, $r=e^{s-k\tau}$, we derive from (4) the estimate

$$\begin{aligned}\psi_i(t)&\leqslant\frac{1}{2e^{k\tau}}\int_0^t e^{2s}\sigma_i^2(s)ds\\&+\iint_{0\mathbb{R}}^{t}e^s\left[\gamma_i(s,z)-\ln(1+\gamma_i(s,z))\right]\Pi_1(dz)ds\\&+\iint_{0\mathbb{R}}^{t}e^s\delta_i(s,z)\Pi_2(dz)ds+\theta e^{k\tau}\ln k,\end{aligned}\tag{5}$$

$i=1,2$. So from (3) and (5) we get for $i=1,2$

$$\begin{aligned}e^t\ln x_i(t)&\leqslant\ln x_{i0}+\int_0^t e^s\Big\{\ln x_i(s)\\&+(-1)^{i-1}\left[a_i(s)-\frac{c_i(s)x_{3-i}(s)}{1+m(s)x_1(s)}\right]-b_i(s)x_i(s)\\&-\frac{\sigma_i^2(s)}{2}\left(1-e^{s-k\tau}\right)+\int_{\mathbb{R}}\delta_i(s,z)\Pi_2(dz)\Big\}ds\\&+\theta e^{k\tau}\ln k\leqslant\ln x_{i0}+\int_0^t e^s[\ln x_i(s)-b_{i\,\inf}x_i(s)\\&+K_i]ds+\theta e^{k\tau}\ln k\leqslant\ln x_{i0}+L(e^t-1)\\&+\theta e^{k\tau}\ln k,\forall k\geqslant k_0(\omega),0\leqslant t\leqslant k\tau,\end{aligned}$$

for some constant $L>0$, where

$$K_1=\alpha_{1\sup},K_2=\sup_{t\geqslant0}\int_{\mathbb{R}}\delta_2(t,z)\Pi_2(dz)+\frac{c_{2\sup}}{m_{\inf}}.$$

So for any $(k-1)\tau\leqslant t\leqslant k\tau$, $\forall k\geqslant k_0(\omega)$ we have

$$\frac{\ln x_i(t)}{\ln t}\leqslant e^{-t}\frac{\ln x_{i0}}{\ln t}+\frac{L}{\ln t}(1-e^{-t}) +\frac{\theta e^{k\tau}\ln k}{e^{(k-1)\tau}\ln(k-1)\tau},i=1,2\quad\text{a.s.}$$

Therefore

$$\limsup_{t\to\infty}\frac{\ln x_i(t)}{\ln t}\leqslant\theta e^\tau,\quad i=1,2,\ \forall\theta>1,\ \forall\tau>0,\quad\text{a.s.}$$

If $\theta\downarrow1$, $\tau\downarrow0$, then we obtain

$$\limsup_{t\to\infty}\frac{\ln x_i(t)}{\ln t}\leqslant1,\ i=1,2\quad\text{a.s.}$$

So

$$\limsup_{t\to\infty}\frac{\ln x_i(t)}{t}\leqslant0,\ i=1,2\quad\text{a.s.}$$

The lemma is proved.

3 Persistence and extinction

Definition 3.1. The solution $X(t)=(x_1(t),x_2(t))$, $t\geqslant0$ to the system (2) will be said extinct if for every initial data $X_0\in\mathbb{R}^2_+$, we have $\lim_{t\to\infty}x_i(t)=0$ a.s., $i=1,2$.

Theorem 3.1. *If*

$$\bar{q}_i^*=\limsup_{t\to\infty}\frac{1}{t}\int_0^t q_i(s)ds<0,$$

where

$$q_1(t)=a_1(t)-\beta_1(t),\ q_2(t)=-a_2(t)+\frac{c_2(t)}{m(t)}-\beta_2(t)$$

then the solution $X(t)$ to the system (2) with the initial condition $X_0\in\mathbb{R}^2_+$ will be extinct.

Proof.Under Assumption 1 there exists a unique global solution $X(t)$ to the system (2) for any initial value $X(0)=X_0>0$, and $\mathbf{P}\{X(t)\in\mathbb{R}^2_+\}=1$ (Lemma 2.1). By the Itô's formula, we have

$$\begin{aligned}\ln x_i(t)&=\ln x_{i0}+\int_0^t\Big\{(-1)^{i-1}\Big[a_i(s)\\&-\frac{c_i(s)x_{3-i}(s)}{1+m(s)x_1(s)}\Big]-\beta_i(s)-b_i(s)x_i(s)\Big\}ds\\&+M_i(t)\leqslant\ln x_{i0}+\int_0^t q_i(s)ds+M_i(t),\end{aligned}\tag{6}$$

$i = 1, 2$, where the martingale

$$M_i(t) = \int_0^t \sigma_i(s) dw_i(s) + \int_0^t\int_{\mathbb{R}} \ln(1+\gamma_i(s,z))\tilde{\nu}_1(ds,dz) + \int_0^t\int_{\mathbb{R}} \ln(1+\delta_i(s,z))\tilde{\nu}_2(ds,dz), \ i=1,2, \quad (7)$$

has quadratic variation

$$\langle M_i, M_i\rangle(t) = \int_0^t \sigma_i^2(s)ds + \int_0^t\int_{\mathbb{R}} \ln^2(1+\gamma_i(s,z))\Pi_1(dz)ds + \int_0^t\int_{\mathbb{R}} \ln^2(1+\delta_i(s,z))\Pi_2(dz)ds \leqslant Kt, \ i=1,2.$$

Then from the strong law of large numbers for local martingales ([6]) we have

$$\lim_{t\to\infty} M_i(t)/t = 0, i = 1,2 \text{ a.s.}$$

Therefore, from (6) we obtain

$$\limsup_{t\to\infty} \frac{\ln x_i(t)}{t} \leqslant \limsup_{t\to\infty} \frac{1}{t}\int_0^t q_i(s)ds < 0, \quad \text{a.s.}$$

So $\lim_{t\to\infty} x_i(t) = 0, i = 1,2$ a.s. The theorem is proved.

Definition 3.2. ([7]) The solution $X(t) = (x_1(t), x_2(t))$, $t \geqslant 0$ to the system (2) will be said non-persistent in the mean if for every initial data $X_0 \in \mathbb{R}^2_+$, we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)ds = 0 \text{ a.s.}, \ i = 1,2.$$

Theorem 3.2. *If $\bar{q}_i^* = 0$, $i = 1,2$, then the solution $X(t) = (x_1(t), x_2(t))$ to the system (2) with the initial value $X_0 \in \mathbb{R}^2_+$ will be non-persistent in the mean.*

Proof. From the first equality in (6) we have

$$\ln x_i(t) \leqslant \ln x_{i0} + \int_0^t q_i(s)ds - b_{i\inf}\int_0^t x_i(s)ds + M_i(t), \ i=1,2, \quad (8)$$

where martingales $M_i(t), i = 1,2$ are defined in (7). From the definition of $\bar{q}_i^*, i = 1,2$ and the strong law of large numbers for $M_i(t), i = 1,2$ it follows, that $\forall\varepsilon > 0$, $\exists t_0 \geqslant 0$, and $\exists \Omega_\varepsilon \subset \Omega$, $\mathbf{P}(\Omega_\varepsilon) \geqslant 1-\varepsilon$ such that

$$\frac{1}{t}\int_0^t q_i(s)ds \leqslant \bar{q}_i^* + \frac{\varepsilon}{2}, \quad \frac{M_i(t)}{t} \leqslant \frac{\varepsilon}{2},$$
$$i = 1,2, \ \forall t \geqslant t_0, \ \omega\in\Omega_\varepsilon.$$

So, from (8) we derive

$$\ln x_i(t) - \ln x_{i0} \leqslant t(\bar{q}_i^* + \varepsilon) - b_{i\inf}\int_0^t x_i(s)ds = t\varepsilon - b_{i\inf}\int_0^t x_i(s)ds, \ i=1,2, \quad (9)$$

$\forall t \geqslant t_0, \ \omega\in\Omega_\varepsilon$.

Let $y_i(t) = \int_0^t x_i(s)ds, i = 1,2$, then from (9) we have for $i = 1,2$

$$\ln\left(\frac{dy_i(t)}{dt}\right) \leqslant \varepsilon t - b_{i\inf} y_i(t) + \ln x_{i0}, \forall t \geqslant t_0, \omega\in\Omega_\varepsilon.$$

Therefore

$$e^{b_{i\inf}y_i(t)}\frac{dy_i(t)}{dt} \leqslant x_{i0}e^{\varepsilon t}, \ i=1,2, \ \forall t\geqslant t_0, \ \omega\in\Omega_\varepsilon.$$

By integrating the last inequality from t_0 to t we obtain

$$e^{b_{i\inf}y_i(t)} \leqslant \frac{b_{i\inf}x_{i0}}{\varepsilon}\left(e^{\varepsilon t} - e^{\varepsilon t_0}\right) + e^{b_{i\inf}y_i(t_0)},$$

$i = 1,2, \forall t \geqslant t_0, \ \omega\in\Omega_\varepsilon$. So

$$y_i(t) \leqslant \frac{1}{b_{i\inf}}\ln\left[e^{b_{i\inf}y_i(t_0)} + \frac{b_{i\inf}x_{i0}}{\varepsilon}\left(e^{\varepsilon t} - e^{\varepsilon t_0}\right)\right],$$

$i = 1,2, \ \forall t \geqslant t_0, \ \omega\in\Omega_\varepsilon$, and therefore

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t x_i(s)ds \leqslant \frac{\varepsilon}{b_{i\inf}}, \ i=1,2, \ \forall\omega\in\Omega_\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and $x_i(t) > 0, i = 1,2$ a.s., we have for $i = 1,2$

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t x_i(s)ds = 0 \ a.s.$$

The theorem is proved.

Definition 3.3. ([7]) The population $x_i(t), i = 1,2$ will be said weakly persistent in the mean if for every initial data $x_{i0} > 0, i = 1,2$, we have

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)ds > 0, \ a.s., \ i = 1,2$$

Theorem 3.3. *If $\bar{p}_2^* > 0$, where $p_2(t) = -a_2(t) - \beta_2(t)$, then the predator population density $x_2(t)$ with the initial condition $x_{20} > 0$ will be weakly persistence in the mean.*

Proof. If the assertion of theorem is not true, then $\mathbf{P}\{\omega \in \Omega \mid \bar{x}_2^* = 0\} > 0$. From the first equality in (6) we get

$$\frac{1}{t}(\ln x_2(t) - \ln x_{20}) + \frac{1}{t}\int_0^t b_2(s)x_2(s)ds = \frac{1}{t}\int_0^t p_2(s)ds + \frac{1}{t}\int_0^t \frac{c_2(s)x_1(s)}{1+m(s)x_1(s)}ds + \frac{M_2(t)}{t} \geqslant \frac{1}{t}\int_0^t p_2(s)ds + \frac{M_2(t)}{t},$$

where martingale $M_2(t)$ is defined in (7). For $\forall\omega \in \{\omega \in \Omega | \ \bar{x}_2^* = 0\}$ in virtue of the strong law of large numbers for martingale $M_2(t)$ we have

$$\limsup_{t\to\infty} \frac{\ln x_2(t)}{t} \geqslant \bar{p}_2^* > 0.$$

Therefore

$$\mathbf{P}\left\{\omega \in \Omega | \ \limsup_{t\to\infty} \frac{\ln x_2(t)}{t} > 0\right\} > 0.$$

But from Lemma 2.2 we have

$$\mathbf{P}\left\{\omega \in \Omega \mid \limsup_{t\to\infty} \frac{\ln x_2(t)}{t} \leqslant 0\right\} = 1.$$

This is a contradiction. The theorem is proved.

Theorem 3.4. *If $\bar{q}_1^* > 0$ and $\bar{q}_2^* \leqslant 0$, then the prey population density $x_1(t)$ with initial condition $x_{10} > 0$ will be weakly persistence in the mean.*

Proof. Let $\mathbf{P}\{\bar{x}_1^* = 0\} > 0$. From the first equality in (6) we get

$$\frac{1}{t}(\ln x_1(t) - \ln x_{10}) + \frac{1}{t}\int_0^t b_1(s)x_1(s)ds = \frac{1}{t}\int_0^t q_1(s)ds - \frac{1}{t}\int_0^t \frac{c_1(s)x_2(s)}{m(s)+x_1(s)}ds + \frac{M_1(t)}{t} \geqslant \frac{1}{t}\int_0^t q_1(s)ds - \frac{c_{1\sup}}{m_{\inf}t}\int_0^t x_2(s)ds + \frac{M_1(t)}{t} \quad (10)$$

where martingale $M_1(t)$ is defined in (7). From the strong law of large numbers for martingale $M_1(t)$ and the results of Theorem 3.1 and Theorem 3.2 for the predator population density $x_2(t)$ we obtain from (10)

$$\limsup_{t\to\infty} \frac{\ln x_1(t)}{t} \geqslant \bar{q}_1^* > 0$$

for $\forall\omega \in \{\omega \in \Omega | \ \bar{x}_2^* = 0\}$. Therefore

$$\mathbf{P}\left\{\omega \in \Omega | \ \limsup_{t\to\infty} \frac{\ln x_1(t)}{t} > 0\right\} > 0.$$

But from Lemma 2.2 we have

$$\mathbf{P}\left\{\omega \in \Omega | \ \limsup_{t\to\infty} \frac{\ln x_1(t)}{t} \leqslant 0\right\} = 1.$$

So we have a contradiction. The theorem is proved.

Definition 3.4. ([7]) The population density $x(t)$, $t \geqslant 0$ is said to be strongly persistence in the mean if for every initial data $x(0) > 0$, we have $\bar{x}_* > 0$ a.s.

Theorem 3.5. *If*

$$\bar{p}_{2*} = \liminf_{t\to\infty} \frac{1}{t}\int_0^t p_2(s)\,ds > 0,$$

then for every initial data $x_{20} > 0$ we have

$$\bar{x}_{2*} = \liminf_{t\to\infty} \frac{1}{t}\int_0^t x_2(s)ds \geqslant \frac{\bar{p}_{2*}}{b_{2\sup}}, \quad a.s. \quad (11)$$

Therefore predator population density $x_2(t)$ will be strongly persistence in the mean.

Proof. Applying the Itô's formula to the process $\ln x_2(t)$, we obtain

$$\ln x_2(t) = \ln x_{20} + \int_0^t p_2(s)ds + \int_0^t \frac{c_2(s)x_2(s)}{1+m(s)x_1(s)}ds - \int_0^t b_2(s)x_2(s)ds + M_2(t) \geqslant \ln x_{20} + \int_0^t p_2(s)\,ds - b_{2\sup}\int_0^t x_2(s)\,ds + M_2(t), \quad (12)$$

where martingale $M_1(t)$ is defined in (7).

From definition of $\bar{p}_{2*}$, strong law of large numbers for $M_2(t)$ it follows that $\forall\varepsilon > 0$, $\exists t_0 \geqslant 0$, $\exists\Omega_\varepsilon \subset \Omega$, such that $\mathbf{P}(\Omega_\varepsilon) \geqslant 1-\varepsilon$,

$$\frac{1}{t}\int_0^t p_2(s)\,ds \geqslant \bar{p}_{2*} - \frac{\varepsilon}{2}, \frac{M_2(t)}{t} \geqslant -\frac{\varepsilon}{2},$$

$\forall t \geqslant t_0, \forall \omega \in \Omega_\varepsilon$. So from (12) we have

$$\ln x_2(t) \geqslant \ln x_{20} + t(\bar{p}_{2*} - \varepsilon) - b_{2\sup} \int_0^t x_2(s)\,ds$$

$\forall t \geqslant t_0, \forall \omega \in \Omega_\varepsilon$. Therefore for the process

$$y_2(t) = \int_0^t x_2(s)\,ds$$

we have inequality

$$\ln\left(\frac{dy_2(t)}{dt}\right) \geqslant (\bar{p}_{2*} - \varepsilon)t - b_{2\sup} y_2(t) + \ln x_{20},$$

$\forall t \geqslant t_0, \forall \omega \in \Omega_\varepsilon$.

Hence

$$e^{b_{2\sup} y_2(t)} \frac{dy_2(t)}{dt} \geqslant x_{20} e^{(\bar{p}_{2*} - \varepsilon)t},$$

$\forall t \geqslant t_0, \forall \omega \in \Omega_\varepsilon$. Integrating the last inequality from t_0 to t and using obvious calculations, yields

$$\frac{1}{t}\int_0^t x_2(s)ds \geqslant \frac{1}{b_{2\sup} t} \ln\Big[e^{b_{2\sup} y_2(t_0)} + \frac{b_{2\sup} x_{20}}{\bar{p}_{2*} - \varepsilon}\left(e^{(\bar{p}_{2*}-\varepsilon)t} - e^{(\bar{p}_{2*}-\varepsilon)t_0}\right)\Big]$$

$\forall t \geqslant t_0, \forall \omega \in \Omega_\varepsilon$. So

$$\bar{x}_{2*} = \liminf_{t\to\infty} \frac{1}{t}\int_0^t x_2(s)\,ds \geqslant \frac{\bar{p}_{2*} - \varepsilon}{b_{2\sup}}, \ \forall \omega \in \Omega_\varepsilon.$$

Using the arbitrariness of $\varepsilon > 0$, we get (11). The theorem is proved.

Theorem 3.6. *If $\bar{q}_{1*} > 0$ and $\bar{q}_2^* \leqslant 0$, then*

$$\bar{x}_{1*} = \liminf_{t\to\infty} \frac{1}{t}\int_0^t x_1(s)\,ds \geqslant \frac{\bar{q}_{1*}}{b_{1\sup}} \quad a.s. \qquad (13)$$

Hence the prey population density $x_1(t)$ will be strongly persistence in the mean.

Proof. By the Itô's formula for the process $\ln x_1(t)$, we obtain

$$\begin{aligned}\ln x_1(t) &= \ln x_{10} + \int_0^t q_1(s)\,ds - \int_0^t b_1(s)x_1(s)\,ds \\ &- \int_0^t \frac{c_1(s)x_2(s)}{1+m(s)x_1(s)}\,ds + M_1(t) \geqslant \ln x_{10} \\ &+ \int_0^t q_1(s)\,ds - b_{1\sup}\int_0^t x_1(s)\,ds \\ &- c_{1\sup}\int_0^t x_2(s)\,ds + M_1(t),\end{aligned} \qquad (14)$$

where martingale $M_1(t)$ is defined in (7).

From definition of $\bar{q}_{1*}$, strong law of large numbers for $M_1(t)$, Theorem 3.1 and Theorem 3.2 it follows that $\forall \varepsilon > 0$, $\exists t_0 \geqslant 0$, $\exists \Omega_\varepsilon \subset \Omega$, such that $\mathbf{P}(\Omega_\varepsilon) \geqslant 1 - \varepsilon$,

$$\frac{1}{t}\int_0^t q_1(s)\,ds \geqslant \bar{q}_{1*} - \frac{\varepsilon}{3}, \frac{M_1(t)}{t} \geqslant -\frac{\varepsilon}{3},$$
$$\frac{1}{t}\int_0^t x_2(s)\,ds \leqslant \frac{\varepsilon}{3c_{2\sup}}$$

$\forall t \geqslant t_0, \forall \omega \in \Omega_\varepsilon$. So from (14) we have

$$\ln x_1(t) \geqslant \ln x_{10} + t(\bar{q}_{1*} - \varepsilon) - b_{1\sup}\int_0^t x_1(s)\,ds$$

$\forall t \geqslant t_0, \forall \omega \in \Omega_\varepsilon$. Therefore for the process

$$y_1(t) = \int_0^t x_1(s)\,ds$$

we have inequality

$$\ln\left(\frac{dy_1(t)}{dt}\right) \geqslant (\bar{q}_{1*} - \varepsilon)t - b_{1\sup} y_1(t) + \ln x_{10},$$

$\forall t \geqslant t_0, \forall \omega \in \Omega_\varepsilon$.

Hence

$$e^{b_{1\sup} y_1(t)} \frac{dy_1(t)}{dt} \geqslant x_{10} e^{(\bar{q}_{1*} - \varepsilon)t},$$

$\forall t \geqslant t_0, \forall \omega \in \Omega_\varepsilon$. Integrating the last inequality from t_0 to t we obtaine

$$\frac{1}{t}\int_0^t x_1(s)ds \geqslant \frac{1}{b_{1\sup} t} \ln\Big[e^{b_{1\sup} y_1(t_0)} + \frac{b_{1\sup} x_{10}}{\bar{q}_{1*} - \varepsilon}\left(e^{(\bar{q}_{1*}-\varepsilon)t} - e^{(\bar{q}_{1*}-\varepsilon)t_0}\right)\Big]$$

$\forall t \geqslant t_0, \forall \omega \in \Omega_\varepsilon$. So

$$\bar{x}_{1*} = \liminf_{t\to\infty} \frac{1}{t}\int_0^t x_1(s)\,ds \geqslant \frac{\bar{q}_{1*} - \varepsilon}{b_{1\sup}}, \ \forall \omega \in \Omega_\varepsilon.$$

From the arbitrariness of $\varepsilon > 0$, we get (13). The theorem is proved.

4 Conclusion

In this paper we derive the sufficient conditions for the extinction, non-persistence in the mean, weak and strong persistence in the mean of predator and prey populations in the non-autonomous stochastic density dependent predator-prey model with Holling-type II functional response driven by the system of stochastic differential equations with white noise, centered and non-centered

Poisson noises. So, in this model we take into account levels of predator density dependence and not only "small" jumps, corresponding to the centered Poisson measure, but also the "large" jumps, corresponding to the non-centered Poisson measure. Under the weak persistence in the mean $\liminf_{t\to\infty} \bar{x}_i(t) = 0, i = 1,2$ is allowed but is not allowed under the strong persistence in the mean, which means that the survival in Theorem 3.5 and in Theorem 3.6 is stronger than in Theorem 3.3 and Theorem 3.4.

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